Linear Systems of Conics over Finite Fields

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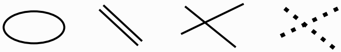
Combinatorial Designs and Codes
(CODESCO'24)
University of Seville

July 8, 2024

Linear Systems of Conics:

Non-empty conics in $\mathrm{PG}(2,\mathbb{F})$:



Linear systems of conics := **Subspaces**(PG(2-forms in the projective plane)).

- a pencil of conic $\mathcal{P} = \langle C_1, C_2 \rangle$ or (f_1, f_2).
- a net of conics $\mathcal{N} = \langle C_1, C_2, C_3 \rangle$ or (f_1, f_2, f_3).
- a web of conics $\mathcal{W} = \langle C_1, C_2, C_3, C_4 \rangle$ or (f_1, f_2, f_3, f_4).
- a squab of conics $\mathcal{W} = \langle C_1, C_2, C_3, C_4, C_5 \rangle$ or $(f_1, f_2, f_3, f_4, f_5)$.

HISTORY:

- Jordan (1906): Classified pencils of conics over $\mathbb{R}$.
- Jordan (1907): Classified pencils of conics over $\mathbb{C}$.
- Wall (1977): Classified nets of conics over $\mathbb{R}$ and $\mathbb{C}$.
- Dickson (1908): Classified pencils of conics over $\mathbb{F}_q$, q odd.
- Wilson (1914): Partially classified rank-one nets of conics (nets with at least a $//$) over $\mathbb{F}_q$, q odd.
- Campbell (1927): Partially classified pencils of conics over $\mathbb{F}_q$, q even.
- Campbell (1928): Partially classified nets of conics over $\mathbb{F}_q$, q even.

For an explanation of some of the shortcomings of Wilson's and Campbell's treatments, we refer to [M. Lavrauw, T. Popiel, J. Sheekey, 2020] for q odd, and to [NA, M. Lavrauw, T. Popiel, 2022] and [NA, M. Lavrauw, 2023] for q even. (Example: Pencils of conics with conic distribution $[0, 0, 1, q]$, q even).

Embracing a New Approach!

- A purely computational approach will unlikely lead to much further progress.
- Projectively inequivalent linear systems of conics in $\mathrm{PG}(2,q) \iff$ representatives of the K-orbits of subspaces of $\mathrm{PG}(5,q)$.
- $K \cong \mathrm{PGL}(3,q)$, $q \neq 2$, is the subgroup of $\mathrm{PGL}(6,q)$ stabilizing the Veronese surface $\mathcal{V}(\mathbb{F}_q)$:

$$\nu : (x_0, x_1, x_2) \mapsto (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2).$$

- $C = \mathcal{Z}(a_{00}X_0^2 + a_{01}X_0X_1 + a_{02}X_0X_2 + a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2)$ $\iff H[a_{00}, a_{01}, a_{02}, a_{11}, a_{12}, a_{22}] \cap \mathcal{V}(\mathbb{F}_q)$.

- a pencil of conic in $\mathrm{PG}(2,q) \iff$ a solid of $\mathrm{PG}(5,q)$.
- a net of conics in $\mathrm{PG}(2,q) \iff$ a plane of $\mathrm{PG}(5,q)$.
- a web of conics in $\mathrm{PG}(2,q) \iff$ a line of $\mathrm{PG}(5,q)$.
- a squab of conics in $\mathrm{PG}(2,q) \iff$ a point of $\mathrm{PG}(5,q)$.

PROGRESS!

- lines, for all q: ✓ [M. Lavrauw, T. Popiel, 2020]
- solids, for q odd: ✓ [M. Lavrauw, T. Popiel, 2020]
- planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially, for q odd: ✓ [M. Lavrauw, T. Popiel, J. Sheekey, 2020]
- solids, for q even: ✓ [NA, M. Lavrauw, T. Popiel, 2022]
- planes meeting $\mathcal{V}(\mathbb{F}_q)$ non-trivially, for q even: ✓ [NA, M. Lavrauw, 2023]

Remaining case:

- Planes meeting $\mathcal{V}(\mathbb{F}_q)$ trivially.
- Nets in $\mathrm{PG}(2, q)$:
  - q odd: Nets having no double lines ($\exists$ a polarity: *the set of conic planes of $\mathcal{V}(\mathbb{F}_q)$ $\rightarrow$ the set of tangent planes of $\mathcal{V}(\mathbb{F}_q)$*).
  - q even: Nets with empty bases.

MORE THAN A CLASSIFICATION!

We seek:

- a representative,
- a uniqueness argument,

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We seek:

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- a uniqueness argument,
- a set of geometric and combinatorial invariants that completely distinguish between different orbits. ✓
- understanding interrelations between different orbits: linear systems/subspaces. ✓ ✓

REPRESENTATIONS:

- Every point $x = (x_0, .., x_5) \in \mathrm{PG}(5, q)$ can be represented by

$$M_x = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{bmatrix}$$

- The line $\ell \subset \mathrm{PG}(5, q)$ spanned by the 1st two points of the standard frame is

$$\ell = \begin{bmatrix} x & y & . \\ y & . & . \\ . & . & . \end{bmatrix} := \left\{ \begin{bmatrix} x & y & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \ (x, y) \in \mathrm{PG}(1, q) \right\}.$$

- We denote by $\mathcal{W} = (f_1, f_2, f_3, f_4)$ the web of conics

$$\mathcal{Z}(af_1 + bf_2 + cf_3 + df_4), (a, b, c, d) \in \mathrm{PG}(3, q).$$

- **Example:** The web of conics associated with the above line is

$$(X_0X_2, X_1^2, X_1X_2, X_2^2).$$

- Its associated **cubic surface** is the zero locus $\mathcal{Z}(\Delta_f)$ in $\mathrm{PG}(3, q)$ of the discriminant $\Delta_f \in \mathbb{F}_q[a, b, c, d]$ of the quadratic form

$$f = af_1 + bf_2 + cf_3 + df_4.$$

$$\Delta_f = 4a_{00}a_{11}a_{22} + a_{01}a_{02}a_{12} - a_{00}a_{12}^2 - a_{11}a_{02}^2 - a_{22}a_{01}^2.$$

K-ORBITS INVARIANTS:

Let W be a subspace of $\mathrm{PG}(5,q)$.
Let U_1, U_2,...,U_m denote the distinct K-orbits of r-spaces in $\mathrm{PG}(5,q)$.

- The **rank distribution of** W is

  $$[r_1, r_2, r_3],$$

  where

  $$r_i = \# \text{ of rank } i \text{ points in } W.$$

- The r**-space orbit-distribution of** W is

  $$[u_1, u_2, \ldots, u_m],$$

  where

  u_i = # of r-spaces incident with W which belong to the orbit U_i.

LINES IN $\mathrm{PG}(5,q)$, q ODD:

Orbits	Point-OD's :$[r_1, r_{2e}, r_{2i}, r_3]$
o_5	$[2, \frac{q-1}{2}, \frac{q-1}{2}, 0]$
o_6	$[1, q, 0, 0]$
$o_{8,1}$	$[1, 1, 0, q-1]$
$o_{8,2}$	$[1, 0, 1, q-1]$
o_9	$[1, 0, 0, q]$
o_{10}	$[0, \frac{q+1}{2}, \frac{q+1}{2}, 0]$
$o_{12,1}$	$[0, q+1, 0, 0]$
$o_{13,1}$	$[0, 2, 0, q-1]$
$o_{13,2}$	$[0, 1, 1, q-1]$
$o_{14,1}$	$[0, 3, 0, q-2]$
$o_{14,2}$	$[0, 1, 2, q-2]$
$o_{15,1}$	$[0, 1, 0, q]$
$o_{15,2}$	$[0, 0, 1, q]$
$o_{16,1}$	$[0, 1, 0, q]$
o_{17}	$[0, 0, 0, q+1]$

Table: K-orbits of lines in $\mathrm{PG}(5,q)$, q odd [M. Lavrauw, T. Popiel, 2020].

Lines in $\mathrm{PG}(5,q)$, q even:

Orbits	Point-OD's : $[r_1, r_{2n}, r_{2s}, r_3]$
o_5	$[2,0,q-1,0]$
o_6	$[1,1,q-1,0]$
$o_{8,1}$	$[1,0,1,q-1]$
$o_{8,3}$	$[1,1,0,q-1]$
o_9	$[1,0,0,q]$
o_{10}	$[0,0,q+1,0]$
$o_{12,1}$	$[0,q+1,0,0]$
$o_{12,3}$	$[0,1,q,0]$
$o_{13,1}$	$[0,1,1,q-1]$
$o_{13,3}$	$[0,0,2,q-1]$
$o_{14,1}$	$[0,0,3,q-2]$
$o_{15,1}$	$[0,0,1,q]$
$o_{16,1}$	$[0,1,0,q]$
$o_{16,3}$	$[0,0,1,q]$
o_{17}	$[0,0,0,q+1]$

Table: K-orbits of lines in $\mathrm{PG}(5,q)$, q even [M. Lavrauw, T. Popiel, 2020].

MAIN RESULTS:

Subspaces of $\mathrm{PG}(5,q)$:

The determination of the distribution of the different types of hyperplanes incident with the K-orbit representatives of points and lines of $\mathrm{PG}(5,q)$.

Linear systems in $\mathrm{PG}(2,q)$:

The determination of the distribution of the different types of conics contained in projectively inequivalent webs and squabs of conics in $\mathrm{PG}(2,q)$.

In the remaining part of the talk, we will discuss various results concerning webs and their connections.

Webs of conics, q odd:

L^K	Webs of Conics	$OD_4(L)$
o_5	$(X_0X_1, X_0X_2, X_1X_2, X_2^2)$	$[1, 2q^2+q, 0, q^3-q^2]$
o_6	$(X_0X_2, X_1^2, X_1X_2, X_2^2)$	$[q+1, \frac{3q^2+q}{2}, \frac{q^2-q}{2}, q^3-q^2]$
$o_{8,1}$	$(X_0X_1, X_0X_2, X_1X_2, X_1^2+X_2^2)$	$[2, q^2+\frac{3q-1}{2}, \frac{q-1}{2}, q^3-q]$
$o_{8,2}$	$(X_0X_1, X_0X_2, X_1X_2, \delta X_1^2+X_2^2)$	$[0, q^2+\frac{3q+1}{2}, \frac{q+1}{2}, q^3-q]$
o_9	$(X_0X_1, X_0X_2-X_1^2, X_1X_2, X_2^2)$	$[1, q^2+q, 0, q^3]$
o_{10}	$(v_0^{-1}X_0^2+uX_0X_1-X_1^2, X_0X_2, X_1X_2, X_2^2)$	$[1, q^2+q, q^2, q^3-q^2]$
$o_{12,1}$	$(X_0^2, X_0X_2, X_1^2, X_2^2)$	$[q+2, q^2+\frac{q-1}{2}, q^2-\frac{q+1}{2}, q^3-q^2]$
$o_{13,1}$	$(X_0^2, X_0X_2, X_1^2+X_2^2, X_1X_2)$	$[3, \frac{q^2+3q-2}{2}, \frac{q^2+q-2}{2}, q^3-q]$
$o_{13,2}$	$(X_0^2, X_0X_2, \delta X_1^2+X_2^2, X_1X_2)$	$[1, \frac{q^2+3q}{2}, \frac{q^2+q}{2}, q^3-q]$
$o_{14,1}$	$(X_0X_1, X_0X_2, X_0^2+X_1^2+X_2^2, X_1X_2)$	$[4, \frac{q^2-1}{2}+2q-1, \frac{q^2-1}{2}+q-1, q^3-2q]$
$o_{14,2}$	$(X_0X_1, X_0X_2, \delta X_0^2+X_1^2+\delta X_2^2, X_1X_2)$	$[0, \frac{q^2+1}{2}+2q, \frac{q^2+1}{2}+q, q^3-2q]$
$o_{15,1}$	$(X_0X_2, X_1X_2, X_0X_1-X_2^2, v_1^{-1}X_0^2+uX_0X_1-X_1^2)$	$[2, \frac{q^2-1}{2}+q, \frac{q^2-1}{2}, q^3]$
$o_{15,2}$	$(X_0X_2, X_1X_2, X_0X_1-X_2^2, v_2^{-1}X_0^2+uX_0X_1-X_1^2)$	$[0, \frac{q^2+1}{2}+q, \frac{q^2+1}{2}, q^3]$
$o_{16,1}$	$(X_0^2, X_0X_1, X_0X_2-X_1^2, X_2^2)$	$[2, \frac{q^2-1}{2}+q, \frac{q^2-1}{2}, q^3]$
o_{17}	$(X_0X_2, X_0X_1-X_2^2, \alpha X_0^2-X_1X_2, \beta X_0X_1-X_1^2-\gamma X_1X_2)$	$[1, \frac{q^2+q}{2}, \frac{q^2-q}{2}, q^3+q]$

WEBS OF CONICS, q EVEN:

L^K	Webs of Conics	$OD_4(L)$
o_5	$(X_0X_1, X_0X_2, X_1X_2, X_2^2)$	$[1, 2q^2+q, 0, q^3-q^2]$
o_6	$(X_0X_2, X_1^2, X_1X_2, X_2^2)$	$[q+1, \frac{3q^2+q}{2}, \frac{q^2-q}{2}, q^3-q^2]$
$o_{8,1}$	$(X_0X_1, X_0X_2, X_1X_2, X_1^2+X_2^2)$	$[1, q^2+\frac{3}{2}q, \frac{q}{2}, q^3-q]$
$o_{8,3}$	$(X_0X_1, X_0X_2, X_1^2, X_2^2)$	$[q+1, q^2+q, 0, q^3-q]$
o_9	$(X_0X_1, X_0X_2+X_1^2, X_1X_2, X_2^2)$	$[1, q^2+q, 0, q^3]$
o_{10}	$(v_0^{-1}X_0^2+uX_0X_1+X_1^2, X_0X_2, X_1X_2, X_2^2)$	$[1, q^2+q, q^2, q^3-q^2]$
$o_{12,1}$	$(X_0^2, X_0X_2, X_1^2, X_2^2)$	$[q^2+q+1, \frac{q^2+q}{2}, \frac{q^2-q}{2}, q^3-q^2]$
$o_{12,3}$	$(X_0^2, X_0X_2, X_0X_1+X_1X_2+X_1^2, X_2^2)$	$[q+1, q^2+\frac{q}{2}, q^2-\frac{q}{2}, q^3-q^2]$
$o_{13,1}$	$(X_0^2, X_0X_2, X_1^2+X_2^2, X_1X_2)$	$[q+1, \frac{q^2}{2}+q, \frac{q^2}{2}, q^3-q]$
$o_{13,3}$	$(X_0^2, X_0X_2, X_1^2+X_0X_1+X_2^2, X_1X_2)$	$[1, \frac{q^2+3q}{2}, \frac{q^2+q}{2}, q^3-q]$
$o_{14,1}$	$(X_0X_1, X_0X_2, X_0^2+X_1^2+X_2^2, X_1X_2)$	$[1, \frac{q^2}{2}+2q, \frac{q^2}{2}+q, q^3-2q]$
$o_{15,1}$	$(X_0X_2, X_1X_2, X_0X_1+X_2^2, v_1^{-1}X_0^2+uX_0X_1+X_1^2)$	$[1, \frac{q^2}{2}+q, \frac{q^2}{2}, q^3]$
$o_{16,1}$	$(X_0^2, X_0X_1, X_0X_2+X_1^2, X_2^2)$	$[q+1, \frac{q^2+q}{2}, \frac{q^2-q}{2}, q^3]$
$o_{16,3}$	$(X_0^2, X_0X_1, X_0X_2+X_1^2, X_1X_2+X_2^2)$	$[1, \frac{q^2}{2}+q, \frac{q^2}{2}, q^3]$
o_{17}	$(X_0X_2, X_0X_1+X_2^2, \alpha X_0^2+X_1X_2, \beta X_0X_1+X_1^2+\gamma X_1X_2)$	$[1, \frac{q^2+q}{2}, \frac{q^2-q}{2}, q^3+q]$

THE $\mathcal{W}_{17}$ CASE:

Theorem

The hyperplane-orbit distribution of a line in o_{17} (lines having $q+1$ rank-3 points) is $[1, \frac{q^2+q}{2}, \frac{q^2-q}{2}, q^3+q]$.

Remarks:

- Initial computations for small q were done using the FinInG package in GAP.
- A purely computational proof presents significant challenges: Let ℓ_{17} be the representative of o_{17} from [M. Lavrauw, T. Popiel, 2020]. Singular conics in $\mathcal{W}_{17}$ correspond to points of the cubic surface in $\mathrm{PG}(3,q)$:

  $$4\alpha bcd + a(b+d\beta)(c+d\gamma) - \alpha c(c+d\gamma)^2 - da^2 - b(b+d\beta)^2 = 0,$$

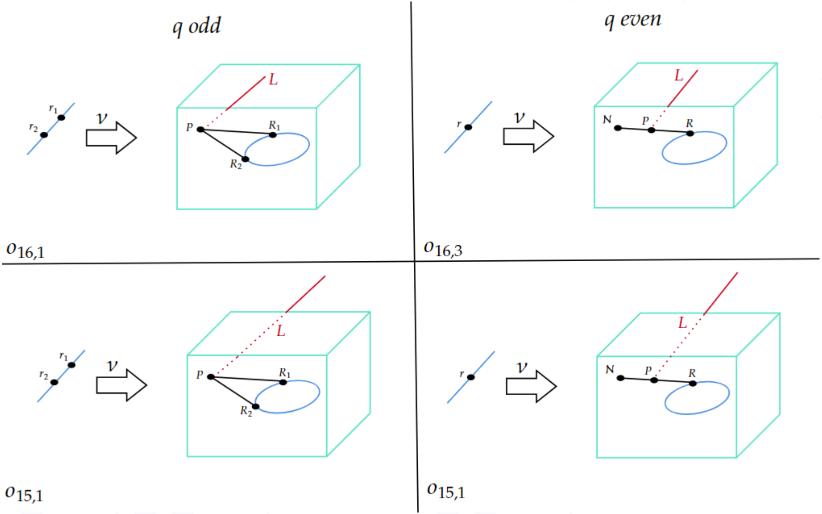
  where $\lambda^3 + \gamma\lambda^2 - \beta\lambda + \alpha \neq 0$ for all $\lambda \in \mathbb{F}_q$.

Sketch of the proof:

- Each conic plane π of $\mathcal{V}(\mathbb{F}_q)$ determines a hyperplane $H_\pi = \langle \pi, \ell_{17} \rangle \in \mathcal{H}_1 \cup \mathcal{H}_{2,r}$.
- Counting flags (π, H_π): $h_1 + 2h_{2,r} = q^2 + q + 1 \Longrightarrow h_1 \geq 1$ and odd.
- Claim $h_1 = 1$: If $H_\pi \neq H_{\pi'} \Longrightarrow H_\pi \cap H_{\pi'} = S \supset \kappa_{\pi \cap \pi'} \Longrightarrow$ ※
- Thus, $h_1 = 1 \Longrightarrow h_{2,r} = \frac{q^2+q}{2}$.
- Each tangent plane π of $\mathcal{V}(\mathbb{F}_q)$ determines a hyperplane $H_\pi = \langle \pi, \ell_{17} \rangle \in \mathcal{H}_1 \cup \mathcal{H}_{2,r} \cup \mathcal{H}_{2,i}$.
- By the first part of the proof, exactly one such hyperplane H_π with π a tangent plane of $\mathcal{V}(\mathbb{F}_q)$ belongs to $\mathcal{H}_1$ and $\frac{q^2+q}{2}$ belongs to $\mathcal{H}_{2,r}$.
- Counting flags (ρ, H_π), where ρ and π are tangent planes of $\mathcal{V}(\mathbb{F}_q)$: $q + 1 + \frac{q^2+q}{2} + h_{2,i} = q^2 + q + 1 \Longrightarrow h_{2,i} = \frac{q^2-q}{2}$. □

CONSEQUENCES:

1) **Lemma:** We differentiate between lines/webs that have the same point-orbit/conic distribution using the following geometric configurations:



2) **Theorem:** A line L in $\mathrm{PG}(5, q)$ intersects the secant variety of $\mathcal{V}(\mathbb{F}_q)$ in i points $\iff$ its associated cubic surface has $q^2 + iq + 1$ points, $i \in \{0, 1, 2, 3, q + 1\}$.

3) **Theorem:** The number of lines of type o_i in a fixed $H \in \mathcal{H}_j$ is

$$\frac{|o_i| \times h_j}{|\mathcal{H}_j|}.$$

Orbits	$\mathcal{H}_1$	$\mathcal{H}_{2,r}$	$\mathcal{H}_{2,i}$	$\mathcal{H}_3$
o_5	$\frac{1}{2}q(q+1)$	$2q^2+q$	0	$\frac{1}{2}q(q+1)$
o_6	$(q+1)^2$	$3q+1$	$q+1$	$q+1$
$o_{8,1}$	$q^3(q+1)$	$\frac{1}{2}q^2(2q^2+3q-1)$	$\frac{1}{2}q^2(q+1)$	$\frac{1}{2}q^2(q+1)^2$
$o_{8,2}$	0	$\frac{q^2(q-1)(2q^2+3q+1)}{2(q+1)}$	$\frac{1}{2}q^2(q+1)$	$\frac{1}{2}q^2(q^2-1)$
o_9	$q(q^2-1)$	$2q(q^2-1)$	0	$q^2(q+1)$
o_{10}	$\frac{1}{2}q(q-1)$	$q(q-1)$	q^2	$\frac{1}{2}q(q-1)$
$o_{12,1}$	$q+2$	$\frac{2q^2+q-1}{q(q+1)}$	$\frac{2q^2-q-1}{q(q-1)}$	1
$o_{13,1}$	$\frac{3}{2}q^3(q^2-1)$	$\frac{1}{2}q^2(q-1)(q^2+3q-2)$	$\frac{1}{2}q^2(q+1)(q^2+q-2)$	$\frac{1}{2}q^2(q+1)(q^2-1)$
$o_{13,2}$	$\frac{1}{2}q^3(q^2-1)$	$\frac{1}{2}q^3(q-1)(q+3)$	$\frac{1}{2}q^3(q+1)^2$	$\frac{1}{2}q^2(q+1)(q^2-1)$
$o_{14,1}$	$\frac{1}{6}q^3(q-1)(q^2-1)$	$\frac{1}{24}q^2(q-1)^2(q^2+4q-3)$	$\frac{1}{24}q^2(q^2-1)(q^2+2q-3)$	$\frac{1}{24}q^2(q^2-1)(q^2-2)$
$o_{14,2}$	0	$\frac{1}{8}q^2(q-1)^2(q^2+4q+1)$	$\frac{1}{8}q^2(q^2-1)(q^2+2q+1)$	$\frac{1}{8}q^2(q^2-1)(q^2-2)$
$o_{15,1}$	$\frac{1}{2}q^3(q-1)(q^2-1)$	$\frac{1}{4}q^2(q-1)^2(q^2+2q-1)$	$\frac{1}{4}q^2(q^2-1)^2$	$\frac{1}{4}q^4(q^2-1)$
$o_{15,2}$	0	$\frac{1}{4}q^2(q-1)^2(q^2+2q+1)$	$\frac{1}{4}q^2(q^4-1)$	$\frac{1}{4}q^4(q^2-1)$
$o_{16,1}$	$2q^2(q^2-1)$	$q(q-1)(q^2+2q-1)$	$q(q+1)(q^2-1)$	$q^3(q+1)$
o_{17}	$\frac{1}{3}q^3(q-1)(q^2-1)$	$\frac{1}{3}q^3(q-1)(q^2-1)$	$\frac{1}{3}q^3(q-1)(q^2-1)$	$\frac{1}{3}q^2(q^4-1)$

Table 3: Line-orbits distributions of hyperplanes in $\mathrm{PG}(5,q)$, q odd.

Orbits	$\mathcal{H}_1$	$\mathcal{H}_{2,r}$	$\mathcal{H}_{2,i}$	$\mathcal{H}_3$
o_5	$\frac{1}{2}q(q+1)$	$2q^2+q$	0	$\frac{1}{2}q(q+1)$
o_6	$(q+1)^2$	$3q+1$	$q+1$	$q+1$
$o_{8,1}$	$q^2(q^2-1)$	$(2q+3)(q-1)q^2$	$q^2(q+1)$	$q(q+1)(q^2-1)$
$o_{8,3}$	$q^2(q+1)$	$2q^2$	0	$q(q+1)$
o_9	$q(q^2-1)$	$2q(q^2-1)$	0	$q^2(q+1)$
o_{10}	$\frac{1}{2}q(q-1)$	$q(q-1)$	q^2	$\frac{1}{2}q(q-1)$
$o_{12,1}$	q^2+q+1	1	1	1
$o_{12,3}$	$(q+1)(q^2-1)$	$(q-1)(2q+1)$	$(q+1)(2q-1)$	q^2-1
$o_{13,1}$	$q^2(q+1)(q^2-1)$	$q^2(q-1)(q+2)$	$q^3(q+1)$	$q(q+1)(q^2-1)$
$o_{13,3}$	$q^2(q-1)(q^2-1)$	$q^2(q+3)(q-1)^2$	$q^2(q+1)(q^2-1)$	$q(q^2-1)^2$
$o_{14,1}$	$\frac{1}{6}q^3(q-1)(q^2-1)$	$\frac{1}{6}q^3(q-1)^2(q+4)$	$\frac{1}{6}q^3(q^2-1)(q+2)$	$\frac{1}{6}q^2(q^2-1)(q^2-2)$
$o_{15,1}$	$\frac{1}{2}q^3(q-1)(q^2-1)$	$\frac{1}{2}q^3(q-1)^2(q+2)$	$\frac{1}{2}q^4(q^2-1)$	$\frac{1}{2}q^4(q^2-1)$
$o_{16,1}$	$q(q+1)(q^2-1)$	$q(q^2-1)$	$q(q^2-1)$	$q^2(q+1)$
$o_{16,3}$	$q(q-1)(q^2-1)$	$q(q-1)^2(q+2)$	$q^2(q^2-1)$	$q^2(q^2-1)$
o_{17}	$\frac{1}{3}q^3(q-1)(q^2-1)$	$\frac{1}{3}q^3(q-1)(q^2-1)$	$\frac{1}{3}q^3(q-1)(q^2-1)$	$\frac{1}{3}q^2(q^4-1)$

Table 4: Line-orbits distributions of hyperplanes in $\mathrm{PG}(5,q)$, q even.

MRD codes, Segre Varieties and their secant varieties:

[Sheekey, 2019]

- We can view an $\mathbb{F}_q$-$[n \times m, k, d]$ linear rank-metric code as a subspace in the projective space $\mathrm{PG}(mn-1, q)$.
- Equivalence of $\mathbb{F}_q$-linear rank-metric codes corresponds to equivalence of subspaces of $\mathrm{PG}(mn-1, q)$ under the setwise stabilizer of the Segre variety in $\mathrm{PGL}(mn, q)$.
- The set of elements of rank at most i corresponds to the $(i-1)$-st secant variety of the Segre variety in $\mathrm{PG}(mn-1, q)$.
- An MRD code in $M_{n\times m}(\mathbb{F}_q)$ corresponds to a maximal subspace disjoint from one of the secant varieties of the Segre variety in $\mathrm{PG}(mn-1, q)$.

In particular, 3×3 symmetric MRD-codes over $\mathbb{F}_q$ correspond to solids of $\mathrm{PG}(5, q)$ disjoint from one of the secant varieties of the Veronese variety $\mathcal{V}(\mathbb{F}_q)$.

CONNECTION WITH MRD CODES:

- In [M. Lavrauw, T. Popiel, 2020] and [NA, M. Lavrauw, T. Popiel, 2022] solids were completely classified in $\mathrm{PG}(5,q)$ and the intersection of the different K-orbits of solids with the secant variety $\mathcal{V}^{(2)}(\mathbb{F}_q)$ were computed.
- It follows that there are three equivalence classes of 3×3 symmetric MRD-codes over $\mathbb{F}_q$.
- For q even, these classes correspond to the K-orbits of solids: Ω_7, Ω_{13} and Ω_{14} described in [NA, M. Lavrauw, T. Popiel, 2022].
- For q odd, these classes correspond to the K-orbits of solids: $\Omega_{8,2}$, $\Omega_{14,2}$ and $\Omega_{15,2}$ described in [M. Lavrauw, T. Popiel, 2020].

Over finite fields of odd order, webs in $\mathcal{W}_{8,2}\cup\mathcal{W}_{14,2}\cup\mathcal{W}_{15,2}$ are equivalent to 3×3 symmetric MRD-codes.

Thank you!

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