Hadamard matrices in centraliser algebras of monomial representations

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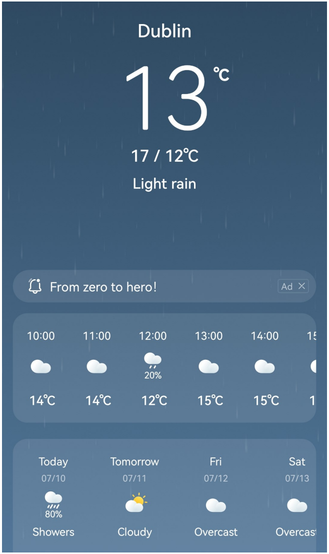
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Hadamard matrices

Theorem (Hadamard, 1893)

An $n \times n$ matrix H with complex entries of modulus no greater than 1 satisfies

$$|\det(H)| \leq n^{n/2}.$$

A matrix attaining this bound is a (complex) *Hadamard matrix*, i.e., the entries all have modulus 1 and the rows are pairwise orthogonal.

Key details:

- every eigenvalue of H has modulus $\sqrt{n}$;
- A pair of monomial matrices (P, Q) such that $PHQ^* = H$ is an automorphism of H.

Motivation

Even up to equivalence, classifying Hadamard matrices at all but small orders is intractable. While there is a unique (real) Hadamard matrix up to equivalence at orders ≤ 12 there are 13,710,027 equivalence classes at order 32, and higher orders remain unclassified.

Many attempt to bring structure to this by restricting consideration to algebraic constructions.

For example, a matrix H is *group-developed* if the rows and columns of H are labelled by the elements of a group G, say $H = [h_{f,g}]_{f,g \in G}$, such that $h_{f,g} = \varphi(fg)$ for some function $\varphi\colon G \to \mathcal{A}$.

Goals

Question: How do I construct matrices (of combinatorial interest) invariant under (a monomial representation of a central extension of) my favourite permutation group G?

We have two goals today:

- summarise and extend previous work of D. G. Higman on monomial group representations and their centraliser algebras;
- apply techniques of computational algebra to search for complex Hadamard matrices in the centraliser of a monomial representation.

Centraliser algebras

Let A be a $\mathbb{C}$–algebra. An n-dimensional *representation* of A is an algebra homomorphism $\rho\colon A \to M_n(\mathbb{C})$. The *character* of ρ is the trace map $\chi_\rho\colon A \to \mathbb{F}$, given by $a \mapsto \mathrm{Tr}(\rho(a))$.

A representation $\rho\colon \mathbb{C}[G] \to M_n(\mathbb{C})$ restricts to a group homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$; this restriction is an n-dimensional (complex) representation of G. A monomial representation is a representation $G \to \mathrm{Mon}_n(\mathbb{C})$.

The centralizer algebra $\mathrm{C}(\rho)$ of a representation ρ is a $\mathbb{C}$-algebra comprised of the set of all matrices in $M_n(\mathbb{C})$ that commute with every element of $\rho(G)$, equipped with matrix multiplication and addition.

Let G be a finite group and let H be a subgroup of G with right transversal $T = \{t_1, \ldots, t_n\}$. Every element $g \in G$ admits a factorisation as

$$g = h_g t_g$$

for uniquely determined $h_g \in H$ and $t_g \in T$. We define the maps $\mathbf{H} : G \to H$ and $\mathbf{T} : G \to T$ by $\mathbf{H}(g) = h_g$ and $\mathbf{T}(g) = t_g$. We assume throughout that $t_1 = 1$.

G acts on pairs of elements of T via $(t_i, t_j)g = (\mathbf{T}(t_i g), \mathbf{T}(t_j g))$, and the orbits under G are called *orbitals*.

Let $\chi\colon H \to \mathbb{C}^\times$ be a 1-dimensional representation of H (commonly refereed to in the literature as a linear character), and extend χ from H to G by

$$\chi^+(g) = \begin{cases} \chi(g) & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$

We write

$$\chi_{\mathbf{H}}(t_i g) = \chi(\mathbf{H}(t_i g))$$

for the χ-value of the H-part of $t_i g$;

Proposition

The monomial representation induced by χ is $\rho_\chi = \chi\uparrow_H^G$, defined by

$$\rho_\chi(g) = \left[\chi^+(t_i g t_k^{-1})\right]_{i,k} \quad \text{for all} \quad g \in G.$$

A matrix M is *centralised* by ρ if and only if $\rho(g)M = M\rho(g)$ for all $g \in G$. The set of all such matrices forms a $\mathbb{C}$-algebra, called the *centraliser algebra* of ρ and denoted by $\mathrm{C}(\rho)$.

Proposition

A matrix M, with rows and columns indexed by the transversal T, is in the centraliser algebra $C(\rho)$ if and only if

$$m(\mathbf{T}(g), \mathbf{T}(tg)) = m(1, t)\chi_{\mathbf{H}}(g)^{-1}\chi_{\mathbf{H}}(tg).$$

for all $g \in G$ and $t \in T$.

This equation defines $m(\mathbf{T}(t_i g), \mathbf{T}(tg))$ in terms of $m(t_i, t)$.

Requirement:

$$m(\mathbf{T}(g), \mathbf{T}(tg)) = m(1, t)\chi_{\mathbf{H}}(g)^{-1}\chi_{\mathbf{H}}(tg).$$

It may happen that distinct $g_1, g_2 \in Ht_i \cap t^{-1}Ht_j$ yield different constants in this equation, in which case every matrix in the centraliser algebra must take the value 0 at $m(1, t)$.

The condition that

$$\chi_{\mathbf{H}}(g_1)^{-1}\chi_{\mathbf{H}}(tg_1) = \chi_{\mathbf{H}}(g_2)^{-1}\chi_{\mathbf{H}}(tg_2).$$

for all $g_1, g_2 \in Ht_i \cap t^{-1}Ht_j$ is necessary and sufficient for the existence of matrices in the centraliser algebra which are non-zero at $m(1, t)$.

Definition

The orbital $\mathcal{O}$ associated with $(1, t)$ is *orientable* if and only if

$$\chi_{\mathbf{H}}(g_1)^{-1}\chi_{\mathbf{H}}(tg_1) = \chi_{\mathbf{H}}(g_2)^{-1}\chi_{\mathbf{H}}(tg_2).$$

for any $g_1, g_2 \in Ht_i \cap t^{-1}Ht_j$, and $t_i, t_j \in T$.

Corollary

The centralizer algebra $C(\rho)$ has a $\mathbb{C}$ basis spanned by the orientable orbital matrices.

Character tables of centraliser algebras

The character table of $C(\rho)$ may be constructed from the character table of G, together with some additional data about double cosets of H in G.

The columns of the character table of $C(\rho)$ correspond to the elements of the $\mathbb{C}$-basis.

If the centralizer algebra is commutative, the character table gives the eigenvalues of a matrix M in the algebra explicitly as a function of the entries.

We follow the instructions of Müller [1] to construct the character table.

[1] J. Müller. On the multiplicity-free actions of the sporadic simple groups. J. Algebra, 320(2):910–926, 2008

General case: Step 1, Covers

Question: How do I construct matrices (of combinatorial interest) invariant under (a monomial representation of a non-split central extension of) my favourite permutation group G?

- Monomial representations are induced from linear characters of the point stabiliser.
- **But** not every monomial representation can be obtained in this way. $\mathrm{PSL}_2(q)$ does **not** act on the Paley matrix: the point stabiliser has odd order, and so no non-trivial real character.
- In general, we need to study central extensions of G by a cyclic group. Sufficient to study **stem** extensions,

$$1 \to \mathbb{C}^* \to \Gamma \to G \to 1\,,$$

these are classified by the Schur multiplier, which is the cohomology group $H^2(G, \mathbb{C}^*)$.

General case: Step 2, Centraliser

Question: How do I construct matrices (of combinatorial interest) invariant under (a monomial representation of a non-split central extension of) my favourite permutation group G?

- $\hat{G}$ is a Schur cover of G, we restrict to studying centralisers. We choose a linear character of the point stabiliser and induce to a monomial representation ρ.
- We find a basis of the space of matrices satisfying $\rho(g)M = M\rho(g)$ for all $g \in G$.
- If the centraliser is commutative, the character table gives eigenvalues of M explicitly as a function of the entries.
- Hadamard matrices correspond to solutions $\mathcal{C}v = \lambda$ where v has all entries of norm 1 and λ has all entries of norm n.

General case: Step 3, Gröbner bases

Question: How do I construct matrices (of combinatorial interest) invariant under (a monomial representation of a non-split central extension of) my favourite permutation group G?

- To solve $\mathcal{C}v = \lambda$, with $v_i v_i^* = 1$ and $\lambda_i \lambda_i^* = n$ with Gröbner bases, we need a system of polynomial equations.
- Norm conditions are not polynomial, so introduce v_{ic} for the conjugate of v_i. The equation is then $v_i v_{ic} - 1$.
- Since $\lambda_i = \sum_{i=1}^{n} c_i \alpha_i$, the variables λ_i can be eliminated, leaving equations
$$\left(\sum_{i=1}^{n} c_i v_i\right)\left(\sum_{i=1}^{n} c_i v_i\right) - n\,.$$
- Feed this system into a Gröbner basis algorithm, carefully exclude degenerate solutions (e.g. with $v_i = d$ and $v_{ic} = d^{-1}$ for real d) and the remaining solutions correspond to Hadamard matrices.

Finding Hadamard matrices in centraliser algebras

We illustrate with an example. Let $G \leq S_{16}$ be the group

$$\langle \sigma = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16),$$
$$\tau = (2,3,5,9,16)(4,7,13,8,15)(6,11,12,10,14)\rangle.$$

This group is a Frobenius group of order 80, with an elementary abelian subgroup of order 16 and a point stabiliser H of order 5.

Let ρ be the permutation representation induced by the trivial character χ of H. The associated centraliser algebra is commutative and spanned by the identity matrix, and three matrices of constant row-sum 5.

The character table of the centraliser algebra is

	M_1	M_2	M_3	M_4
	1	5	5	5
	1	-3	1	1
	1	1	-3	1
	1	1	1	-3

The character table of the centraliser algebra is

$$\left(\begin{array}{rrrr} M_1 & M_2 & M_3 & M_4 \\ \hline 1 & 5 & 5 & 5 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{array}\right) \left(\begin{array}{c} v \\ \hline 1 \\ 1 \\ -1 \\ -1 \end{array}\right) = \left(\begin{array}{c} \lambda \\ \hline -4 \\ -4 \\ 4 \\ 4 \end{array}\right)$$

The following $\{\pm 1\}$-linear combination of basis matrices

$$M = M_1 + M_2 - M_3 - M_4$$

is Hadamard matrix, because its eigenvalues are all of absolute value 4, by virtue of which its determinant achieves the Hadamard bound.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - \\
1 & 1 & - & - & - & - & 1 & - & - & - & 1 & 1 & 1 & - & - & - \\
1 & - & 1 & - & - & - & 1 & 1 & - & - & - & - & - & 1 & 1 & - \\
1 & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & - & - & - & 1 \\
1 & - & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & - & - & - & 1 & - & - & - & 1 & 1 & - & - & - & 1 & 1 \\
- & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & - & - & - & - & 1 \\
- & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & - & - \\
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- & - & 1 & - & 1 & - & - & - & - & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - \\
- & - & - & 1 & - & 1 & 1 & - & - & - & - & - & 1 & 1 & - & 1
\end{bmatrix}
$$

$$M = M_1 + M_2 - M_3 - M_4$$

n	Schur cover of	Stab	$[x_1, \ldots, x_r]$
7	$C_7 \rtimes C_3$	C_3	$[1, 1, \frac{-3+i\sqrt{7}}{4}]$
7	$C_7 \rtimes C_3$	C_3	$[1, 1, \frac{(\sqrt{3}+i)(\sqrt{7}-3i)}{8}]$
11	$C_{11} \rtimes C_5$	C_5	$[1, 1, \frac{-5+i\sqrt{11}}{6}]$
11	$C_{11} \rtimes C_5$	C_5	$[1, 1, \frac{x^2}{3}]^*$
27	$3^3 \rtimes S_4$	S_4	$[1, \zeta_3, 1, \zeta_3^2]$
35	A_7	$(S_3 \times S_4) \cap A_7$	$[1, \frac{15-i\sqrt{31}}{16}, \frac{15-i\sqrt{31}}{16}, -1]$
35	A_7	$(S_3 \times S_4) \cap A_7$	$[1, 1, 1, \frac{17+i\sqrt{35}}{18}]$
49	$C_7^2 \rtimes (D_4 \times C_3)$	$D_4 \times C_3$	$[1, \frac{1-i\sqrt{35}}{6}, \frac{1-i\sqrt{35}}{6}, \frac{1+i\sqrt{35}}{6}]$
63	$G_2(2)'$	$2^{2+1+2} \rtimes C_3$	$[1, 1, 1, \frac{-31+i\sqrt{63}}{32}]$
63	$G_2(2)'$	$2^{2+1+2} \rtimes C_3$	$[1, \frac{29+i\sqrt{59}}{30}, \frac{29+i\sqrt{59}}{30}, -1]$

$*$ x satisfies $x^8 - x^7 - 2x^6 + 5x^5 + x^4 + 15x^3 - 18x^2 - 27x + 81$