Grid-Based Graphs, Linear Realizations, and the Buratti-Horak-Rosa (BHR) Conjecture

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1. Introduction to the Buratti-Horak-Rosa Conjecture

Initial Problem *(by Marco Buratti)*

Listing all the elements in $\mathbb{Z}_v$ with given (multiset of) minimal cyclic step-lengths.

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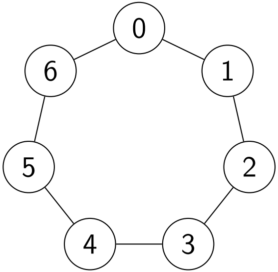

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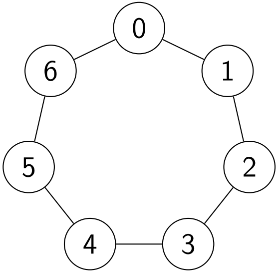

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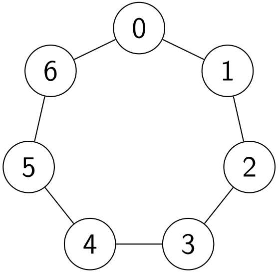

Example multiset: $\{1, 1, 2, 2, 2, 3\} = \{1^2, 2^3, 3^1\}$

Example list: $[0, 2, 3, 1, 5, 6, 4]$

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Finding Hamiltonian paths in K_v with given (multiset of) cyclic edge-lengths.

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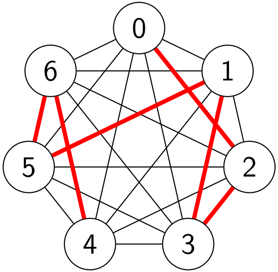

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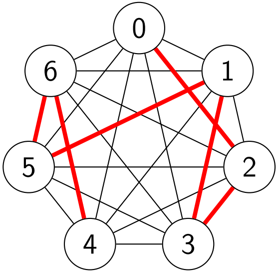

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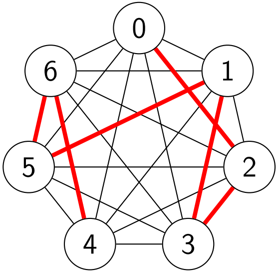

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Example path: $[0, 2, 3, 1, 5, 6, 4]$

2. The Buratti-Horak-Rosa Conjecture

For L multiset of $v-1$ positive integers not exceeding $\lfloor v/2 \rfloor$

a. If v is prime, then K_v has a Hamiltonian path with cyclic edge-lengths as L (i.e. L has a **realization**).

   (conjecture by Marco Buratti in 2007)

2. The Buratti-Horak-Rosa Conjecture

For L multiset of $v-1$ positive integers not exceeding $\lfloor v/2 \rfloor$

a. If v is prime, then K_v has a Hamiltonian path with cyclic edge-lengths as L (i.e. L has a **realization**).

   (conjecture by Marco Buratti in 2007)

b. There is a realization for L if and only if for any divisor d of v, the number of multiples of d in L is at most $v-d$.

   (conjecture by Peter Horak and Alexander Rosa in 2009,
   reformulated by Anita Pasotti and Marco Pellegrini in 2014)

3. Known Results on the Buratti-Horak-Rosa Conjecture

$|U| = 1$: Trivial. (**Note:** U is the underlying set of L)

$|U| = 2$: Solved!

For prime v: Jeff Dinitz and Susan Janiszewski (2009)

For prime v & all v: Peter Horak and Alexander Rosa (2009)

$|U| = 3$: Complete solutions for specific choices of U:

Stefano Capparelli and Alberto Del Fra (2010):
$\{1, 2, 3\}$

Anita Pasotti and Marco Pellegrini (2014):
$\{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 2, 8\}, \{1, 3, 5\}$

Matt Ollis, A. Pasotti, M. Pellegrini, John Schmitt (2022):
$\{1, 4, 5\}$

Pranit Chand and Matt Ollis (2022):
$\max(U) \leq 7$

$|U| = 3$: Partial families for many U:

Anita Pasotti and Marco Pellegrini (2014):

$L = \{1^a, 2^b, x^c\}$ when x is even and $a + b \geq x - 1$

Adrian Vazquez Avila (2022):

$L = \{1^a, 2^{x-1}, x\}$

Matt Ollis, A. Pasotti, M. Pellegrini, John Schmitt (2021):

$L = \{1^a, x^b, (x+1)^c\}$ when x is odd and
either $a \geq \min(3x-3, b+2x-3)$ or $a \geq 2x-2$ and $c \geq 4b/3$

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$L = \{1^a, x^b, y^c\}$ when x is even, $x < y$ and
either y is even and $a \geq y-1$ or y is odd and $a \geq 3y-4$

Peter Horak and Alexander Rosa (2009), followed by
Matt Ollis, A. Pasotti, M. Pellegrini, John Schmitt (2022):

$L = \{1^a, x^b, y^c\}$ where $x < y$ and $a \geq x + 4y - 5$

3. Known Results on the Buratti-Horak-Rosa Conjecture

$|U| > 3$: Another partial list:

Peter Horak and Alexander Rosa (2009):
$v = 2m + 1$ is prime and $L = (\{1^2, 2^2, \ldots, m^2\} \cup \{x\}) \setminus \{y\}$

Peter Horak and Alexander Rosa (2009), followed by
Matt Ollis, A. Pasotti, M. Pellegrini, John Schmitt (2021):
$L \cup \{1^s\}$ for some constant s depending only on U

Anita Pasotti and Marco Pellegrini (2014), followed by
Matt Ollis, A. Pasotti, M. Pellegrini, John Schmitt (2021):
$U = \{1, 2, 3, 4\}, \{1, 2, 3, 5\}$

Matt Ollis, A. Pasotti, M. Pellegrini, John Schmitt (2021):
Partial results for $U \subseteq \{1, 2, 4, \ldots, 2x, 2x + 1\}$;
$\{1^{a_1}, 2^{a_2}, \ldots, x^{a_x}\}$ when $a_1 \geq a_2 \geq \cdots \geq a_x$

Brendon McKay and Tim Peters (2022):
$|L| < 37$

4. Recent Results on the Buratti-Horak-Rosa Conjecture

$|U| = 3$: A., Ollis (2024):

$U = \{1, x, x+1\}$ and $v \geq 2x^2 + 13x + 11$ with $\gcd(v, x) = \gcd(v, x+1) = 1$

$U = \{1, 2, x\}$ and $v > 4x$ with $\gcd(v, x) = 1$
(except possibly when x is odd and $a \in \{1, 2\}$ for $L = \{1^a, 2^b, x^c\}$)

For the first time:
Infinitely many U for which there are infinitely many values of v where the conjecture holds.

Maybe a first step towards solutions for infinitely many U?

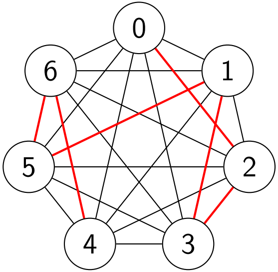
(Similar techniques also provide partial families for more U.)

$|U| > 3$: A., Ollis (2024):

$L = \{1^a, x^b, (x+1)^c, y^d\}$ when $a \geq x + y + 1$

$L = \{1^a, x^b, (x+1)^c, y^d, (y+1)^e\}$ when $a \geq x + y + 2$

Back to Our Example



Given multiset: $L = \{1, 1, 2, 2, 2, 3\} = \{1^2, 2^3, 3^1\}$

A realization like $\Gamma = [0, 2, 3, 1, 5, 6, 4]$ gives rise to many!

Induced Realizations from the Given Realization for $L = \{1^2, 2^3, 3^1\} = \{1, 1, 2, 2, 2, 3\}$

Directed Edge Labels		+2	+1	-2	-3	+1	-2	
Our realization, Γ	0	2	3	1	5	6	4	
Translations	1	3	4	2	6	0	5	
Translations	2	4	5	3	0	1	6	
Translations	3	5	6	4	1	2	0	*"Linear"*
Translations	4	6	0	5	2	3	1	
Translations	5	0	1	6	3	4	2	
Translations	6	1	2	0	4	5	3	

Directed Edge Labels		+2	-1	+3	+2	-1	-2	
Reverse of Γ *(Converse/Transpose)*	4	6	5	1	3	2	0	
Translations	5	0	6	2	4	3	1	
Translations	6	1	0	3	5	4	2	
Translations	0	2	1	4	6	5	3	*"Standard"*
Translations	1	3	2	5	0	6	4	
Translations	2	4	3	6	1	0	5	
Translations	3	5	4	0	2	1	6	

Directed Edge Labels		-2	-1	+2	+3	-1	+2	
Complement of Γ	6	4	3	5	1	0	2	
Translations	5	3	2	4	0	6	1	
Translations	4	2	1	3	6	5	0	
Translations	3	1	0	2	5	4	6	*"Linear"*
Translations	2	0	6	1	4	3	5	
Translations	1	6	5	0	3	2	4	
Translations	0	5	4	6	2	1	3	

Directed Edge Labels		-2	+1	-3	-2	+1	+2	
Complement & Reverse	2	0	1	5	3	4	6	
Translations	1	6	0	4	2	3	5	
Translations	0	5	6	3	1	2	4	
Translations	6	4	5	2	0	1	3	*"Linear"*
Translations	5	3	4	1	6	0	2	
Translations	4	2	3	0	5	6	1	
Translations	3	1	2	6	4	5	0	

Other Realizations for the Same L

Different Directed Edge Labels		+2	+1	-2	-3	-1	+2	
Another Realization	0	2	3	1	5	4	6	***etc.***

A Special Type of Realization (for a Different L)

Different Directed Edge Labels		+2	-1	+3	+1	-2	**+3**	
A Special Realization	0	2	1	4	5	3	6	*"Perfect"*

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Theorem

For all $v \geq 2x^2 + 13x + 11$ *with* $\gcd(v, x) = \gcd(v, x+1) = 1$,
the Buratti-Horak-Rosa Conjecture holds for multisets with support $\{1, x, x+1\}$.

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The theorem follows from the construction via straightforward modular arithmetic, as demonstrated by the following example.

6. Main Result for $U = \{1, x, x+1\}$

Example

Let $v = 101$ and let $U = \{1, 7, 8\}$.

In mod 101:

$7^{-1} = 29$ and $\{1, 7, 8\} \xmapsto{\times 29} \{29, 1, 30\}$

$8^{-1} = 38$ and $\{1, 7, 8\} \xmapsto{\times 38} \{38, 64, 1\} \equiv \{38, 37, 1\}$

Thus, $\{1^a, 7^b, 8^c\}$, $\{29^a, 1^b, 30^c\}$, and $\{38^a, 37^b, 1^c\}$ are equivalent.

Hence, we can construct a realization whenever $a \geq 8$, $b \geq 30$, or $c \geq 38$.

Therefore, a counterexample would need $a + b + c < 8 + 30 + 38 = 76$.

This contradicts with $a + b + c = v - 1 = 100$.

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Therefore, a counterexample would need $a + b + c < 8 + 30 + 38 = 76$.

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So what remains is to describe the constructions of needed the standard linear realizations.

7. Construction Tool: Using Grid Graphs to "Optimize" for $U = \{1, x\}$

$\omega(x, b)$: the smallest possible number of 1-edges used in a standard linear realization with b x-edges and support $U = \{1, x\}$

Method: To find $\omega(x, b)$, we work on the subgraph of K_v with only edges of length 1 & x, drawn on a grid, where vertex labels differ horizontally by 1 and vertically by x.

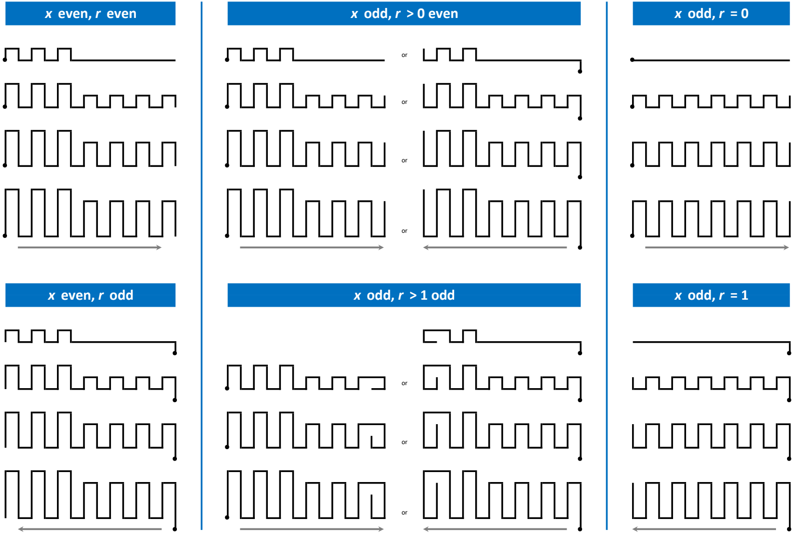
Theorem

For given $b \geq 0$ and $x > 1$ with Euclidean division of $b = qx + r$,

$$\omega(x, b) = \begin{cases} x, & \text{if } x \text{ and } r \text{ are both odd and } r > 1 \\ x - 1, & \text{otherwise.} \end{cases}$$

Notes: The constructions yield **growable** realizations.
The exceptional case is due to a "tail-curl" that is needed.

Optimal Constructions (i.e. minimal a) for Standard Linear Realizations of the Multisets $\{1^a, x^c\}$, where $c = qx + r$ with $r < x$



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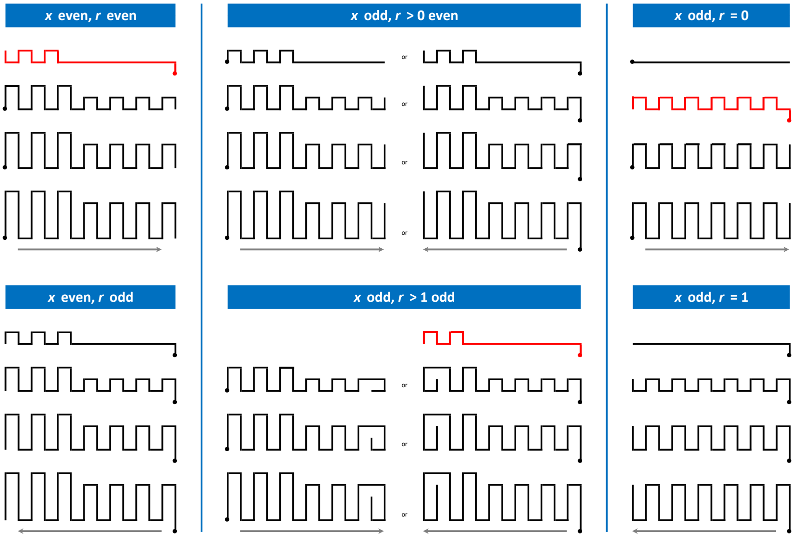

8. Construction Technique I: Path Concatenations

g $\oplus$ **h**: The complement of **g** (with v vertices) and the translation of **h** by $v-1$, identified at the end-vertices labeled with $v-1$.

Use: Realizations on K_v and K_w to a realization on K_{v+w-1}:

1. standard $\oplus$ standard $\implies$ linear
2. standard $\oplus$ perfect $\implies$ standard
3. perfect $\oplus$ perfect $\implies$ perfect

Note: For the multiset $\{1^s\}$, $[0, 1, \ldots, s-1]$ is a perfect realization.

Lemma: If L has a standard (respectively perfect) realization then $L \cup \{1^s\}$ has a standard (respectively perfect) realization $\forall s \geq 0$.

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A perfect realization can be constructed for $L = \{x^x, (x+1)^{x+1}\}$.

Technique

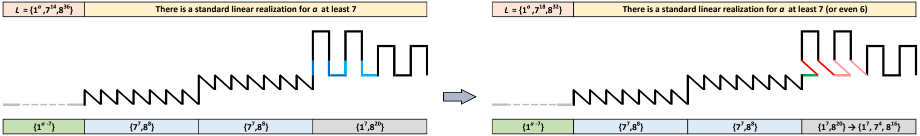
To construct realizations with $a \geq x+1$ for $L = \{1^a, x^b, (x+1)^c\}$ with given b and c, use concatenated perfect realizations to reduce to b' and c', followed by modifications on the optimal constructions for either $U = \{1, x\}$ or $U = \{1, x+1\}$, as needed.

Constructing realizations with $a \geq 7$ for $L = \{1^a, 7^{18}, 8^{32}\}$

$$\{1^a, 7^{18}, 8^{32}\} = \{7^7, 8^8\} \cup \{7^7, 8^8\} \cup \{1^a, 7^4, 8^{16}\}$$

Since $4 < 16$, we use modifications on the optimal construction for $\{1^a, 8^{4+16}\}$:

$$\{1^7, 8^{4+16}\} \setminus \{1^1, 8^2\} \cup \{1^1, 7^2\} \setminus \{1^1, 8^2\} \cup \{1^1, 7^2\} = \{1^7, 7^4, 8^{16}\}$$

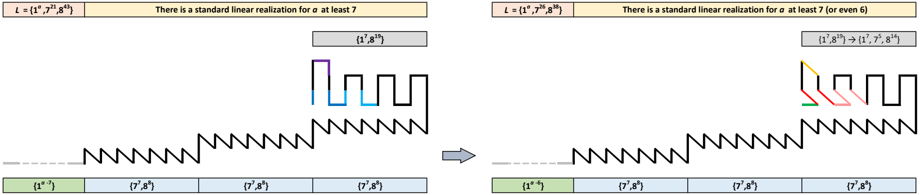


Constructing realizations with $a \geq 6$ for $L = \{1^a, 7^{26}, 8^{38}\}$

$$\{1^a, 7^{26}, 8^{38}\} = \{7^7, 8^8\} \cup \{7^7, 8^8\} \cup \{7^7, 8^8\} \cup \{1^a, 7^5, 8^{14}\}$$

Since $5 < 14$, we use modifications on the optimal construction for $\{1^a, 8^{5+14}\}$:

$$\{1^7, 8^{5+14}\} \setminus \{1^1, 8^2\} \cup \{1^1, 7^2\} \setminus \{1^1, 8^2\} \cup \{1^1, 7^2\} \setminus \{1^1, 8^1\} \cup \{7^1\} = \{1^6, 7^5, 8^{14}\}$$

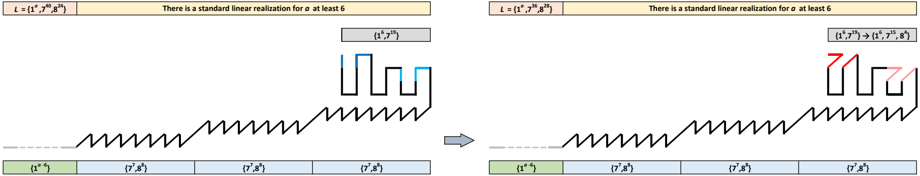


Constructing realizations with $a \geq 6$ for $L = \{1^a, 7^{36}, 8^{28}\}$

$$\{1^a, 7^{36}, 8^{28}\} = \{7^7, 8^8\} \cup \{7^7, 8^8\} \cup \{7^7, 8^8\} \cup \{1^a, 7^{15}, 8^4\}$$

Since 15 > 4, we use modifications on the optimal construction for $\{1^a, 7^{15+4}\}$

$$\{1^6, 7^{15+4}\} \setminus \{1^1, 7^2\} \cup \{1^1, 8^2\} \setminus \{1^1, 7^2\} \cup \{1^1, 8^2\} = \{1^6, 7^{15}, 8^4\}$$

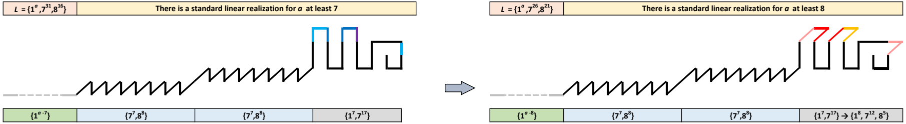


Constructing realizations with $a \geq 8$ for $L = \{1^a, 7^{26}, 8^{21}\}$

$$\{1^a, 7^{26}, 8^{21}\} = \{7^7, 8^8\} \cup \{7^7, 8^8\} \cup \{1^a, 7^{12}, 8^5\}$$

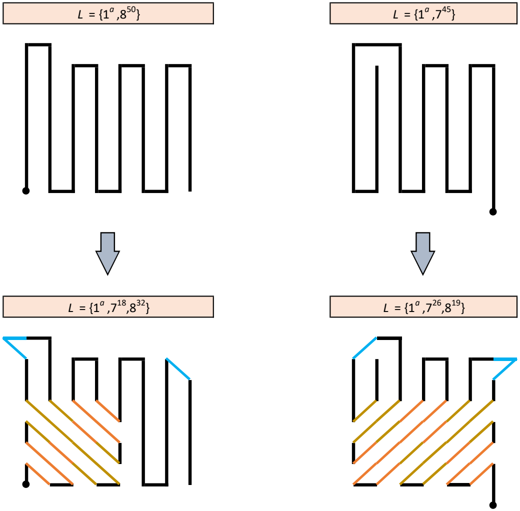
Since 12 > 5, we use modifications on the optimal construction for $\{1^a, 7^{12+5}\}$

$$\{1^7, 7^{12+5}\} \setminus \{1^1, 7^2\} \cup \{1^1, 8^2\} \setminus \{1^1, 7^2\} \cup \{1^1, 8^2\} \setminus \{7^1\} \cup \{1^1, 8^1\} = \{1^8, 7^{15}, 8^4\}$$



Strategy

Use block modifications on the optimal constructions for either $U = \{1, x\}$ (if $a \geq b$) or $U = \{1, x + 1\}$ (if $b \geq$ a).


L = {1^a ,8^50}
L = {1^a ,7^45}
L = {1^a ,7^18,8^32}
L = {1^a ,7^26,8^19}

10. Extensions to Other Cases of $|U| = 3$

Conjectures

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1. A concatenated linear realization can be constructed for $L = \{1^a, (y-k)^b, y^c\}$ whenever $a \geq y + k - 1$.
   (The realization is standard if k is odd.)
2. A non-concatenated standard linear realization can be constructed for $L = \{1^a, (y-k)^b, y^c\}$ and $L = \{1^a, k^b, y^c\}$ whenever $a \geq y$.

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For $k \leq \lfloor v/2 \rfloor$,

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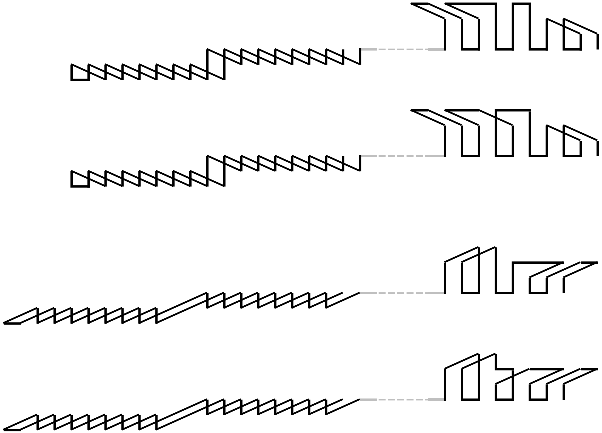
   (The realization is standard if k is odd.)

2. A non-concatenated standard linear realization can be constructed for $L = \{1^a, (y-k)^b, y^c\}$ and $L = \{1^a, k^b, y^c\}$ whenever $a \geq y$.

Current progress towards these conjectures :

$U = \{1, y-2, y\}$ (mostly resolved), $U = \{1, y-3, y\}$ (partial results),
$U = \{1, 2, y\}$ (mostly resolved), $U = \{1, 3, y\}$ (partial results),
$U = \{1, x, 2x+1\}$ (mostly resolved), $U = \{1, x, 2x-1\}$ (partial results).

Examples of Concatenated Constructions for U = {1, *y* - 2, *y* }



¡Muchas Gracias y Vámonos!