A linear programming bound for sum-rank metric codes

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based on joint work with Aida Abiad, Antonina Khramova and Ilia Ponomarenko

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Sum-rank metric space

A **sum-rank metric space** is

$$\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}} := \mathbb{F}_q^{n_1\times m_1} \times \cdots \times \mathbb{F}_q^{n_t\times m_t},$$

where $\mathbf{N} = [n_1, \ldots, n_t]$ and $\mathbf{M} = [m_1, \ldots, m_t]$, with **sum-rank distance** between two tuples $\mathrm{A} := (A_1, \ldots, A_t)$ and $\mathrm{B} := (B_1, \ldots, B_t)$:

$$\mathrm{srkd}(\mathrm{A},\mathrm{B}) = \sum_{i=1}^{t} \mathrm{rk}(A_i - B_i).$$

This generalizes both Hamming metric ($\mathbf{N} = \mathbf{M} = [1, \ldots, 1]$) and rank metric ($t = 1$).

A **sum-rank metric code** $\mathcal{C}$ **with minimum distance** d is a subset of $\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}}$ such that:

$$\min_{X,Y\in\mathcal{C}} \mathrm{srkd}(X, Y) = d.$$

Question: What is the maximum size of a sum-rank metric code with minimum distance d?

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Example

Here $\mathbf{N} = [3,2,1,1]$ and $\mathbf{M} = [3,3,2,1]$, $q = 2$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, 1 \qquad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, 0$$

$$\mathrm{rk}\left(\begin{bmatrix} 1 & 0 & 0 & & & & & & \\ 0 & 1 & 1 & & & & & & \\ 1 & 1 & 1 & & & & & & \\ & & & 1 & 0 & 0 & & & \\ & & & 1 & 0 & 1 & & & \\ & & & & & & 0 & 1 & \\ & & & & & & & & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & & & & & & \\ 1 & 0 & 0 & & & & & & \\ 1 & 0 & 1 & & & & & & \\ & & & 0 & 0 & 0 & & & \\ & & & 1 & 1 & 1 & & & \\ & & & & & & 0 & 1 & \\ & & & & & & & & 0 \end{bmatrix} \right) = 6$$

Code-theoretical and geometric bounds

Question: What is the maximum size of a sum-rank metric code with minimum distance d?

Some upper bounds were introduced in

E. Byrne, H. Gluesing-Luerssen, and A. Ravagnani. Fundamental properties of sum-rank-metric codes. *IEEE Trans. Inf. Theory*, 67(10):6456–6475, 2021.

- Bounds *induced* by Singleton, Hamming, Plotkin, and Elias bounds via embedding a sum-rank metric code into a Hamming space.
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They introduced a **sum-rank metric graph** $\Gamma := \Gamma(\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}})$:

- the *vertex set* of Γ = { all the t-tuples of matrices from $\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}}$ };
- $\mathrm{A} := (A_1, \ldots, A_t)$ and $\mathrm{B} := (B_1, \ldots, B_t)$ form an *edge* iff the sum-rank distance is 1:

$$\mathrm{A} \sim \mathrm{B} \quad \Longleftrightarrow \quad \mathrm{srkd}(\mathrm{A},\mathrm{B}) = \sum_{i=1}^{t} \mathrm{rk}(A_i - B_i) = 1.$$

Geodesic distance between A and B in Γ = sum-rank distance $\mathrm{srkd}(\mathrm{A}, \mathrm{B})$.

Then the maximum size of a sum-rank metric code in $\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}}$ with minimum distance d equals $\alpha_{d-1}(\Gamma)$, the $(d-1)$-**independence number** of Γ.
Now, α_{d-1} can be bounded via eigenvalues of the adjacency matrix of Γ.

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Eigenvalue bounds on independence numbers

Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A of a regular graph G.

Ratio bound (Hoffman, 1974): $\alpha_1 \leq n\dfrac{-\lambda_n}{\lambda_1 - \lambda_n}$.

Ratio-type bound (Abiad, Coutinho, Fiol, 2019): $\alpha_{d-1} \leq n\dfrac{W(p) - \min_{i\in[2,n]} p(\lambda_i)}{p(\lambda_1) - \min_{i\in[2,n]} p(\lambda_i)}$, for some polynomial $p \in \mathbb{R}_{d-1}[x]$, where $W(p)$ is the largest element of the diagonal of $p(A)$.

Question: How to find the best polynomial $p \in \mathbb{R}_{d-1}[x]$ to optimize the bound?
The best polynomial for the Ratio-type bound:

- $d = 3$: Abiad, Coutinho, Fiol (2019);
- $d = 4$: Kavi, Newman (2023);
- $d \geq 5$: a linear programming problem (Fiol, 2020); *no* explicit closed formula is known.

Question: How to compute the eigenvalues of the sum-rank metric graph Γ?

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Γ as a Cartesian product

Let $\mathbf{N} = [n_1, \ldots, n_t]$, $\mathbf{M} = [m_1, \ldots, m_t]$ and $\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}} := \mathbb{F}_q^{n_1\times m_1} \times \cdots \times \mathbb{F}_q^{n_t\times m_t}$.
Abiad, Khramova, and Ravagnani (2023) observed:

- The sum-rank-metric graph $\Gamma(\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}})$ is the Cartesian product $\square$ of graphs $\Gamma(\mathbb{F}_q^{n_i\times m_i})$:

$$\Gamma(\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}}) = \Gamma(\mathbb{F}_q^{n_1\times m_1}) \,\square \ldots \square\, \Gamma(\mathbb{F}_q^{n_t\times m_t})$$

- Each graph $\Gamma(\mathbb{F}_q^{n_i\times m_i})$ is a *bilinear forms graph*.

- The bilinear forms graph $\Gamma(\mathbb{F}_q^{n\times m})$ is distance-regular, and its eigenvalues are given by

$$\theta_i = \frac{(q^{n-i}-1)(q^m-q^i)-q^i+1}{q-1}, \quad i = 0, \ldots, n.$$

- The eigenvalues of the Cartesian product are all possible sums of eigenvalues of the product's factors.

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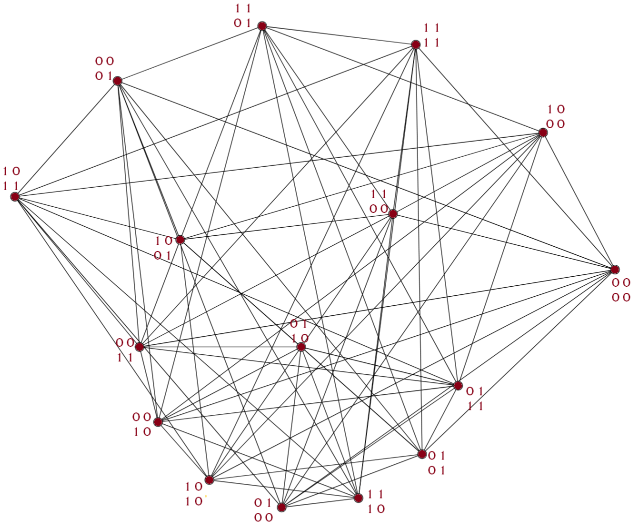
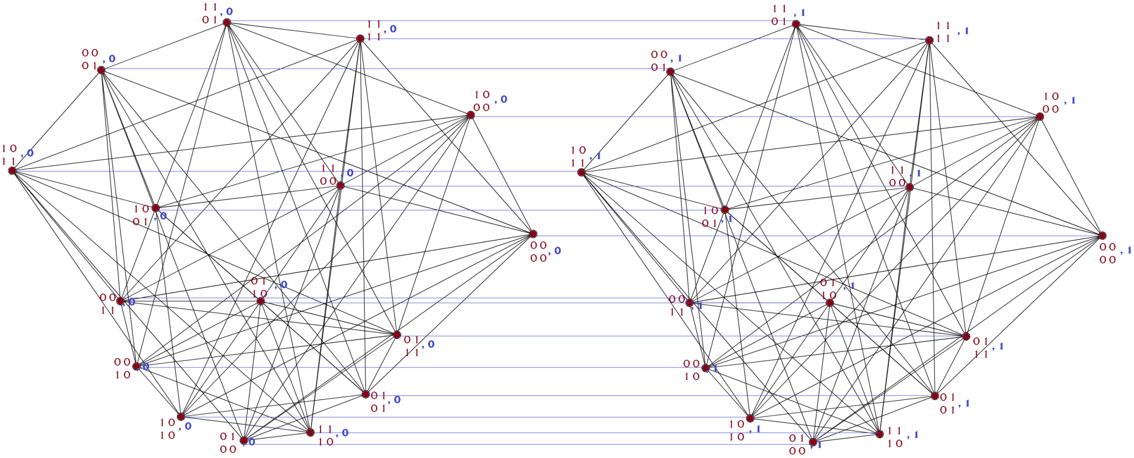
Γ as a Cartesian product of bilinear forms graphs

The bilinear forms graph $\Gamma(\mathbb{F}_2^{2\times 2})$: the vertices are 2×2 matrices over $\mathbb{F}_2$:



Γ as a Cartesian product of bilinear forms graphs

The graph $\Gamma(\mathbb{F}_2^{2\times 2} \times \mathbb{F}_2^{1\times 1})$: each vertex is a $((2 \times 2), (1 \times 1))$ tuple of matrices over $\mathbb{F}_2$:



Delsarte's LP

Computing the eigenvalues of Γ as those of the Cartesian product, Abiad, Khramova and Ravagnani showed that the Ratio-type bound sometimes outperforms previously known bounds.

- Delsarte's linear programming method (1973) is one of the most powerful tools for bounding the sizes of codes in association schemes:
  - Hamming scheme (Hamming distance),
  - Johnson scheme (constant weight codes),
  - Grassmann scheme (subspace metric),
  - bilinear forms scheme (rank-metric codes),
  - Lee distance, etc.
- Abiad, Khramova and Ravagnani (RICCOTA, Rijeka, 2023) asked if this method can be adopted to bound the size of sum-rank metric codes.
  - An obvious obstacle here is that a sum-rank-metric graph is not distance-regular, i.e., it does not explicitly generate an association scheme.

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Association schemes and Delsarte's LP

A **symmetric association scheme** $\mathcal{A} = (X, \mathcal{R})$ on a set X with relations $\mathcal{R} = \{R_0, \dots, R_D\}$:

- $\mathcal{R}$ is a partition of $X \times X$;
- R_0 consists of (x, x) for all $x \in X$.
- $(x, y) \in R_i$ implies $(y, x) \in R_i$ for all x, y and R_i.
- for all $(x, y) \in R_k$, the number of z s.t. $(x, z) \in R_i$ and $(y, z) \in R_j$ is a constant that does not depend on the choice of x, y.

The Hamming scheme:

- $X = \{\text{words of length } n \text{ over } \{1, \dots, q\}\}$,
- $R_i = \{\text{all pairs with Hamming distance } i\}$.

The bilinear forms scheme:

- $\{\text{all matrices from } \mathbb{F}_q^{n \times m}\}$,
- $\{\text{all pairs of matrices with } \mathrm{rk}(A - B) = i\}$.

The relation R_1 in both cases defines the (Hamming, bilinear forms) distance-regular graph, and all relations R_i are the "distance" relations in this graph.

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- for all $(x, y) \in R_k$, the number of z s.t. $(x, z) \in R_i$ and $(y, z) \in R_j$ is a constant that does not depend on the choice of x, y.

In general, an association scheme gives rise to a commutative matrix algebra.

The idea of Delsarte's LP method is to:

- consider a code C as a subset of X of an association scheme,
- then formulate an optimization problem where the objective is to maximize $|C|$ subject to linear constraints imposed by the properties of this matrix algebra.

By solving this linear program, one can obtain upper bounds on the number $|C|$ of "codewords".

Association schemes and Delsarte's LP

A **symmetric association scheme** $\mathcal{A} = (X, \mathcal{R})$ on a set X with relations $\mathcal{R} = \{R_0, \ldots, R_D\}$:

- $\mathcal{R}$ is a partition of $X \times X$;
- R_0 consists of (x, x) for all $x \in X$.
- $(x, y) \in R_i$ implies $(y, x) \in R_i$ for all x, y and R_i.
- for all $(x, y) \in R_k$, the number of z s.t. $(x, z) \in R_i$ and $(y, z) \in R_j$ is a constant that does not depend on the choice of x, y.

Question: How to define an association scheme for the sum-rank metric graph $\Gamma(\mathbb{F}_q^{\mathbf{N} \times \mathbf{M}})$?

Partial order on association schemes

Let $\mathcal{X} = (X, \mathcal{R})$ and $\mathcal{Y} = (X, \mathcal{S})$ be association schemes on the same set X. We say that

$$\mathcal{X} \leq \mathcal{Y}$$

if and only if *every* relation of $\mathcal{X}$ is a union of some relations of $\mathcal{Y}$.

In other words, the partition $\mathcal{S}$ is a refinement of the partition $\mathcal{R}$.

In this case, then $\mathcal{X}$ is said to be a **fusion** (scheme) of $\mathcal{Y}$ and $\mathcal{Y}$ is a **fission** (scheme) of $\mathcal{X}$.

The **trivial** scheme $(X, \{R_0, R_1\})$ is $\leq$ any other scheme on X.

For every graph G, there exists the *smallest* (w.r.t. $\leq$) association scheme* $\mathrm{WL}(G)$ such that the edge set of G is a union of some relations of $\mathrm{WL}(G)$.

$\mathrm{WL}(G)$ is called the **Weisfeiler-Leman (coherent) closure** of G.

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WL(G) is called the **Weisfeiler-Leman (coherent) closure** of G.

Direct product of association schemes

Given two association schemes $\mathcal{A}_i = (X_i, \mathcal{R}_i)$ with $D_i + 1$ relations R^i_j, $j = 0, \dots, D_i$, $i = 1, 2$, their **direct product** $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the association scheme $(X_1 \times X_2, \mathcal{R})$ such that:

- $\mathcal{R} = \{R_{0,0}, R_{0,1}, \dots, R_{0,D_2}, R_{1,0}, \dots, R_{D_1,D_2}\}$;
- $((x_1, x_2), (y_1, y_2)) \in R_{i,j} \iff (x_1, y_1) \in R^1_i$ and $(x_2, y_2) \in R^2_j$.

Lemma

$\mathrm{WL}(G_1 \square G_2) \leq \mathrm{WL}(G_1) \otimes \mathrm{WL}(G_2)$.

$\mathrm{WL}(G_1 \square G_2)$ is an association scheme whenever $\mathrm{WL}(G_1)$ and $\mathrm{WL}(G_2)$ are.

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Association scheme of a sum-rank-metric graph

The coherent closure of the sum-rank metric graph $\Gamma(\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}})$ is contained in the direct product of bilinear forms schemes corresponding to $\Gamma(\mathbb{F}_q^{n_1\times m_1}),\dots,\Gamma(\mathbb{F}_q^{n_t\times m_t})$:

$$\mathrm{WL}(\Gamma(\mathbb{F}_q^{\mathbf{N}\times\mathbf{M}})) \leq \mathrm{WL}(\Gamma(\mathbb{F}_q^{n_1\times m_1}))\otimes\dots\otimes\mathrm{WL}(\Gamma(\mathbb{F}_q^{n_t\times m_t})).$$

Question: When do we have equality?

- Note: the Hamming scheme $<$ the direct product of the trivial schemes.
- We conjecture that equality happens whenever the factors of the Cartesian product are pairwise non-isomorphic.
- Nevertheless, applying the Delsarte's LP method to the *larger* (fission) scheme still gives an upper bound on the size of a code.

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Computational results

bold = best performing bound; <u>underlined</u> = Ratio-type bound outperforms coding bounds.

t	q	**N**	**M**	d	$\|V\|$	Ratio-type	Delsarte LP	iS_d	iH_d	iE_d	S_d	SP_d	PSP_d
2	2	[2, 2]	[2, 2]	3	256	<u>11</u>	**10**	16	19	34	16	13	13
3	2	[2, 2, 1]	[2, 2, 1]	3	512	25	**20**	64	64	151	32	25	25
3	2	[2, 2, 1]	[2, 2, 1]	4	512	10	**6**	16	64	27	8	25	18
3	2	[2, 2, 1]	[2, 2, 2]	3	1024	<u>38</u>	**34**	64	64	151	64	46	46
3	2	[2, 2, 1]	[2, 2, 2]	4	1024	<u>15</u>	**8**	16	64	27	16	46	36
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	3	512	<u>28</u>	**24**	64	64	151	32	30	30
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	4	512	11	**6**	16	64	27	8	30	32
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	3	1024	<u>44</u>	**42**	64	64	151	64	53	53
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	4	1024	18	**10**	16	64	27	16	53	64
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	3	1024	<u>46</u>	**40**	256	215	529	64	48	48
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	4	1024	19	**12**	64	215	119	16	48	36
5	2	[2, 1, 1, 1, 1]	[2, 1, 1, 1, 1]	5	256	5	**2**	16	26	19	4	4	3
5	2	[2, 1, 1, 1, 1]	[3, 1, 1, 1, 1]	5	1024	8	**2**	64	336	240	4	6	3
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	3	1024	56	**49**	256	215	529	64	56	56
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	4	1024	22	**13**	64	215	119	16	56	64
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	4	512	16	**12**	256	512	407	16	34	32
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	5	512	8	**4**	64	77	99	8	6	5
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	5	1024	11	**6**	64	77	99	8	9	8
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	6	1024	7	**2**	16	77	14	4	9	3