Weaving High-Dimension (affine) classes of (Complex) Hadamard matrices

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- But Sylvester was WRONG! Very wrong ... in this claim.

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$h_{ij} \in U$. Taking all $r_{ij} = 0$ must give $H_0 \in \mathcal{H}_n$. The factors $x_{ij} = e^{ir_{ij}}$ are regarded as variables, $0 < i, j < n$.

Families as solutions

The variables r_{ij} make $H \in \mathcal{H}_n$ if

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Roughly: a solution set to this system of $n^2 - n$ equations in $(n-1)^2$ variables r_{ij}, $0 < i, j < n$,in $\mathbb{R}^{(n-1)^2}$ containing $(0, 0, \ldots, 0)$

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The variables r_{ij} make $H \in \mathcal{H}_n$ if

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And roughly speaking such a family can be expressed as an array with d free multiplicative factors $x = e^{ai} \in U$ among the entries.

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Then $W \in \mathcal{H}_{mn}$ (easy exercise).

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(Alternatively we could take $A_1 = B_1 = B_2$ and the second column of A_2 multiplied by λ.)

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- The woven matrix W may have $(m-1)(n-1)$ parameters introduced as scalar multiples of blocks, and remain dephased.

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Warp & woof matrices are unrelated. So each may carry into the weaving construction its own independent parameters.

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Theorem (RC 2022) Under the above conditions $W \in \mathcal{H}_{mn}$ is a member of a family of affine dimension at least

$$(m-1)(n-1) + \sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j.$$

Inheriting non-affine parameters

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Not fully described here: parametric families of Hadamard matrices

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  All three equivalent to constrained versions of weaving

Some calculations

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$(m-1)(n-1) + na + mb$: number of parameters in woven matrix

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m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 4 = ab$, weave matrix $\in \mathcal{H}_4$					
2	2	0	0	1	1

Some calculations

$N = mn \in \{4, 6, 8\}$

Weaving doesn't affect prime orders (no nontrivial factorization)

a = max # parameters for warp matrices $\in \mathcal{H}_m$

b = max # parameters for woof $\in \mathcal{H}_n$

$(m-1)(n-1) + na + mb$: number of parameters in woven matrix

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m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 4 = ab$, weave matrix $\in \mathcal{H}_4$					
2	2	0	0	1	1
$N = 6 = ab$, weave matrix $\in \mathcal{H}_6$					
2	3	0	0	2	2; 4^{NA}

Some calculations

$N = mn \in \{4, 6, 8\}$

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a = max # parameters for warp matrices $\in \mathcal{H}_m$

b = max # parameters for woof $\in \mathcal{H}_n$

$(m-1)(n-1) + na + mb$: number of parameters in woven matrix

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m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 4 = ab$, weave matrix $\in \mathcal{H}_4$					
2	2	0	0	1	1
$N = 6 = ab$, weave matrix $\in \mathcal{H}_6$					
2	3	0	0	2	2; 4^{NA}
$N = 8 = ab$, weave matrix $\in \mathcal{H}_8$					
2	4	0	1	5	5

Calculations

$N = mn \in \{9, 10, 12, 14, 15, 16\}$

m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 9 = ab$, weave matrix $\in \mathcal{H}_9$					
3	3	0	0	4	4

Calculations

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m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 9 = ab$, weave matrix $\in \mathcal{H}_9$					
3	3	0	0	4	4
$N = 10 = ab$, weave matrix $\in \mathcal{H}_{10}$					
2	5	0	0	4	7

Calculations

$N = mn \in \{9, 10, 12, 14, 15, 16\}$

m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 9 = ab$, weave matrix $\in \mathcal{H}_9$					
3	3	0	0	4	4
$N = 10 = ab$, weave matrix $\in \mathcal{H}_{10}$					
2	5	0	0	4	7
$N = 12 = ab$, weave matrix $\in \mathcal{H}_{12}$					
3	4	0	1	9	9

Calculations

$N = mn \in \{9, 10, 12, 14, 15, 16\}$

m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 9 = ab$, weave matrix $\in \mathcal{H}_9$					
3	3	0	0	4	4
$N = 10 = ab$, weave matrix $\in \mathcal{H}_{10}$					
2	5	0	0	4	7
$N = 12 = ab$, weave matrix $\in \mathcal{H}_{12}$					
3	4	0	1	9	9
2	6	0	4^{NA}	13^{NA}	9

Calculations

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$N = 12 = ab$, weave matrix $\in \mathcal{H}_{12}$					
3	4	0	1	9	9
2	6	0	4^{NA}	13^{NA}	9
$N = 14 = ab$, weave matrix $\in \mathcal{H}_{14}$					
2	7	0	1	8*	7

Calculations

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$N = 9 = ab$, weave matrix $\in \mathcal{H}_9$					
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$N = 12 = ab$, weave matrix $\in \mathcal{H}_{12}$					
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2	6	0	4^{NA}	13^{NA}	9
$N = 14 = ab$, weave matrix $\in \mathcal{H}_{14}$					
2	7	0	1	8*	7
$N = 15 = ab$, weave matrix $\in \mathcal{H}_{15}$					
3	5	0	0	8	8

Calculations

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$N = 9 = ab$, weave matrix $\in \mathcal{H}_9$					
3	3	0	0	4	4
$N = 10 = ab$, weave matrix $\in \mathcal{H}_{10}$					
2	5	0	0	4	7
$N = 12 = ab$, weave matrix $\in \mathcal{H}_{12}$					
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2	6	0	4^{NA}	13^{NA}	9
$N = 14 = ab$, weave matrix $\in \mathcal{H}_{14}$					
2	7	0	1	8*	7
$N = 15 = ab$, weave matrix $\in \mathcal{H}_{15}$					
3	5	0	0	8	8
$N = 16 = ab$, weave matrix $\in \mathcal{H}_{16}$					
2	8	0	5	17	17

Calculations

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$N = 9 = ab$, weave matrix $\in \mathcal{H}_9$					
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2	5	0	0	4	7
$N = 12 = ab$, weave matrix $\in \mathcal{H}_{12}$					
3	4	0	1	9	9
2	6	0	4^{NA}	13^{NA}	9
$N = 14 = ab$, weave matrix $\in \mathcal{H}_{14}$					
2	7	0	1	8*	7
$N = 15 = ab$, weave matrix $\in \mathcal{H}_{15}$					
3	5	0	0	8	8
$N = 16 = ab$, weave matrix $\in \mathcal{H}_{16}$					
2	8	0	5	17	17
4	4	1	1	17	17

Calculations ($N = mn \in \{18, 20, 21, 22, 24, 25\}$)

m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 18 = ab$, weave matrix $\in \mathcal{H}_{18}$					
2	9	0	4	16*	n/a

Calculations ($N = mn \in \{18, 20, 21, 22, 24, 25\}$)

m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 18 = ab$, weave matrix $\in \mathcal{H}_{18}$					
2	9	0	4	16*	n/a
3	6	0	4	22*[NA]	n/a

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m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 18 = ab$, weave matrix $\in \mathcal{H}_{18}$					
2	9	0	4	16*	n/a
3	6	0	4	22*NA	n/a
$N = 20 = ab$, weave matrix $\in \mathcal{H}_{20}$					
2	10	0	7	23*	n/a

Calculations ($N = mn \in \{18, 20, 21, 22, 24, 25\}$)

m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 18 = ab$, weave matrix $\in \mathcal{H}_{18}$					
2	9	0	4	16*	n/a
3	6	0	4	22*[NA]	n/a
$N = 20 = ab$, weave matrix $\in \mathcal{H}_{20}$					
2	10	0	7	23*	n/a
4	5	1	0	17*	n/a

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m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 18 = ab$, weave matrix $\in \mathcal{H}_{18}$					
2	9	0	4	16*	n/a
3	6	0	4	22*^{NA}	n/a
$N = 20 = ab$, weave matrix $\in \mathcal{H}_{20}$					
2	10	0	7	23*	n/a
4	5	1	0	17*	n/a
$N = 21 = ab$, weave matrix $\in \mathcal{H}_{21}$					
3	7	0	1	15*	0

Calculations ($N = mn \in \{18, 20, 21, 22, 24, 25\}$)

m	n	a	b	woven dimension	CHM Catalog max (2022)
$N = 18 = ab$, weave matrix $\in \mathcal{H}_{18}$					
2	9	0	4	16*	n/a
3	6	0	4	$22^{*\mathrm{NA}}$	n/a
$N = 20 = ab$, weave matrix $\in \mathcal{H}_{20}$					
2	10	0	7	23*	n/a
4	5	1	0	17*	n/a
$N = 21 = ab$, weave matrix $\in \mathcal{H}_{21}$					
3	7	0	1	15*	0
$N = 22 = ab$, weave matrix $\in \mathcal{H}_{22}$					
2	11	0	0	10*	n/a

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3	6	0	4	22*[NA]	n/a
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4	5	1	0	17*	n/a
$N = 21 = ab$, weave matrix $\in \mathcal{H}_{21}$					
3	7	0	1	15*	0
$N = 22 = ab$, weave matrix $\in \mathcal{H}_{22}$					
2	11	0	0	10*	n/a
$N = 24 = ab$, weave matrix $\in \mathcal{H}_{24}$					
2	12	0	13	37*	n/a

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2	11	0	0	10*	n/a
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2	12	0	13	37*	n/a
3	8	0	5	29*	n/a

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2	11	0	0	10*	n/a
$N = 24 = ab$, weave matrix $\in \mathcal{H}_{24}$					
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3	8	0	5	29*	n/a
4	6	1	4	31*NA	n/a

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2	9	0	4	16*	n/a
3	6	0	4	22*^{NA}	n/a
$N = 20 = ab$, weave matrix $\in \mathcal{H}_{20}$					
2	10	0	7	23*	n/a
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$N = 21 = ab$, weave matrix $\in \mathcal{H}_{21}$					
3	7	0	1	15*	0
$N = 22 = ab$, weave matrix $\in \mathcal{H}_{22}$					
2	11	0	0	10*	n/a
$N = 24 = ab$, weave matrix $\in \mathcal{H}_{24}$					
2	12	0	13	37*	n/a
3	8	0	5	29*	n/a
4	6	1	4	31*^{NA}	n/a
$N = 25 = ab$, weave matrix $\in \mathcal{H}_{25}$					
5	5	0	0	16*	n/a

Calculations ($N = mn \in \{18, 20, 21, 22, 24, 25\}$)

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2	9	0	4	16*	n/a
3	6	0	4	22*[NA]	n/a
$N = 20 = ab$, weave matrix $\in \mathcal{H}_{20}$					
2	10	0	7	23*	n/a
4	5	1	0	17*	n/a
$N = 21 = ab$, weave matrix $\in \mathcal{H}_{21}$					
3	7	0	1	15*	0
$N = 22 = ab$, weave matrix $\in \mathcal{H}_{22}$					
2	11	0	0	10*	n/a
$N = 24 = ab$, weave matrix $\in \mathcal{H}_{24}$					
2	12	0	13	37*	n/a
3	8	0	5	29*	n/a
4	6	1	4	31*[NA]	n/a
$N = 25 = ab$, weave matrix $\in \mathcal{H}_{25}$					
5	5	0	0	16*	n/a

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Each of the variables we introduce in the Weaving construction for $\mathcal{H}(N)$, $N = mn$ affect exactly N entries.

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More complete bibliography of theory and contributing "catalog" constructions https://chaos.if.uj.edu.pl/