CODESCO 2024

Seville July 8-12

Additive Combinatorial Designs

Marco Buratti

Sapienza Università di Roma

ITALY

Joint works with

Anamari Nakic and Anita Pasotti:

MB, A. Nakic, Super-regular Steiner 2-designs,
FFA 2023

MB, A. Nakic, Additivity of symmetric subspace designs,
~~to appear in~~ DCC 2024

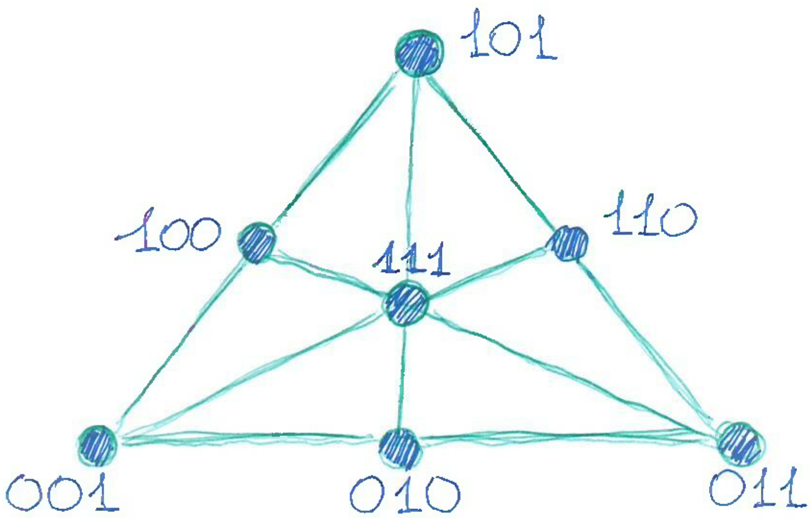
M.B, A. Pasotti, Heffter Spaces,
~~to appear in~~ FFA 2024

ADDITIVE DESIGNS

[Caggegi Falcone Pavone JACO 2017]

A 2-(v, k, λ) design $(V, \mathcal{B})$ is **additive** under an abelian group G if:

$V \subseteq G$ and every $B \in \mathcal{B}$ is ZERO-SUM



The Fano plane is **additive** under $\mathbb{Z}_2^3$

Existence results for 2-(v, k, 1) designs with k "small"

k	v	Reference
3	$6n+1$, $6n+3$ $\forall n$	Kirkman 1847
4	$12n+1$, $12n+4$ $\forall n$	HANANI 1975
5	$20n+1$, $20n+5$ $\forall n$	
6	$15n+1$, $15n+6$ with three exceptions (16, 36, 46) and 29 open cases	MANY AUTHORS
7	$42n+1$, $42n+7$ with one exception (43) and 21 open cases	
8	$56n+1$, $56n+8$ with 28 open cases	
9	$72n+1$, $72n+9$ with 91 open cases	

Existence results for 2-$(v, k, 1)$ designs with k general

k = prime power q: the point-line 2-$(q^n, q, 1)$ design
of $AG(n, q)$ $\forall n$.

k = prime power $q+1$: the point-line 2-$\left(\frac{q^n-1}{q-1}, q+1, 1\right)$ design
of $PG(n-1, q)$ $\forall n$.

For any admissible v $\left(\left\{\frac{v-1}{k-1}, \frac{v(v-1)}{k(k-1)}\right\} \subset \mathbb{N}\right)$ SUFFICIENTLY LARGE

[Wilson 1972]

= HUGE 

N.B.: **76231** was the first v for which a 2-$(v, 15, 1)$ design was found (Wilson 1972).
A smaller v (**13441**) was found by me in 1995.

Existence results for **additive** 2-$(v,k,1)$ designs

For $k=3$ [CFP 2017]:
ONLY the point-line 2-$(3^n,3,1)$ design of $AG(n,3)$ $\forall n$
and the point-line 2-$(2^n-1,3,1)$ design of $PG(n-1,2)$ $\forall n$

For $k>2$ a prime power we **OBVIOUSLY** have
the point-line 2-$(k^n,k,1)$ design of $AG(n,k)$ $\forall n$

For $k=q+1$, q a prime power, the point-line
2-$\left(\frac{q^n-1}{q-1}, q+1, 1\right)$ design of $PG(n-1,q)$ is **additive** $\forall n$
[MB, Nakic 2024]

There is a $\mathbb{F}_{5^3}$-**additive** 2-$(124, 4, 1)$ design

[Nakic, unpublished]

There is a $\mathbb{F}_{2^{13}}$-**additive** 2-$(8191, 7, 1)$ design, that is the famous 2-$(13, 3, 1)_2$ design by

Braun, Etzion, Ostergard, Vardy, Wassermann Forum Math PI 2016

N.B.: Every 2-design over a finite field is **additive** in view of [MB, Nakic 2024]

Main result in [MB, Nakic 2023]

THM If $k \not\equiv 2 \pmod 4$, $k \neq 2^t \cdot 3$, and $p^\alpha > 3$ is an odd prime power divisor of k, then there are infinitely many n for which there exists an **additive** 2-$(p^n k, k, 1)$ design.

The first v for which our construction gives an **additive** 2-$(v, 15, 1)$ design is $3 \cdot 5^{187}$

With a slight different construction we got it with $V = 3 \cdot 5^{31}$

Maybe a computer search could lead to $V = 3 \cdot 5^7$

OPEN PROBLEM

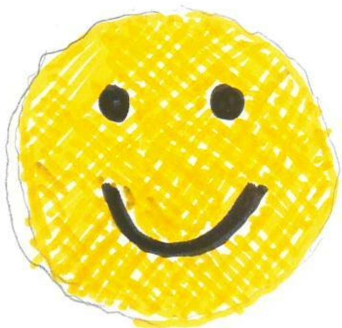
If $k \equiv 2 \pmod 4$ or $k = 2^n \cdot 3$ and k is not a prime power $+ 1$, then there is no v for which an additive 2-$(v, k, 1)$ design is known to exist.

The first ks for which the problem is open are

22, 34, 46, 54, 58, 66, 70, 78,
86, 94, 96 . . .

MOTIVATION:

I LIKE IT!!!



HEFFTER
SPACES

DEFINITION [MB, Pasotti 2024]

A $(v, k; r)$ **Heffter Space** is a partial 2-$(v, k, 1)$ design $(V, \mathcal{B})$ s.t.:

1. V is a "half-set" of an abelian group G:
$$G \setminus \{0\} = V \,\dot{\cup}\, -V;$$
2. it is **additive**;
3. every point has degree r (= belongs to exactly r blocks);
4. it is **resolvable** (= the blocks can be partitioned into r "parallel classes" each of which is a partition of V);

$(v,k;1)$ HSs = Heffter Systems

$(v,k;2)$ HSs = Square Heffter Arrays

For a survey on Heffter Arrays see [Pasotti, Dimitz, FIC 2024]

THM. There exists a SIMPLE $(v,k;1)$-HS for every pair (v,k) with $k \geq 3$ and $\frac{v}{k} \in \mathbb{N}$.

It is a consequence of the existence of a CYCLIC k-cycle decomposition of K_{2v+1} for any pair (v,k) as above [MB, Del Fra 2003 and Bryant, Gavlas, Ling 2003]

THM. There exists a $(v,k;2)$-HS iff $k \geq 3$ and $v \geq k^2$

[Archdeacon, Dimitz, Donovan, Yazici 2015 +
Dimitz, Wanless 2017 +
Cavenagh, Dimitz, Donovan, Yazici 2019]

An example of a $(20, 4; 3)$ Heffter Space over $G = \mathbb{Z}_{41}$.

Points:

1, 2, 3, 4, 5, 6, 7, −8, 9, 10, 11, −12, 13, −14, 15, −16, −17, −18, 19, −20

Parallel classes:

1	3	4	−8
2	5	13	−20
6	−12	−17	−18
7	9	10	15
11	−14	−16	19

1	2	9	−12
3	6	13	19
4	5	7	−16
−8	11	15	−18
10	−14	−17	−20

1	6	7	−14
2	4	11	−17
3	5	10	−18
−8	9	19	−20
−12	13	15	−16

The DENSITY of a $(v, k; r)$ Heffter Space is the number

$$\boxed{\delta = \frac{r(k-1)}{v-1}}$$

that is the density of its "collinear graph" (V, E) where $\{x, y\} \in E \iff x, y$ are collinear.

Since $\delta \leq 1$, we have $r \leq \left\lfloor \frac{v-1}{k-1} \right\rfloor$.

Note that

$$\boxed{\delta = 1 \iff r = \frac{v-1}{k-1} \iff \text{the HS is a } 2\text{-}(v, k, 1) \text{ design}}$$

A $(35, 5; 5)$ Heffter space over $\mathbb{Z}_{71}$:

Point-set: $\mathbb{F}_{71}^{\square}$;

Parallel classes:

1	24	25	43	49
45	15	60	18	4
37	36	2	29	38
32	58	19	27	6
20	54	3	8	57
48	16	64	5	9
30	10	40	12	50

49	40	18	48	58
4	25	29	30	54
38	60	27	1	16
6	2	8	45	10
57	19	5	37	24
9	3	12	32	15
50	64	43	20	36

58	43	30	9	2
54	18	1	50	19
16	29	45	49	3
10	27	37	4	64
24	8	32	38	40
15	5	20	6	25
36	12	48	57	60

2	48	50	15	27
19	30	49	36	8
3	1	4	58	5
64	45	38	54	12
40	37	6	16	43
25	32	57	10	18
60	20	9	24	29

27	9	36	25	45
8	50	58	60	37
5	49	54	2	32
12	4	16	19	20
43	38	10	3	48
18	6	24	64	30
29	57	15	40	1

Its <u>density</u> is $\delta = \dfrac{r(k-1)}{v-1} = \dfrac{10}{17} = 0.5882\ldots$

The most intriguing problem:
does there exist a Heffter 2-$(v,k,1)$ design?
We should need a 2-$(v,k,1)$ design s.t.:

① its point set is a half-set of G for some G;

② it is additive under G;

③ it is resolvable.

①+②: point-line design of $PG(2n,3)$;
①+③: as many as you want!;
②+③: point-line design of $AG(n,q)$ or $PG(2n+1,q)$.

①+②+③: we find this very hard!

Our DENSEST HS has parameters

$$(121, 11; 9).$$

Its density is $\frac{9 \cdot (11-1)}{121-1} = 0.75.$

It is additive under $\mathbb{F}_{3^5}$.

Very recently, A. Nakic found a

$(105, 5; 21)$-HS

of density $\delta > 0.8$

Our main result:

THEOREM [MB-A. Pasotti 2024]

$\forall$ k ODD ≥ 3 and $\forall$ $r \geq 1$, there exists a SIMPLE $(v, k; r)$ Heffter space whenever v is odd and $2v+1 > 8k^5 \lceil \frac{r}{k} \rceil$ is a prime power.

COROLLARY. $\forall$ k ODD ≥ 3, $\forall$ $r \geq 1$, there are infinitely many v s.t. $\underbrace{\text{a } (v}_{\text{SIMPLE}}, k; r)$ Heffter space exists

APPLICATION TO MUTUALLY ORTHOGONAL CYCLE SYSTEMS

Two (K_v, C_k) designs are ORTHOGONAL if every cycle of one system shares at most one edge with every cycle of the other system

What about the maximum number $\mu(k, v)$ of mutually ORTHOGONAL (K_v, C_k) designs?

CURRENT RESULTS

$\mu(k, v) \geq 2$ for many instances of (k, v)

[Dinitz - Mattern AJC 2017]

[Costa - Morini - Pasotti - Pellegrini AJC 2018]

[Burrage - Donovan - Cavenagh - Yazici DM 2020]

[Kucukcifci - Yazici JCD 2024]

$$\mu(k, 2kn+1) \geq \begin{cases} 4n & \text{if } k=4 \\ \dfrac{n}{4k-2} - 1 & \text{if } k \equiv 0 \pmod 4 \\ \dfrac{n}{24k-18} - 1 & \text{if } k \equiv 2 \pmod 4 \end{cases}$$

[Burgess Cavenagh Pike ElJC 2023]

$$\mu(k, 2kn+1) \geq \left\lceil \frac{n}{4k^4} \right\rceil \text{ whenever}$$

$2kn+1$ is a prime power with kn odd

[MB - A. Pasotti 2024]