Critical sets based on non-trivial autoparatopisms of Latin squares

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(Partial) Latin squares

$\mathcal{PL}(n) := \{n \times n$ partial Latin squares with entries in $[n] \cup \{\cdot\}\}$.

Example :

1	2	·	4
·	3	4	·
3	·	1	2
4	1	·	·

$\in \mathcal{PL}(4)$

$\mathcal{L}(n) := \{n \times n$ Latin squares with entries in $[n]\}$.

Example :

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

$\in \mathcal{L}(4)$

Paratopisms in Latin squares.

Definition

Two Latin squares $L_1, L_2 \in \mathcal{L}(n)$ are **paratopic** if there exist an action σ of the wreath product $\mathcal{P}_n = \mathcal{S}_n \wr \mathcal{S}_3$ such that $L_1^\sigma = L_2$. Then σ is called **paratopism** and $\sigma = (\alpha, \beta, \gamma; \delta)$

- **Isotopism**: $\Theta = (\alpha, \beta, \gamma) \in \mathcal{S}_n^3$.
- **Autotopism**: $L^\Theta = L$.
- **Autoparatopism**: $L^\sigma = L$.

Paratopisms in Latin squares.

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- **Isotopism**: $\Theta = (\alpha, \beta, \gamma) \in \mathcal{S}_n^3$.
- **Autotopism**: $L^\Theta = L$.
- **Autoparatopism**: $L^\sigma = L$.

n	$\lvert\mathcal{L}(n)\rvert$
1	1
2	2
3	12
4	576
5	161280
6	812851200
7	61479419904000
8	108776032459082956800
9	5524751496156892842531225600
10	9982437658213039871725064756920320000
11	776966836171770144107444346734230682311065600000

Representative Latin squares of main classes.

1	2
2	1

L_2

1	2	3
2	3	1
3	1	2

L_3

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$L_{4.1}$

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

$L_{4.2}$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L_{5.1}$

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

$L_{5.2}$

Θ-Orbits and Θ-critical sets.

Let $L \in \mathcal{L}(5)$ and $\Theta \in Atop(L)$:

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$$\Theta = ((2354), (1243), (1243))$$

Let $L \in \mathcal{L}(5)$ and $\Theta \in Atop(L)$:

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1	2	3	4	5
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5	1	2	3	4

$$\Theta = ((2354), (1243), (1243))$$

Θ-Orbits and Θ-critical sets.

Let $L \in \mathcal{L}(5)$ and $\Theta \in Atop(L)$:

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$$\Theta = ((2354),(1243),(1243))$$

·	·	·	·	·
·	3	4	·	1
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

Θ-Orbits and Θ-critical sets.

Let $L \in \mathcal{L}(5)$ and $\Theta \in Atop(L)$:

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$$\Theta = ((2354), (1243), (1243))$$

·	·	·	·	·
·	3	4	·	1
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

$$\xrightarrow{\Theta\text{-forced}}$$

·	·	·	·	·
·	3	4	·	1
3	·	·	1	2
4	·	·	2	3
·	1	2	·	4

Θ-Orbits and Θ-critical sets.

Let $L \in \mathcal{L}(5)$ and $\Theta \in Atop(L)$:

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$$\Theta = ((2354), (1243), (1243))$$

·	·	·	·	·
·	3	4	·	1
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

$\xrightarrow{\Theta\text{-forced}}$

·	·	·	·	·
·	3	4	·	1
3	·	·	1	2
4	·	·	2	3
·	1	2	·	4

$\xrightarrow[\Theta\text{-forced}]{\mathcal{L}\text{-forced}}$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Θ-Orbits and Θ-critical sets.

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Research article

A census of critical sets based on non-trivial autotopisms of Latin squares of order up to five

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Abstract: This paper delves into the study of critical sets of Latin squares having a given isotopism in their autotopism group. Particularly, we prove that the sizes of these critical sets only depend on both the main class of the Latin square and the cycle structure of the isotopism under consideration. Keeping then in mind that the autotopism group of a Latin square acts faithfully on the set of entries of the latter, we enumerate all the critical sets based on autotopisms of Latin squares of order up to five.

Keywords: Latin square; autotopism; cycle structure; critical set; enumeration
Mathematics Subject Classification: 05B15

Θ-Orbits and Θ-critical sets.

L	$\Theta \in \mathrm{Atop}(L)$	z_Θ	$\lvert \mathrm{CS}_\Theta(L)\rvert$	$\mathrm{scs}_\Theta(L)$	$\mathrm{lcs}_\Theta(L)$
L_2	$(\mathrm{Id}_2, \mathrm{Id}_2, \mathrm{Id}_2)$	$(1^2, 1^2, 1^2)$	4	1	1
	$((12), (12), \mathrm{Id}_2)$	$(2, 2, 1^2)$	4	1	1
L_3	$(\mathrm{Id}_3, \mathrm{Id}_3, \mathrm{Id}_3)$	$(1^3, 1^3, 1^3)$	27	2	3
	$((12), (12), (13))$	$(21, 21, 21)$	14	1	2
	$((123), (132), \mathrm{Id}_3))$	$(3, 3, 1^3)$	27	2	2
	$((123), (123), (132))$	$(3, 3, 3)$	9	1	1
$L_{4.1}$	$(\mathrm{Id}_4, \mathrm{Id}_4, \mathrm{Id}_4)$	$(1^4, 1^4, 1^4)$	576	5	7
	$((12)(34), (12)(34), \mathrm{Id}_4)$	$(2^2, 2^2, 1^4)$	192	4	4
	$((23), (14), (14))$	$(21^2, 21^2, 21^2)$	256	4	4
	$((12)(34), (13)(24), (14)(23))$	$(2^2, 2^2, 2^2)$	256	3	3
	$((243), (134), (134))$	$(31, 31, 31)$	90	2	2
	$((1234), (1234), (24))$	$(4, 4, 21^2)$	64	2	2
$L_{4.2}$	$(\mathrm{Id}_4, \mathrm{Id}_4, \mathrm{Id}_4)$	$(1^4, 1^4, 1^4)$	736	4	6
	$((12)(34), (12)(34), \mathrm{Id}_4)$	$(2^2, 2^2, 1^4)$	192	4	4
	$((13)(24), (14)(23), (34))$	$(2^2, 2^2, 21^2)$	224	3	3
	$((12), (12), (34))$	$(21^2, 21^2, 21^2)$	256	4	4
	$((1324), (1324), (12)(34))$	$(4, 4, 2^2)$	64	2	2
	$((1423), (1324), \mathrm{Id}_4)$	$(4, 4, 1^4)$	256	3	3
$L_{5.1}$	$(\mathrm{Id}_5, \mathrm{Id}_5, \mathrm{Id}_5)$	$(1^5, 1^5, 1^5)$	53250	6	10
	$((12)(35), (13)(45), (14)(23))$	$(2^21, 2^21, 2^21)$	3088	3	5
	$((2354), (1243), (1243))$	$(41, 41, 41)$	832	3	3
	$((12345), (15432), \mathrm{Id}_5)$	$(5, 5, 1^5)$	3125	4	4
	$((12345), (12345), (13524))$	$(5, 5, 5)$	250	2	2
$L_{5.2}$	$(\mathrm{Id}_5, \mathrm{Id}_5, \mathrm{Id}_5)$	$(1^5, 1^5, 1^5)$	48462	7	11
	$((13)(45), (25)(34), (13)(45))$	$(2^21, 2^21, 2^21)$	2896	3	5
	$((345), (345), (345))$	$(31^2, 31^2, 31^2)$	8424	5	6

CONTENTS

Let $L \in \mathcal{L}(5)$:

$L =$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Let $L \in \mathcal{L}(5)$:

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

- Has the following conjugates:

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$\pi \in \{Id, (12)\}$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

$\pi \in \{(13), (123)\}$

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

$\pi \in \{(23), (132)\}$

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354),(5324),(2354)),(123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

- (1,1,1) $\rightarrow$

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

- $(1,1,1) \rightarrow (1,1,1) \rightarrow$

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

- (1,1,1) $\rightarrow (1, 1, 1) \rightarrow$ (1,1,1)

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

- $(1,1,1) \rightarrow (1,1,1) \rightarrow (1,1,1)$
- $(1,2,2) \rightarrow$

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354),(5324),(2354)),(123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

- (1,1,1) $\rightarrow (1,1,1) \rightarrow$ (1,1,1)
- (1,2,2) $\rightarrow (2,1,2) \rightarrow$

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

- $(1,1,1) \to (1,1,1) \to (1,1,1)$
- $(1,2,2) \to (2,1,2) \to (3,1,3) \to \ldots$

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

·	·	·	·	·
·	3	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^{π} its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^{\pi} \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

·	·	·	·	·
·	3	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

$\xrightarrow{\sigma - forced}$

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354),(5324),(2354)),(123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

·	·	·	·	·
·	3	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

$\xrightarrow{\sigma - forced}$

·	·	·	·	·
·	3	4	5	·
·	4	5	·	2
·	5	·	2	3
·	·	2	3	4

σ-orbits and σ-critical sets.

Let $L \in \mathcal{L}(5)$, L^{π} its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^{\pi} \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

·	·	·	·	·
·	3	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

$\xrightarrow{\sigma - forced}$

·	·	·	·	·
·	3	4	5	·
·	4	5	·	2
·	5	·	2	3
·	·	2	3	4

$\xrightarrow[\mathcal{L} - forced]{\sigma - forced}$

Let $L \in \mathcal{L}(5)$, L^π its conjugate by $\pi = (123)$ and $\sigma \in Par(L)$:

$$\sigma = (((2354), (5324), (2354)), (123))$$

$L \equiv$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L^\pi \equiv$

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

·	·	·	·	·
·	3	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

$\xrightarrow{\sigma - forced}$

·	·	·	·	·
·	3	4	5	·
·	4	5	·	2
·	5	·	2	3
·	·	2	3	4

$\xrightarrow[\mathcal{L} - forced]{\sigma - forced}$

·	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

L	π	Θ	$\mathrm{scs}_\sigma(L)$	$\mathrm{lcs}_\sigma(L)$
L_2	(12)	(Id, Id, Id)	1	1
		(Id, (12), (12))	1	1
		((12), Id, (12))	1	1
		((12), (12), Id)	1	1
	(13)	(Id, Id, Id)	1	1
		(Id, (12), (12))	1	1
		((12), Id, (12))	1	1
		((12), (12), Id)	1	1
	(23)	(Id, Id, Id)	1	1
		(Id, (12), (12))	1	1
		((12), Id, (12))	1	1
		((12), (12), Id)	1	1
	(123)	(Id, Id, Id)	1	1
		(Id, (12), (12))	1	1
		((12), Id, (12))	1	1
		((12), (12), Id)	1	1
	(132)	(Id, Id, Id)	1	1
		(Id, (12), (12))	1	1
		((12), Id, (12))	1	1
		((12), (12), Id)	1	1

L	π	Θ	$\mathrm{scs}_\sigma(L)$	$\mathrm{lcs}_\sigma(L)$
L_3	(12)	(Id, Id, Id)	2	2
		(Id, (123), (123))	1	1
		((23), (23), (23))	1	1
		((123), Id, (123))	1	1
		((123), (123), (132))	1	1
		((123), (132), Id)	2	2
	(13)	(Id, (32), Id)	1	1
		(Id, (12), (123))	1	1
		((23), Id, (23))	2	2
		((23), (123), (12))	1	1
		((123), (32), (123))	1	1
		((123), (13), Id)	1	1
	(23)	(Id, (32), (32))	2	2
		((23), Id, Id)	1	1
		((23), (123), (123))	1	1
		((12), Id, (123))	1	1
		((12), (132), Id)	1	1
		((123), (32), (12))	1	1
	(123)	(Id, (32), Id)	1	1
		(Id, (12), (123))	1	1
		((23), Id, (23))	1	1
		((23), (123), (12))	1	1
		((123), (32), (123))	1	1
		((123), (13), Id)	1	1
	(132)	(Id, (32), (32))	1	1
		((23), Id, Id)	1	1
		((23), (123), (123))	1	1
		((12), Id, (123))	1	1
		((12), (132), Id)	1	1
		((123), (32), (12))	1	1

L	π	Θ	$\mathrm{scs}_\sigma(L)$	$\mathrm{lcs}_\sigma(L)$
$L_{4.1}$	(12)	(Id, Id, Id)	4	6
		(Id, (12)(34), (12)(34))	3	3
		((34), (34), (34))	3	4
		((34), (1324), (1324))	2	2
		((234), (234), (234))	1	1
		((12)(34), Id, (12)(34))	3	3
		((12)(34), (12)(34), Id)	4	6
		((12)(34), (13)(24), (14)(23))	3	3
		((1234), (24), (1234))	2	2
		((1234), (1234), (24))	3	3
	(13)	(Id, Id, Id)	4	6
		(Id, (12)(34), (12)(34))	3	3
		((34), (34), (34))	3	4
		((34), (1324), (1324))	2	2
		((234), (234), (234))	1	1
		((12)(34), Id, (12)(34))	4	6
		((12)(34), (12)(34), Id)	3	3
		((12)(34), (13)(24), (14)(23))	3	3
		((1234), (24), (1234))	3	3
		((1234), (1234), (24))	2	2
	(23)	(Id, Id, Id)	4	6
		(Id, (12)(34), (12)(34))	4	6
		((34), (34), (34))	3	4
		((34), (1324), (1324))	3	3
		((234), (234), (234))	1	1
		((12)(34), Id, (12)(34))	3	3
		((12)(34), (12)(34), Id)	3	3
		((12)(34), (13)(24), (14)(23))	3	3
		((1234), (24), (1234))	2	2
		((1234), (1234), (24))	2	2
	(123)	(Id, Id, Id)	3	3
		(Id, (12)(34), (12)(34))	3	3
		((34), (34), (34))	2	2
		((34), (1324), (1324))	2	2
		((234), (234), (234))	2	3
		((12)(34), Id, (12)(34))	3	3
		((12)(34), (12)(34), Id)	3	3
		((12)(34), (13)(24), (14)(23))	3	3
		((1234), (24), (1234))	2	2
		((1234), (1234), (24))	2	2
	(132)	(Id, Id, Id)	3	3
		(Id, (12)(34), (12)(34))	3	3
		((34), (34), (34))	2	2
		((34), (1324), (1324))	2	2
		((234), (234), (234))	2	3
		((12)(34), Id, (12)(34))	3	3
		((12)(34), (12)(34), Id)	3	3
		((12)(34), (13)(24), (14)(23))	3	3
		((1234), (24), (1234))	2	2
		((1234), (1234), (24))	2	2

L	π	Θ	$\mathrm{scs}_\sigma(L)$	$\mathrm{lcs}_\sigma(L)$
$L_{4.2}$	(12)	(Id, Id, Id)	3	4
		(Id, (12)(34), (12)(34))	3	3
		(Id, (1324), (1324))	1	1
		((34), (34), (34))	3	4
		((34), (13)(24), (13)(24))	2	2
		((12)(34), Id, (12)(34))	3	3
		((12)(34), (12)(34), Id)	3	4
		((12)(34), (1324), (1423))	1	1
		((13)(24), (34), (13)(24))	2	2
		((13)(24), (13)(24), (12))	3	4
		((1324), Id, (1324))	1	1
		((1324), (12)(34), (1423))	1	1
		((1324), (1324), (12)(34))	3	3
		((1324), (1423), Id)	3	4
	(13)	(Id, (43), Id)	3	4
		(Id, (12), (12)(34))	3	3
		(Id, (13)(24), (1324))	2	2
		((34), Id, (34))	3	4
		((34), (12)(43), (12))	3	3
		((34), (1324), (13)(24))	1	1
		((12)(34), (43), (12)(34))	3	4
		((12)(34), (12), Id)	3	3
		((12)(34), (13)(24), (1423))	2	2
		((13)(24), Id, (13)(24))	3	4
		((13)(24), (12)(43), (14)(23))	3	3
		((13)(24), (1324), (12))	1	1
		((1324), (43), (1324))	3	3
		((1324), (13)(24), (12)(34))	2	2
		((1324), (14)(23), Id)	2	2
	(23)	(Id, (43), (43))	3	4
		(Id, (13)(24), (13)(24))	3	4
		((34), Id, Id)	3	4
		((34), (12)(43), (12)(43))	3	4
		((34), (1324), (1324))	3	3
		((12), Id, (12)(43))	3	3
		((12), (12)(43), Id)	3	3
		((12)(34), (43), (12))	3	3
		((12)(34), (13)(24), (14)(23))	3	3
		((13)(24), Id, (1324))	2	2
		((13)(24), (12)(43), (1423))	2	2
		((13)(24), (1324), (12)(43))	2	2
		((13)(24), (1423), Id)	2	2
		((1324), (43), (13)(24))	1	1
		((1324), (13)(24), (12))	1	1
	(123)	(Id, (43), Id)	2	2
		(Id, (12), (12)(34))	2	2
		(Id, (13)(24), (1324))	2	2
		((34), Id, (34))	2	3
		((34), (12)(43), (12))	2	3
		((34), (1324), (13)(24))	2	3
		((12)(34), (43), (12)(34))	2	2
		((12)(34), (12), Id)	2	2
		((12)(34), (13)(24), (1423))	2	2
		((13)(24), Id, (13)(24))	2	3
		((13)(24), (12)(43), (14)(23))	2	3
		((13)(24), (1324), (12))	2	3
		((1324), (43), (1324))	2	2
		((1324), (13)(24), (12)(34))	2	2
		((1324), (14)(23), Id)	2	2
	(132)	(Id, (43), (43))	2	3
		(Id, (13)(24), (13)(24))	2	3
		((34), Id, Id)	2	2
		((34), (12)(43), (12)(43))	2	2
		((34), (1324), (1324))	2	2
		((12), Id, (12)(43))	2	2
		((12), (12)(43), Id)	2	2
		((12)(34), (43), (12))	2	3
		((12)(34), (13)(24), (14)(23))	2	3
		((13)(24), Id, (1324))	2	2
		((13)(24), (12)(43), (1423))	2	2
		((13)(24), (1324), (12)(43))	2	2
		((13)(24), (1423), Id)	2	2
		((1324), (43), (13)(24))	2	3
		((1324), (13)(24), (12))	2	3

L	π	Θ	$\mathrm{scs}_\sigma(L)$	$\mathrm{lcs}_\sigma(L)$
$L_{5.1}$	(12)	(Id, Id, Id)	4	6
		(Id, (12345), (12345))	1	1
		((2354), (2354), (2354))	3	3
		((25)(34), (25)(34), (25)(34))	3	4
		((12345), Id, (12345))	1	1
		((12345), (12345), (13524))	1	1
		((12345), (15432), Id)	4	6
	(13)	(Id, (52)(43), Id)	3	4
		(Id, (12)(53), (12345))	2	2
		((2354), (5324), (2354))	3	3
		((25)(34), Id, (25)(34))	4	6
		((25)(34), (12345), (12)(35))	1	1
		((12345), (52)(43), (12345))	2	2
		((12345), (15)(42), Id)	2	2
	(23)	(Id, (52)(43), (52)(43))	4	6
		((2354), (5324), (5324))	3	3
		((25)(34), Id, Id)	3	4
		((25)(34), (12345), (12345))	2	2
		((12)(35), Id, (12345))	2	2
		((12)(35), (15432), Id)	2	2
		((12345), (52)(43), (12)(53))	1	1
	(123)	(Id, (52)(43), Id)	1	1
		(Id, (12)(53), (12345))	1	1
		((2354), (5324), (2354))	1	1
		((25)(34), Id, (25)(34))	2	4
		((25)(34), (12345), (12)(35))	2	4
		((12345), (52)(43), (12345))	1	1
		((12345), (15)(42), Id)	1	1
	(132)	(Id, (52)(43), (52)(43))	2	4
		((2354), (5324), (5324))	1	1
		((25)(34), Id, Id)	1	1
		((25)(34), (12345), (12345))	1	1
		((12)(35), Id, (12345))	1	1
		((12)(35), (15432), Id)	1	1
		((12345), (52)(43), (12)(53))	2	4

L	π	Θ	$\mathrm{scs}_\sigma(L)$	$\mathrm{lcs}_\sigma(L)$
$L_{5.2}$	(12)	((12)(45), (12)(45), (34))	4	6
		((123), (15342), (1354))	3	3
		((12345), (15432), (13))	4	6
		((12354), (152), (1345))	3	3
	(13)	((45), (43), (45))	4	6
		((1345), (2534), (1345))	3	3
	(23)	((45), (12)(53), (12)(53))	4	6
		((13), (15432), (12345))	4	6
		((1345), (152), (12354))	3	3
		((1354), (15342), (123))	3	3
	(123)	(Id, (12)(543), (12)(345))	4	5
		((345), (12), (12)(354))	3	6
		((354), (12)(534), (12))	3	5
		((13)(45), (1532), (1235))	2	2
		((134), (1542), (123)(45))	3	5
		((135), (152)(34), (1234))	3	6
	(132)	((12), (12)(534), (543))	3	6
		((12)(345), (12)(543), Id)	4	5
		((12)(354), (12), (534))	3	5
		((123)(45), (1542), (134))	3	6
		((1234), (152)(34), (135))	3	5
		((1235), (1532), (13)(54))	2	2

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Critical sets based on non-trivial autoparatopisms of Latin squares

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