Exponentiation of Graphs

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Outline

1. Exponentiation of graphs
   - Definition
   - Order, Degree and Diameter
2. Connectivity
3. Hamiltonicity
4. Applications
   - Multiexponential bounded-degree networks with logarithmic diameter

Graph Operations

- There are many graph operations defined so far.

◆ Binary operations:

1. Union
2. Join
3. Cartesian product
4. Lexicographic product
5. Strong product

⋮

◆ Unary operations:

1. Complementation
2. Line graph operation
3. Subdivided-line graph operation

⋮

- We newly introduce exponential operation on graphs.
- Our motivation is to construct very large-scale networks with logarithmic diameter.

Exponentiation of graphs

G: graph of order p

H: graph of order q with $V(H) = \{w_1, w_2, \dots, w_q\}$

Definition of G^H:

- $V(G^H) = \{(u_1, u_2, \dots, u_q; w_j) \mid u_i \in V(G), 1 \leq i, j \leq q\}$
- $(u_1, u_2, \dots, u_q; w_j)(v_1, v_2, \dots, v_q; w_k) \in E(G^H)$

$\Longleftrightarrow$

- $j = k, u_j v_j \in E(G)$, and $u_i = v_i$ for $i \neq j$

  or

- $j \neq k, w_j w_k \in E(H)$, and $u_i = v_i$ for $1 \leq i \leq q$

G: graph of order p

H: graph of order q with $V(H) = \{w_1, w_2, \ldots, w_q\}$

Definition of G^H:

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- $(u_1, u_2, \ldots, u_q; w_j)(v_1, v_2, \ldots, v_q; w_k) \in E(G^H)$

⟺

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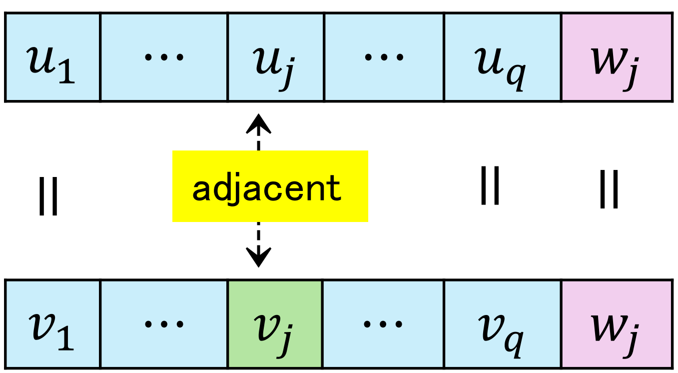
$u_i = v_i$ for $i \ne j$



j-th G-edge

G-edge of dimension j

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H: graph of order q with $V(H) = \{w_1, w_2, \ldots, w_q\}$

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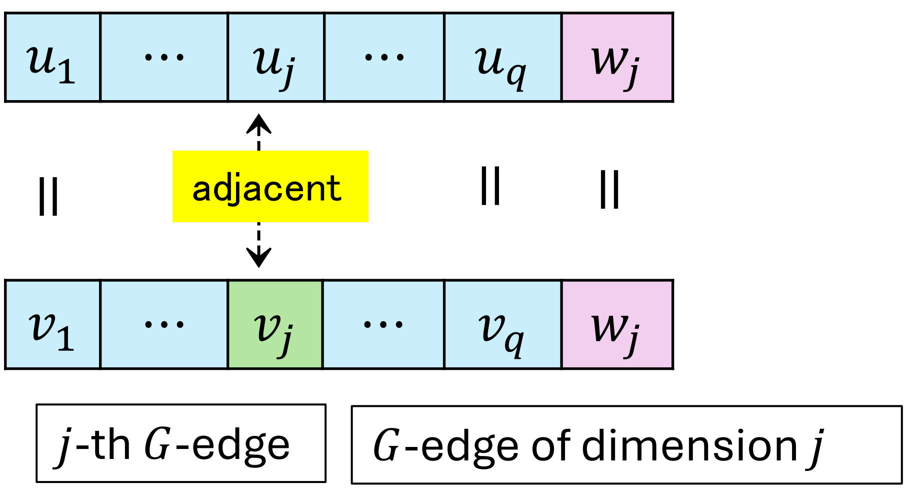
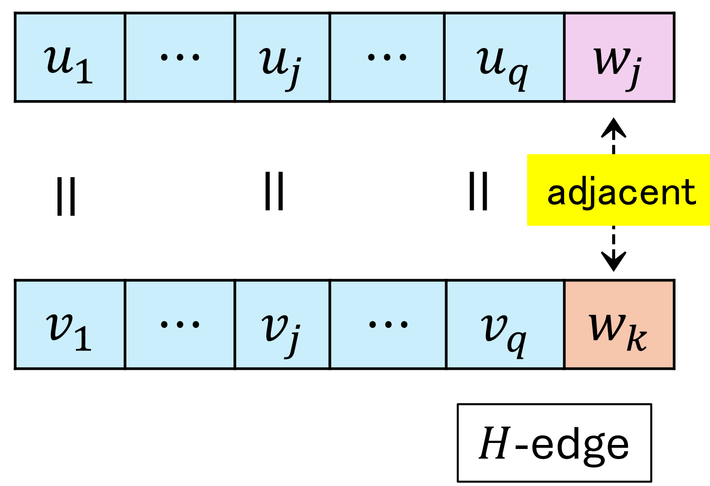
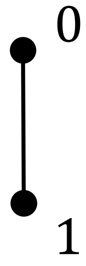




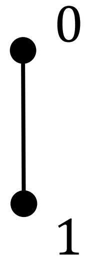
Examples

K_2:



$$K_2^{K_2} = G^H \quad \begin{cases} G = K_2 \\ H = K_2 \end{cases}$$

K_2:



(0,0; 0) (1,0; 0)

(0,0; 1) (1,0; 1)

(0,1; 0) (1,1; 0)

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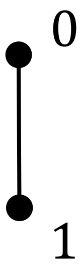

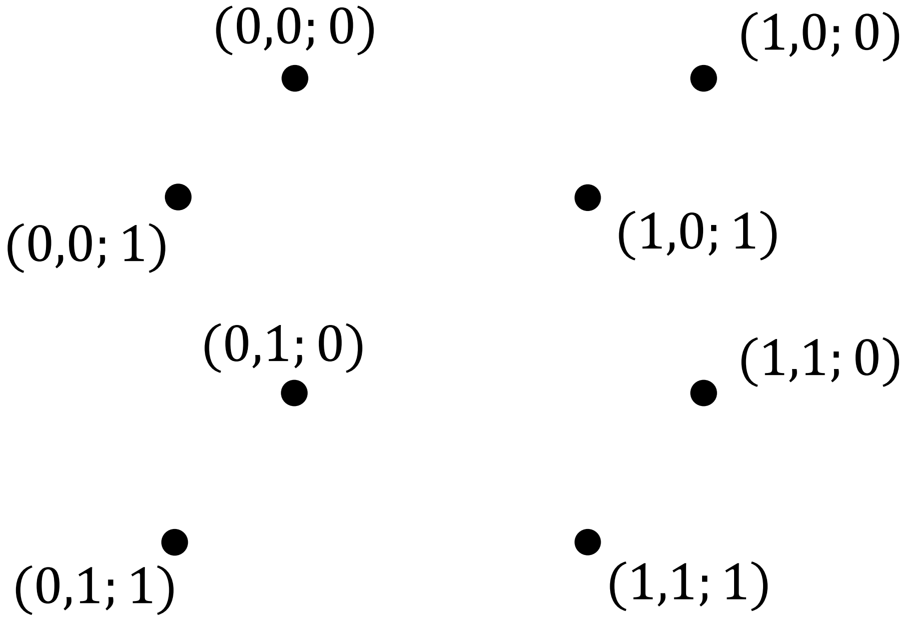

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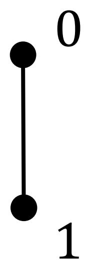
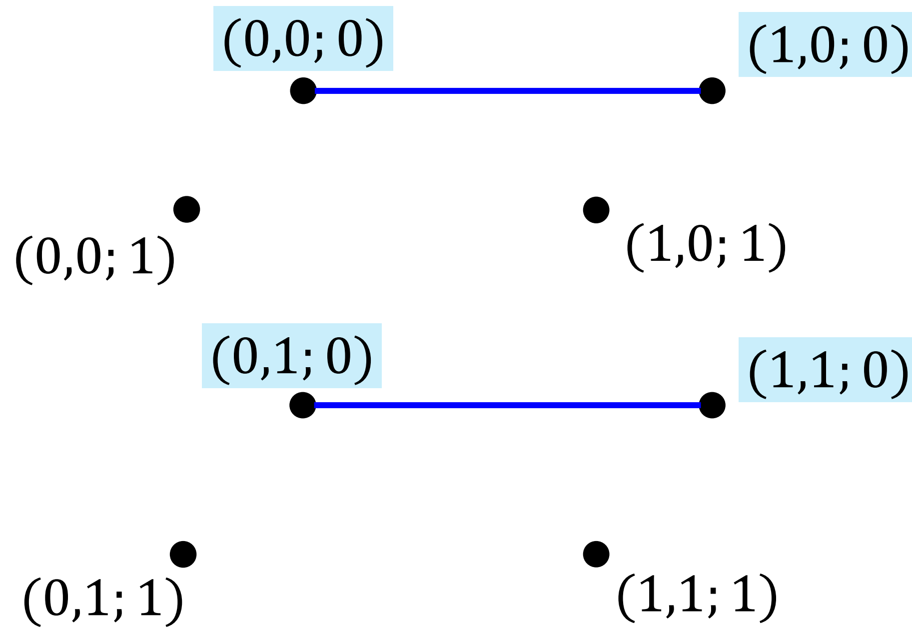
1-st G-edges

K_2:





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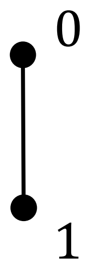
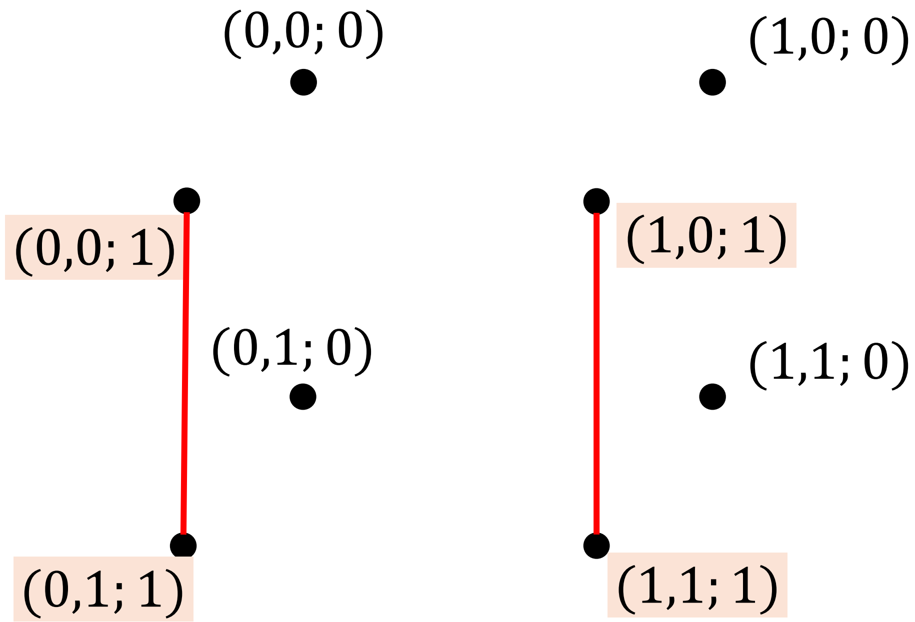
2-nd G-edges

K_2:

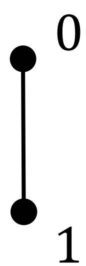


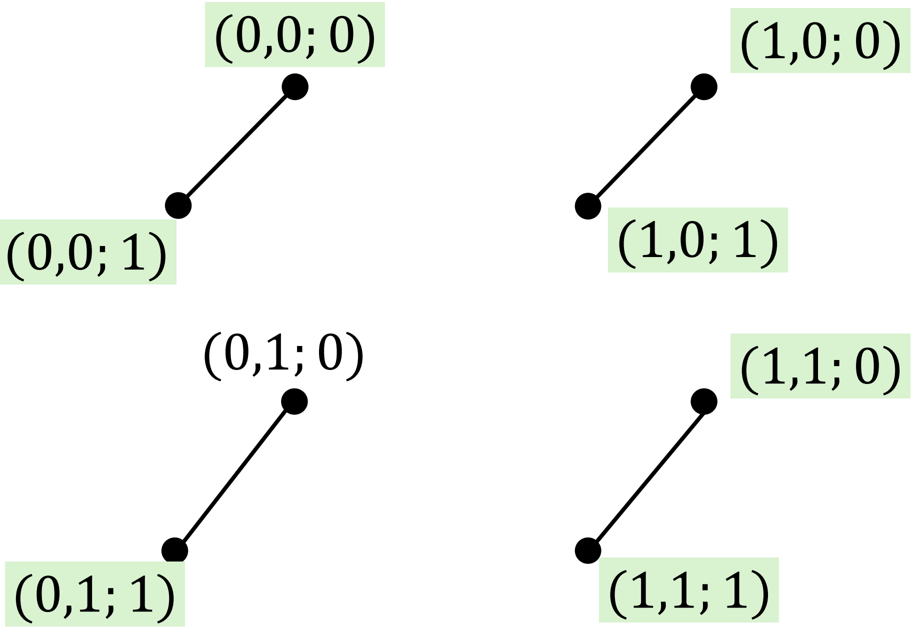


$$K_2^{K_2} = G^H \begin{cases} G = K_2 \\ H = K_2 \end{cases}$$

H-edges

K_2:





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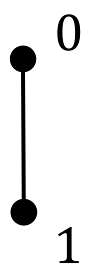

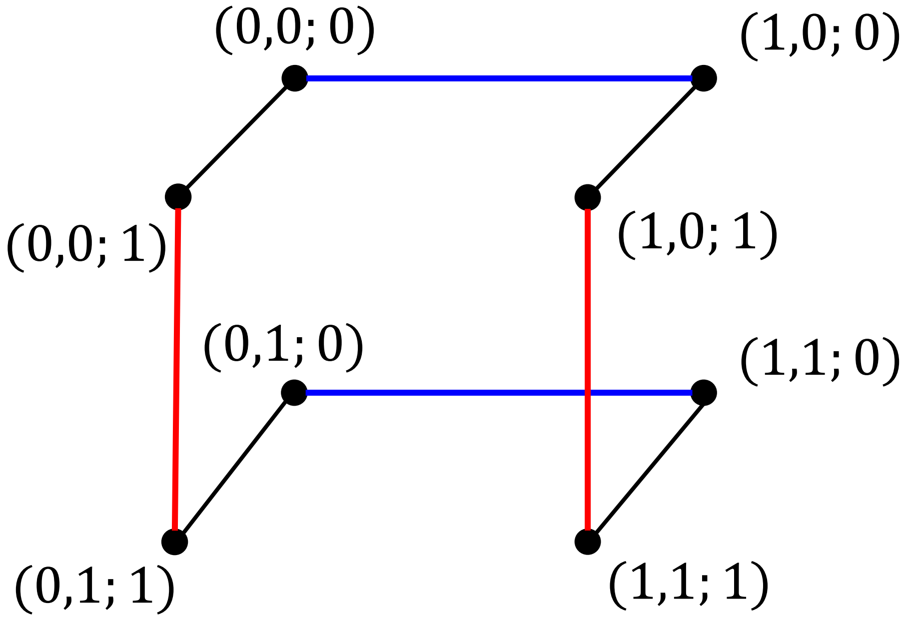

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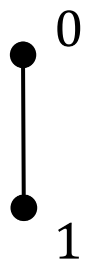
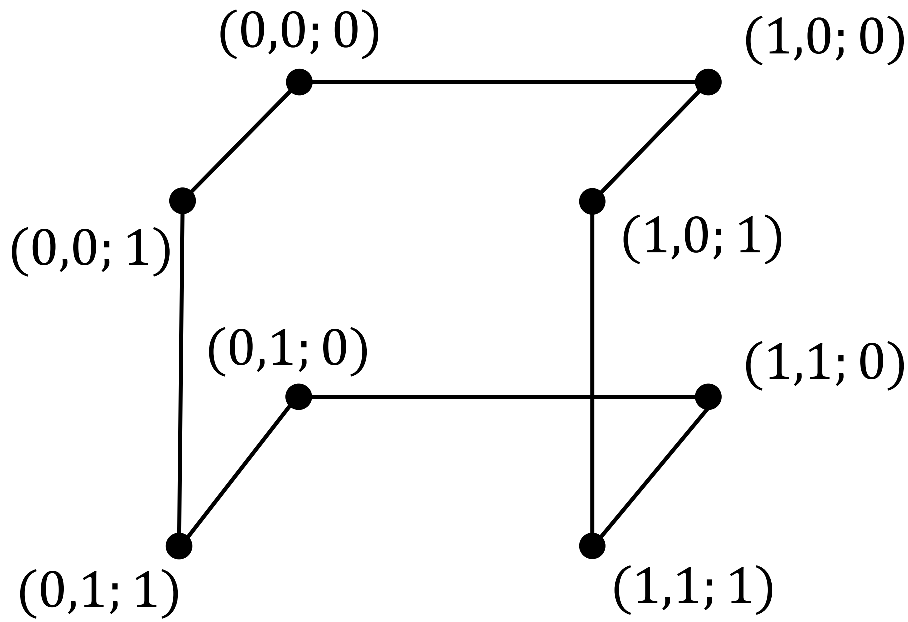
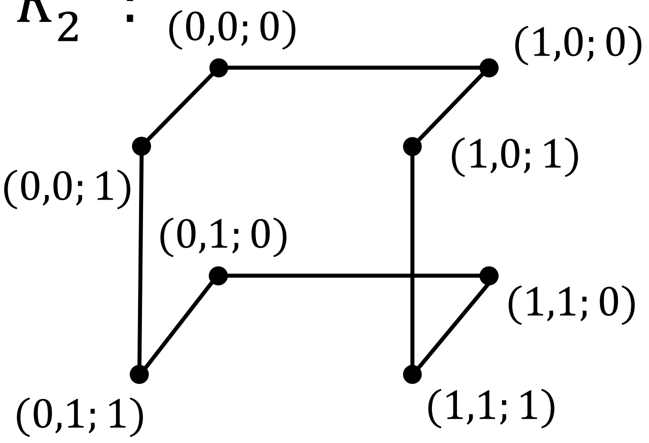




$K_2^{K_2}$:



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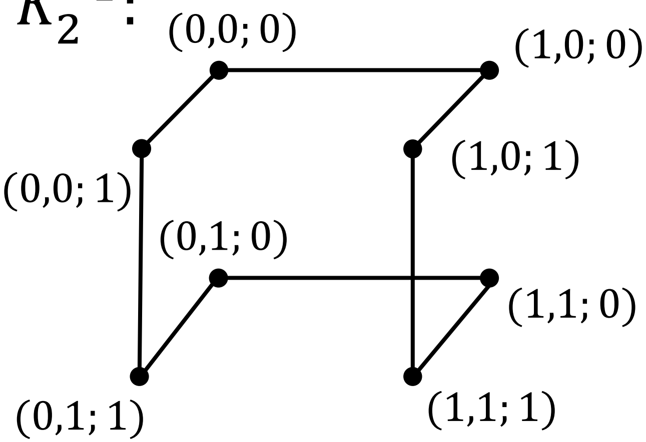
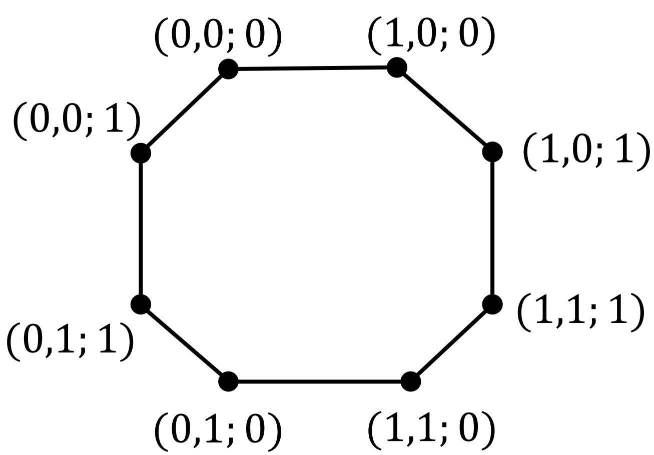
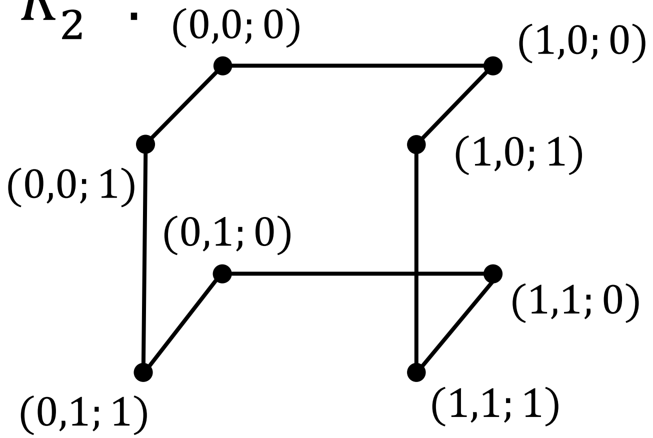
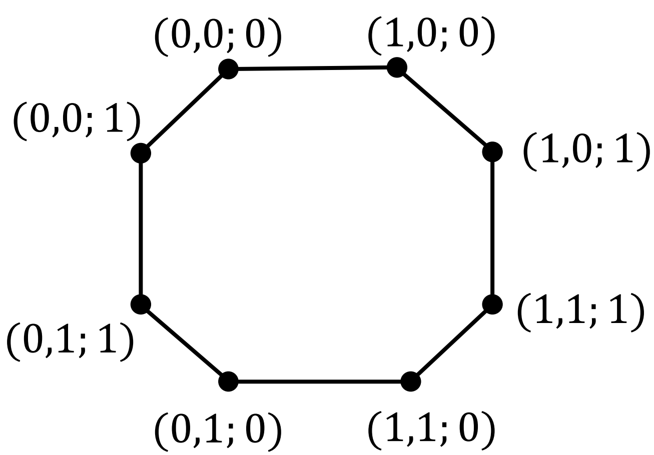


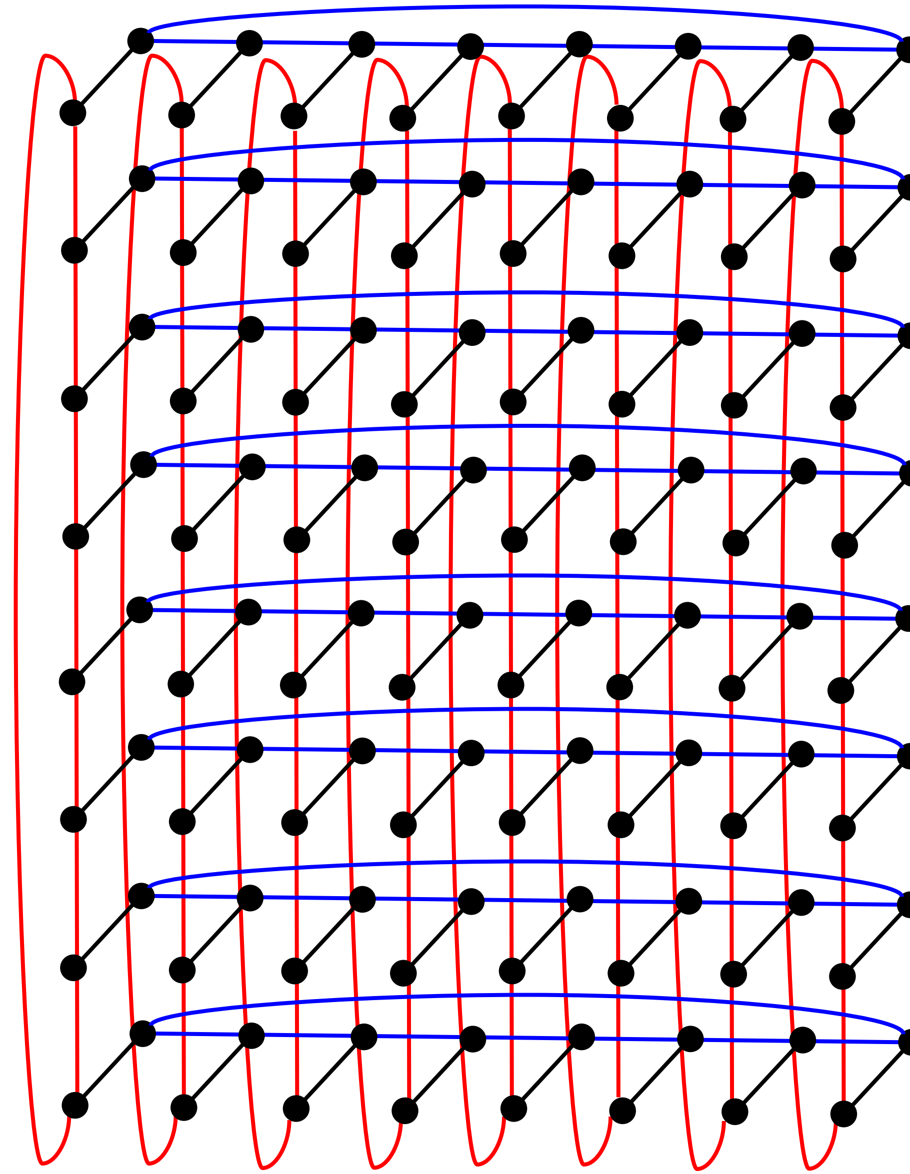


$$\left(K_2^{K_2}\right)^{K_2} \cong C_8^{K_2}:$$

$K_2^{K_2}$:







G: graph of order p

H: graph of order q with $V(H) = \{w_1, w_2, \dots, w_q\}$

Order
$|V(G^H)| = p^q q$

Definition of G^H:

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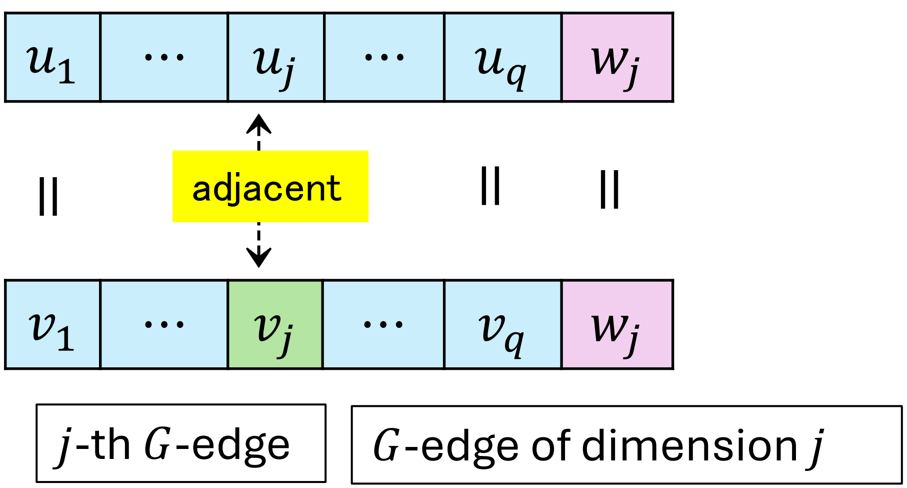

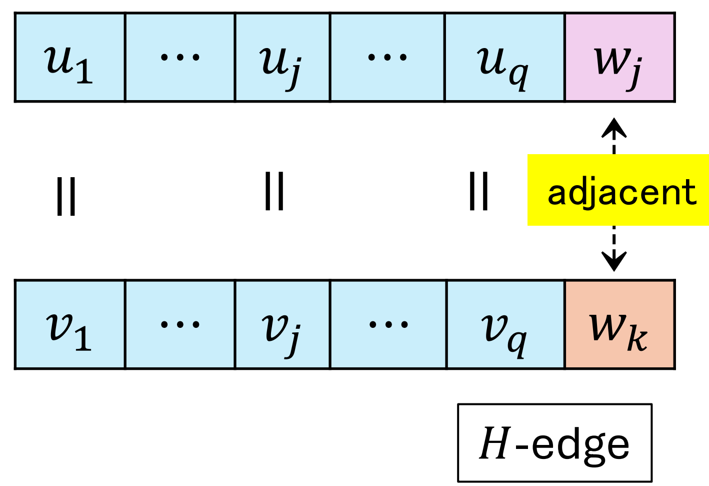

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Maximum degree
$\Delta(G^H) = \Delta(G) + \Delta(H)$

Minimum degree
$\delta(G^H) = \delta(G) + \delta(H)$

⟺

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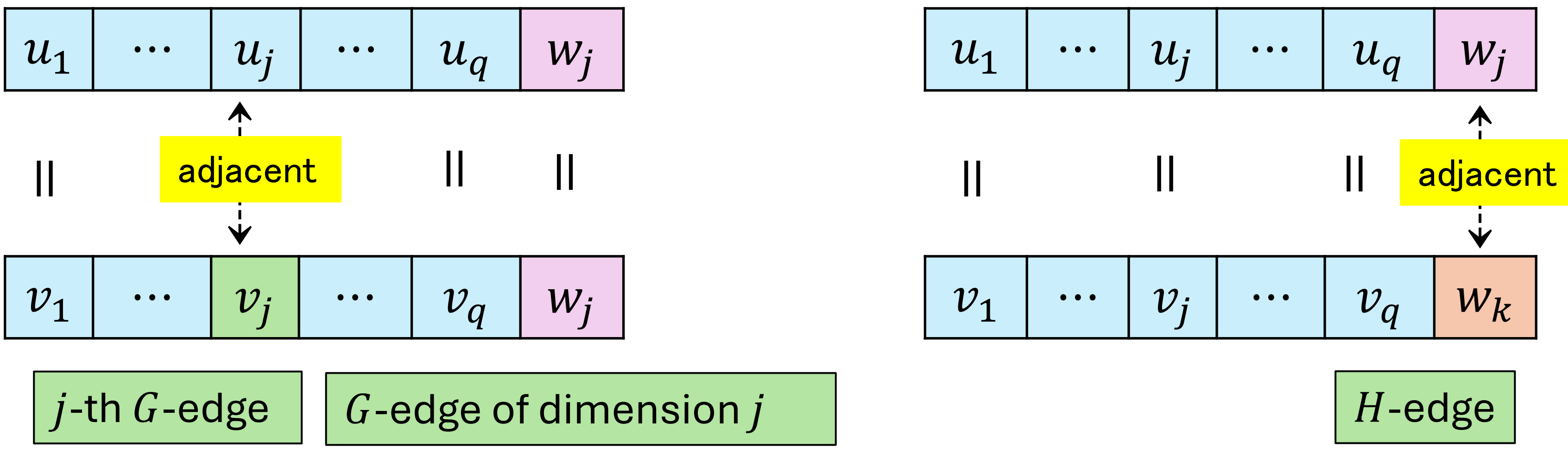
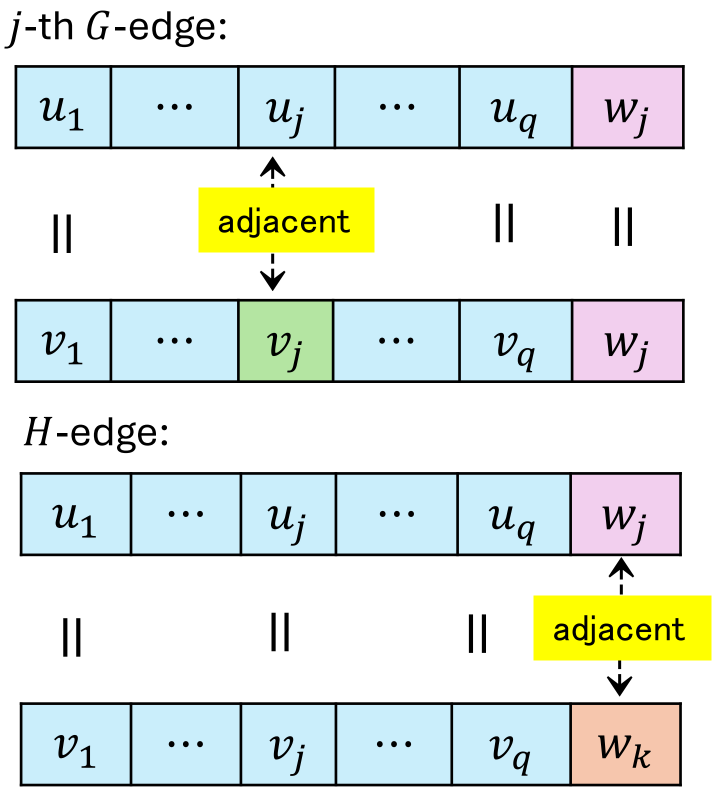
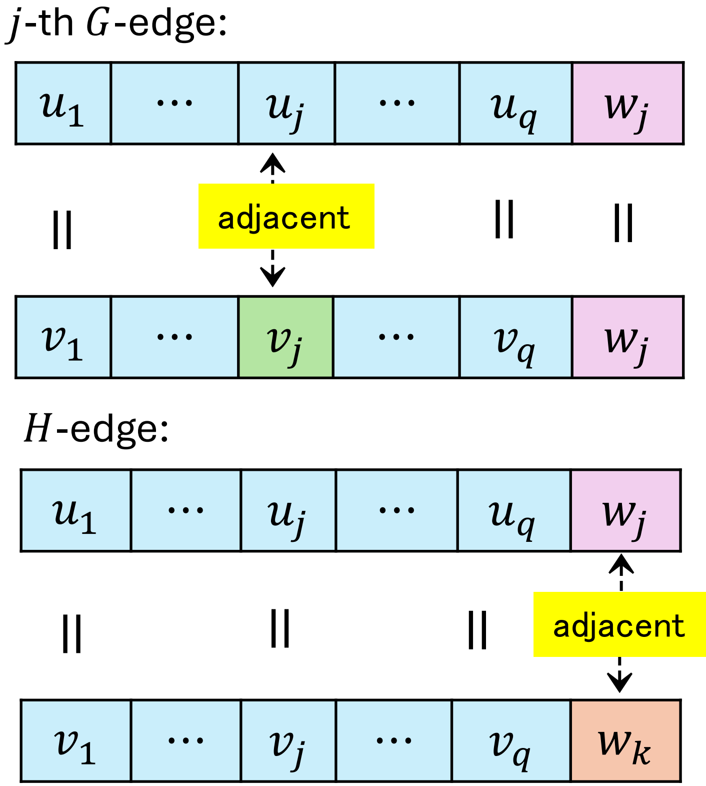




Let $u = (u_1, \ldots, u_q; w_j)$ and $v = (v_1, \ldots, v_q; w_k)$.

Both G and H are connected.



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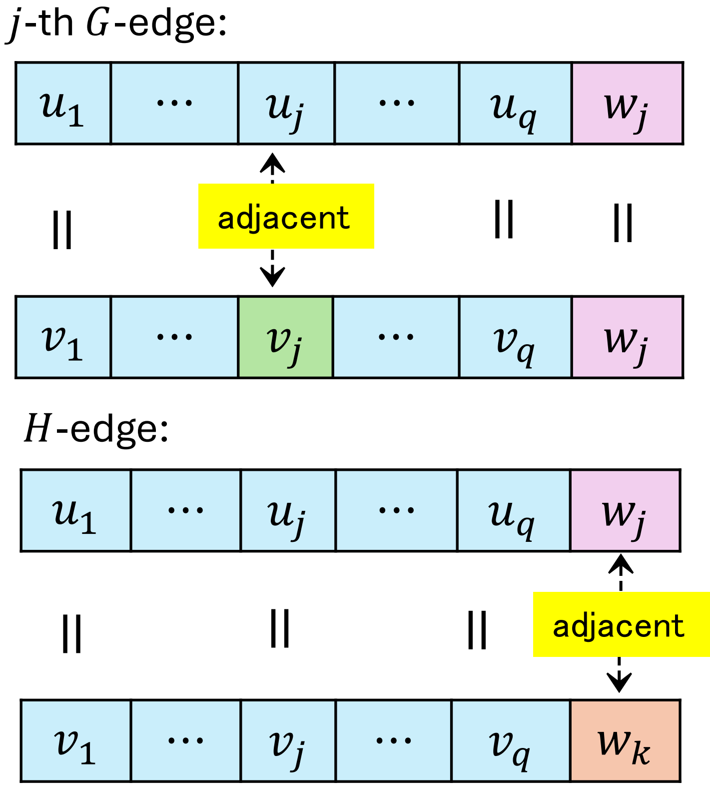
Let $W = \{w_i \in V(H) \mid u_i \neq v_i\}$.



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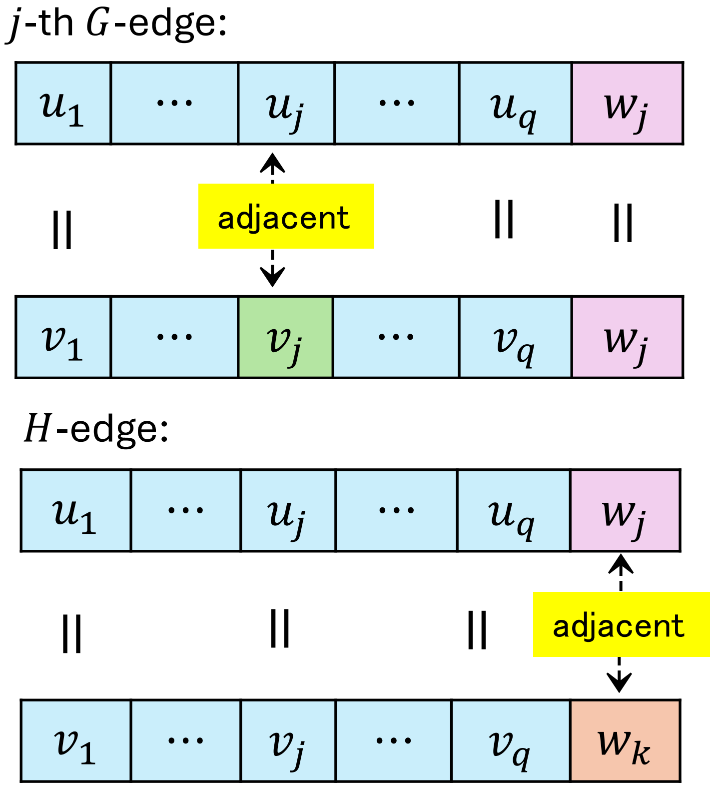
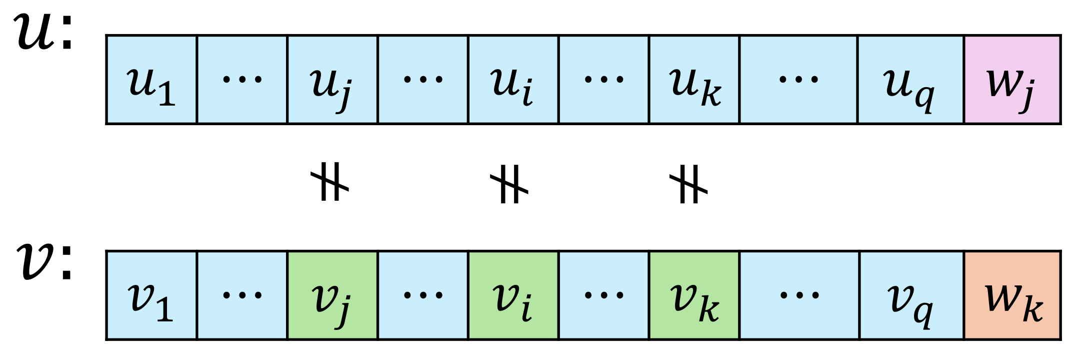
Let $W = \{w_i \in V(H) \mid u_i \neq v_i\}$.

Let P be a walk from w_j to w_k which goes through every vertex in W.





$$W = \{w_j, w_i, w_k\}$$

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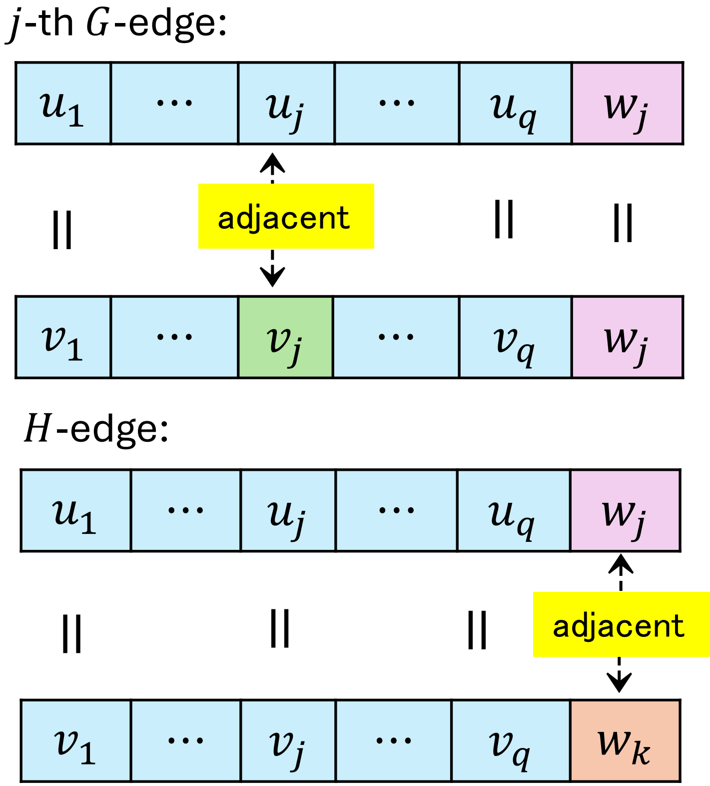

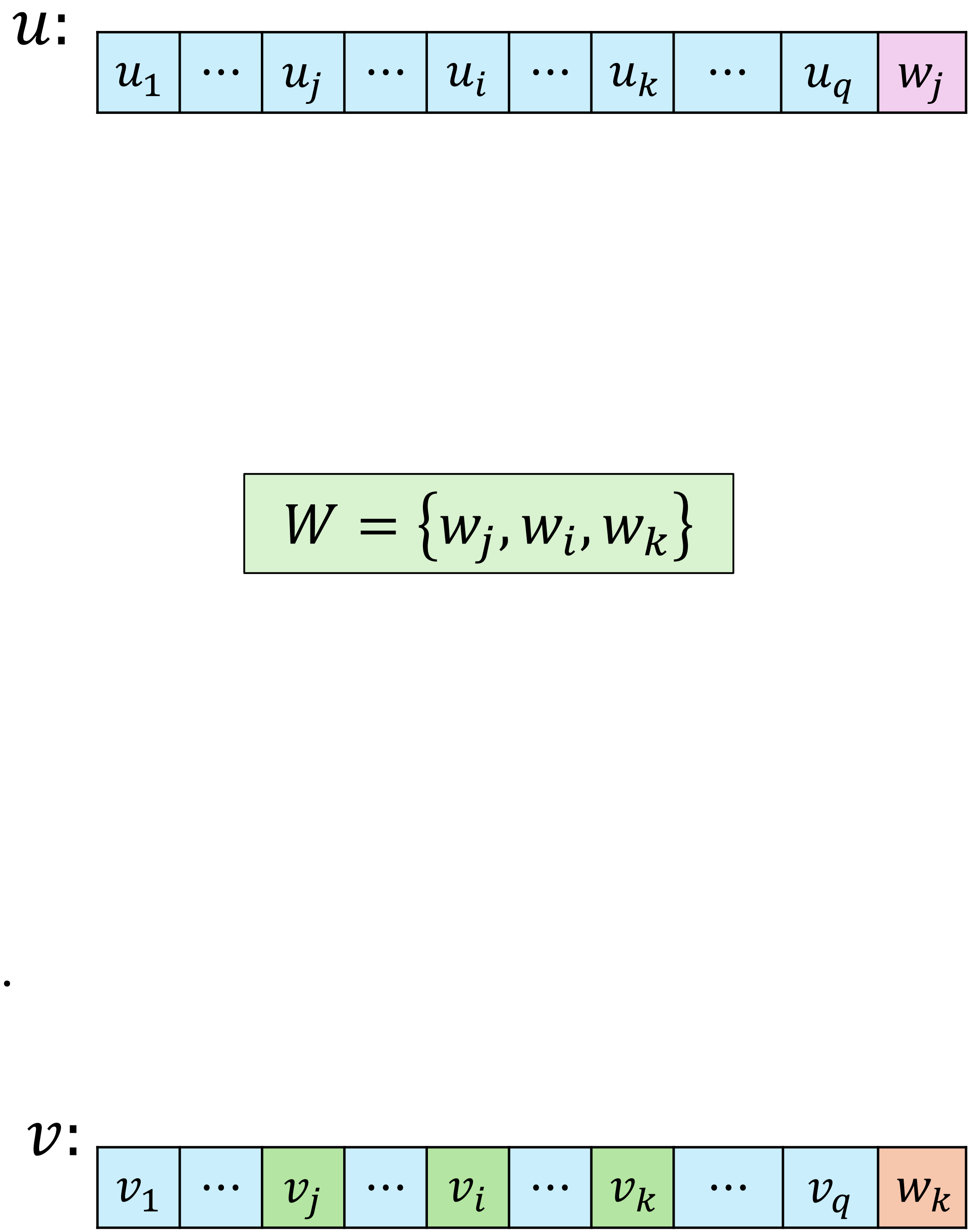

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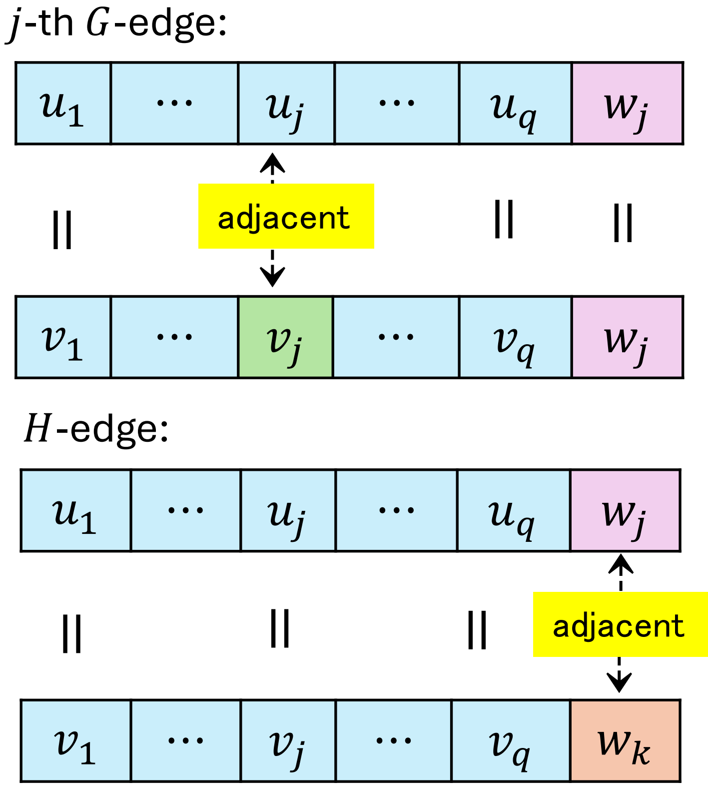

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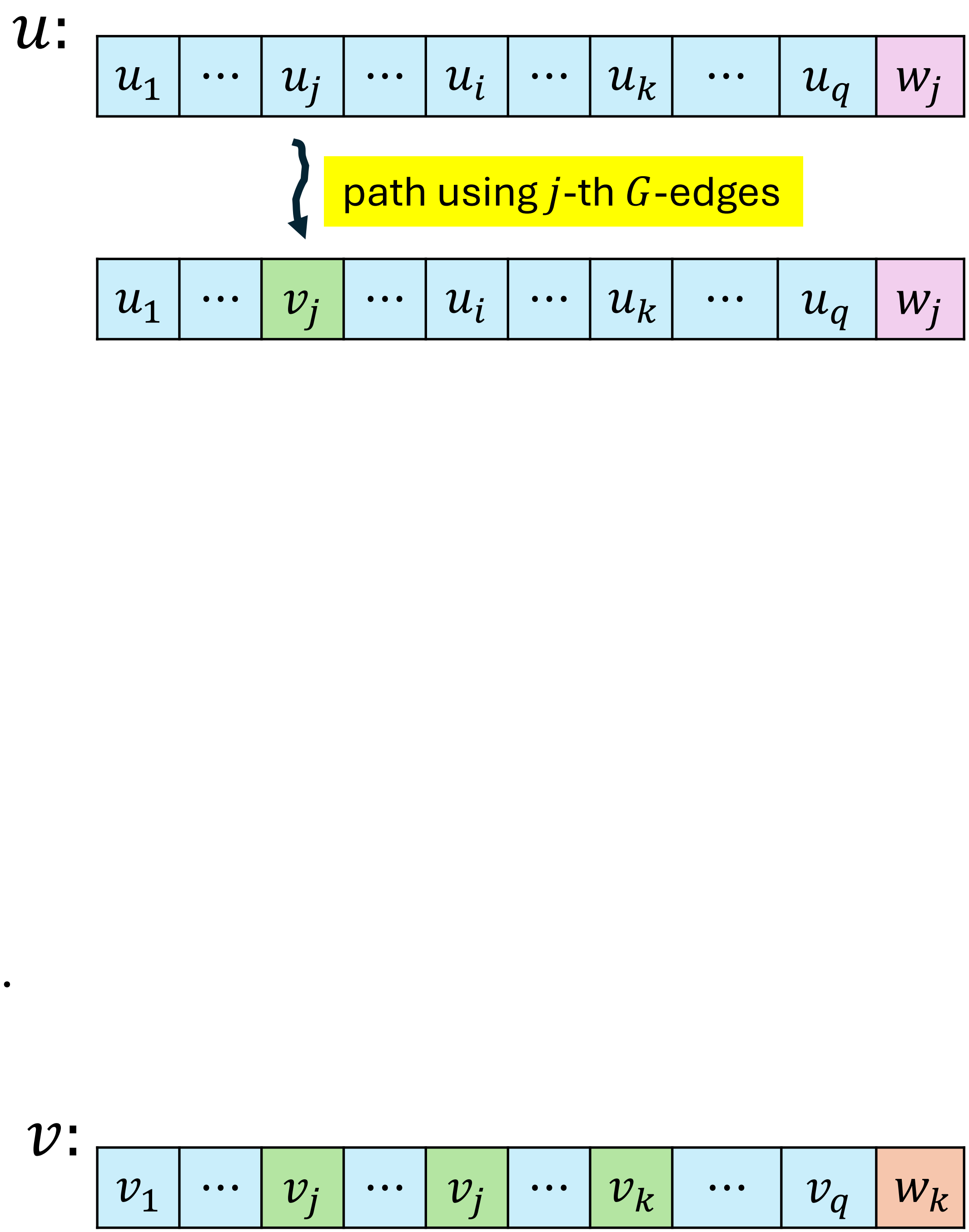

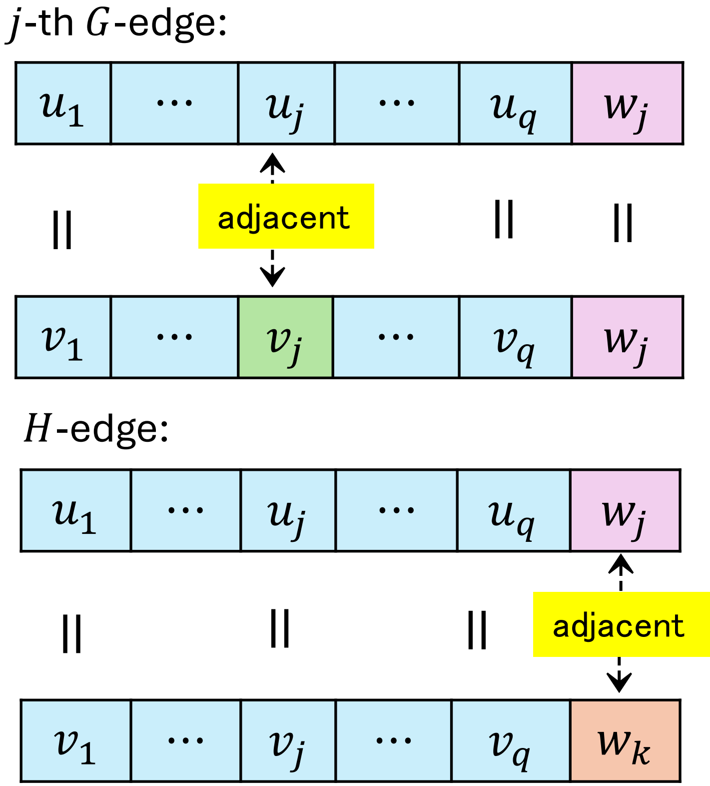

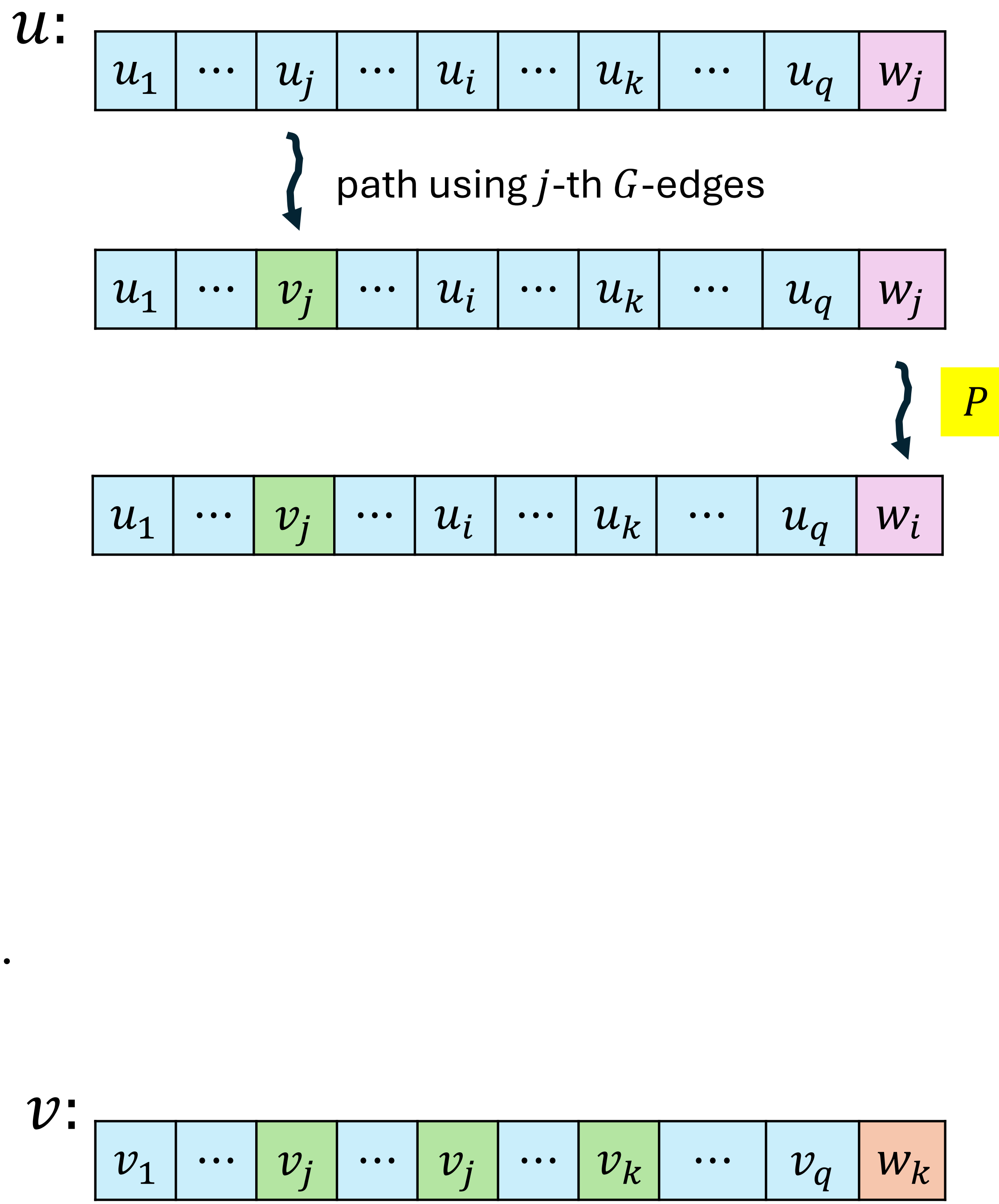

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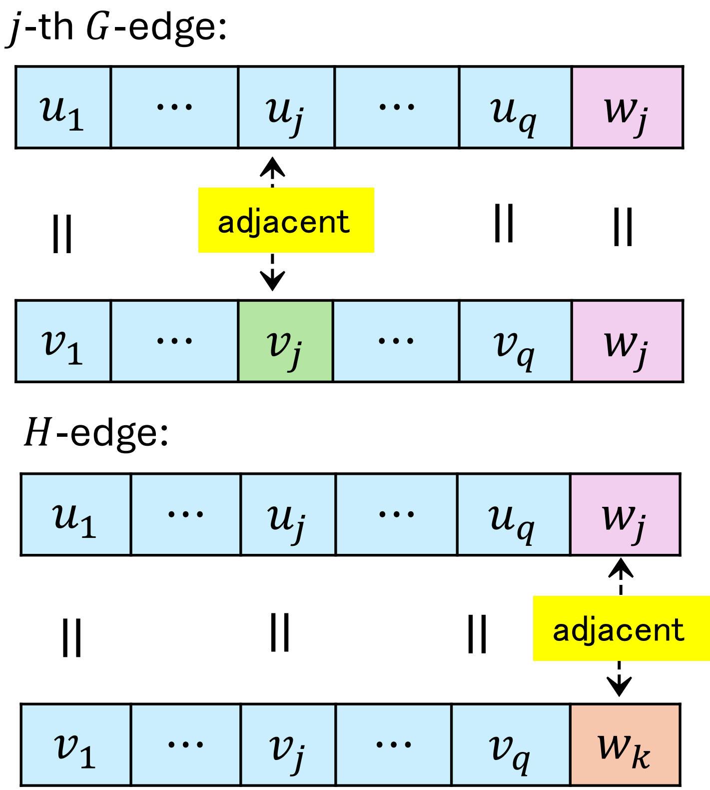

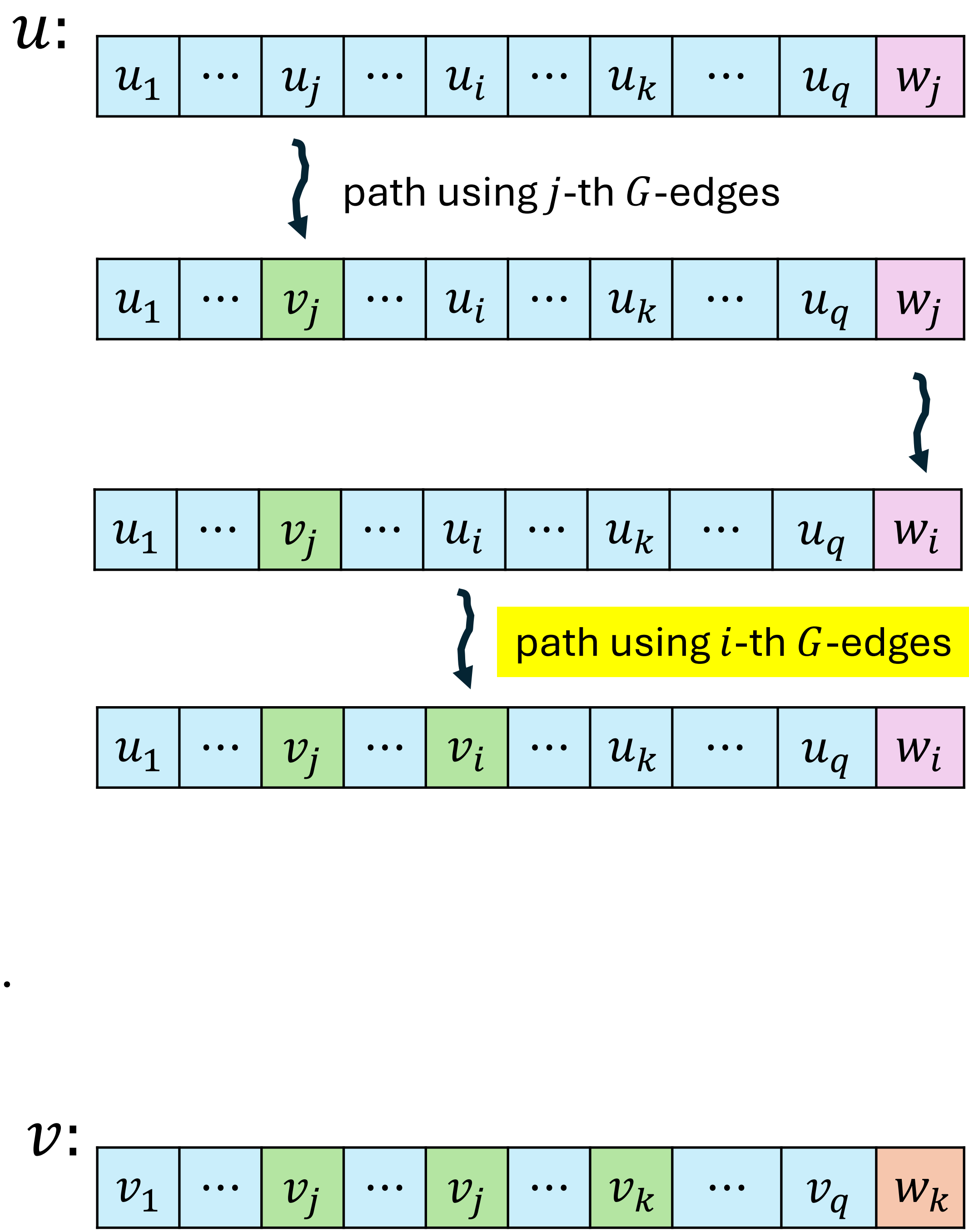

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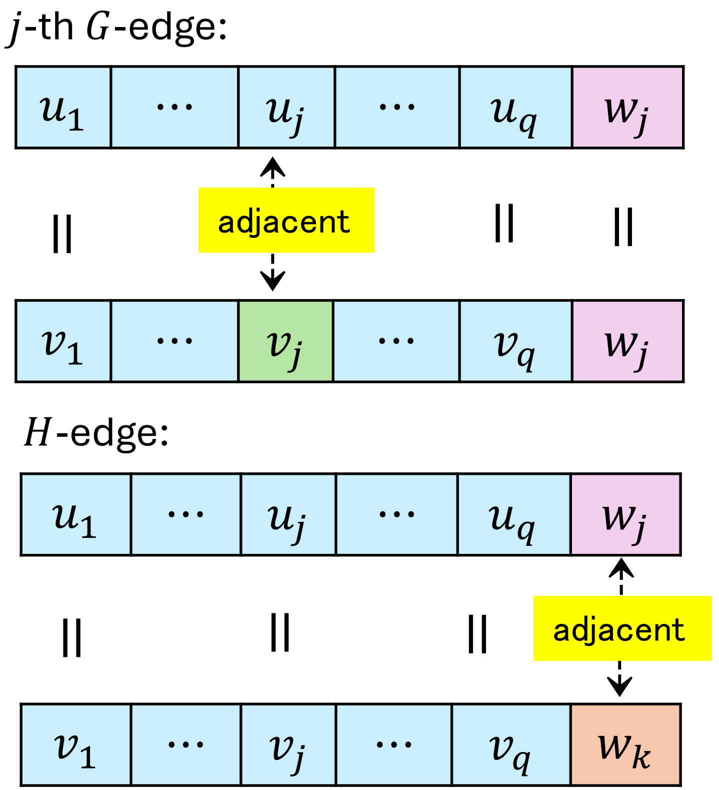

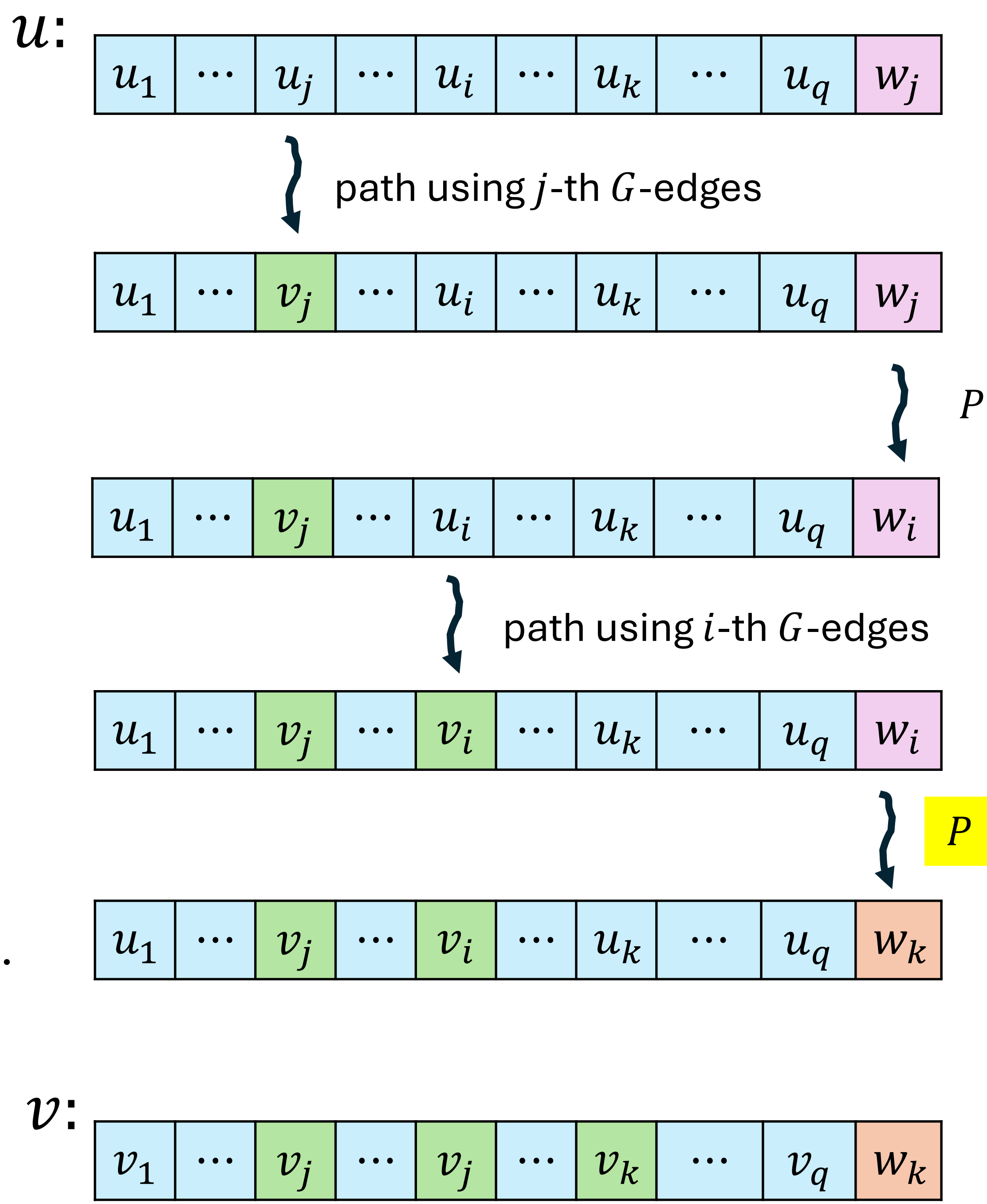

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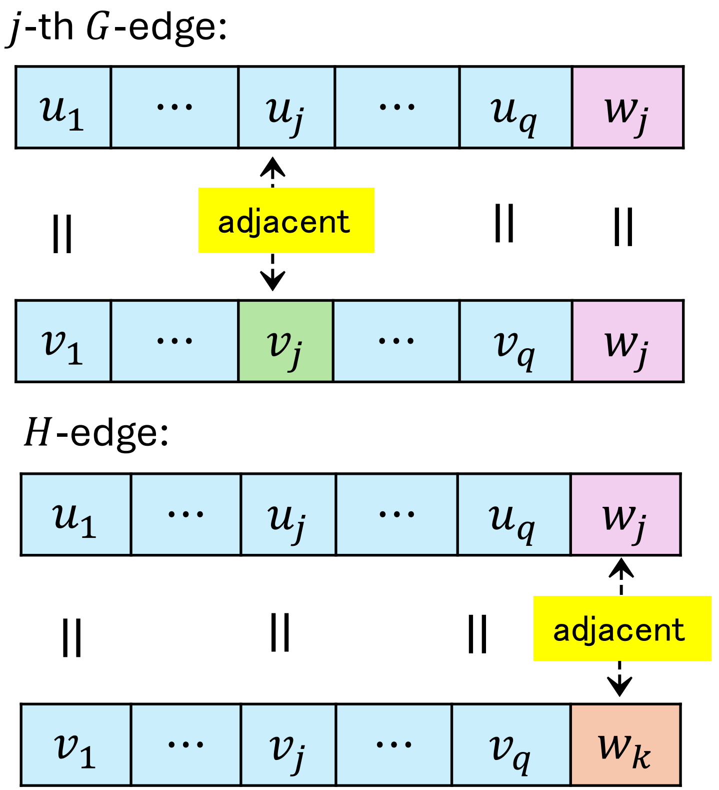

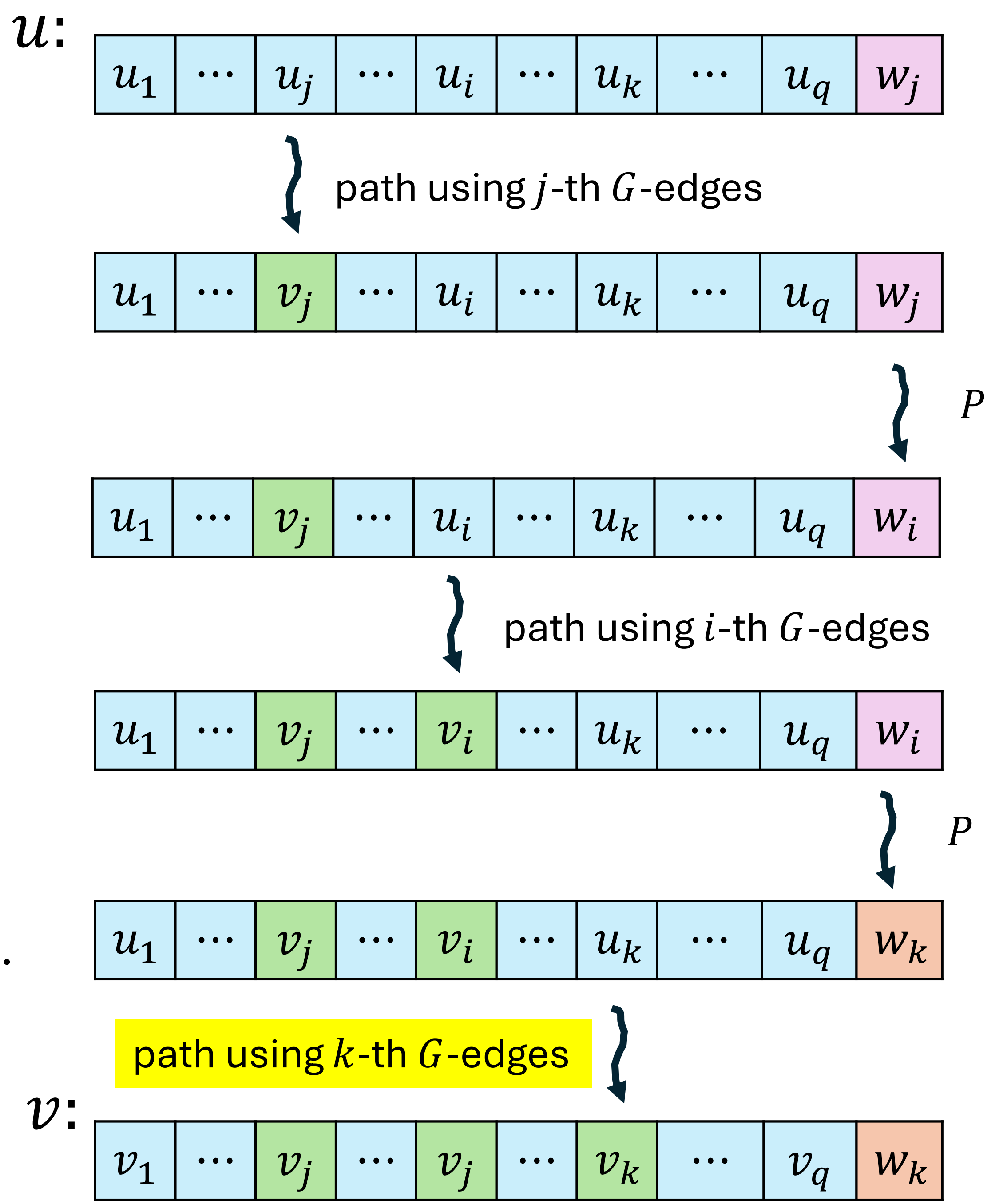

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Let $W = \{w_i \in V(H) \mid u_i \neq v_i\}$.

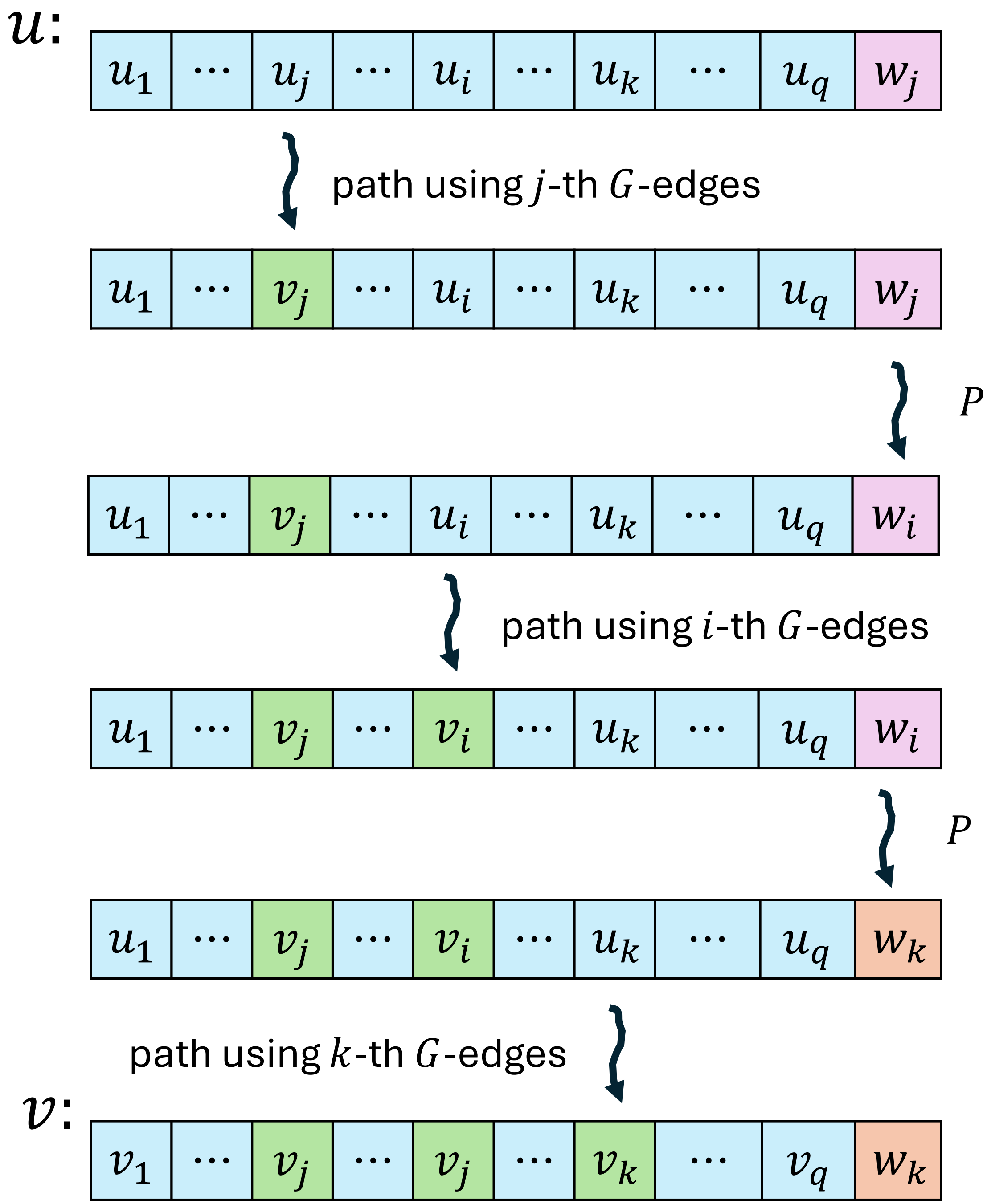
Let P be a walk from w_j to w_k which goes through every vertex in W.

This path from u to v has length is at most $\mathrm{diam}(G) \cdot q + \ell(P)$.

$\ell(P)$: length of P

$$\mathrm{diam}(G) = \max_{u,v \in V(G)} \mathrm{dist}_G(u,v)$$

$\mathrm{dist}_G(u,v)$: length of shortest path from u to v



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$\ell(P)$: length of P

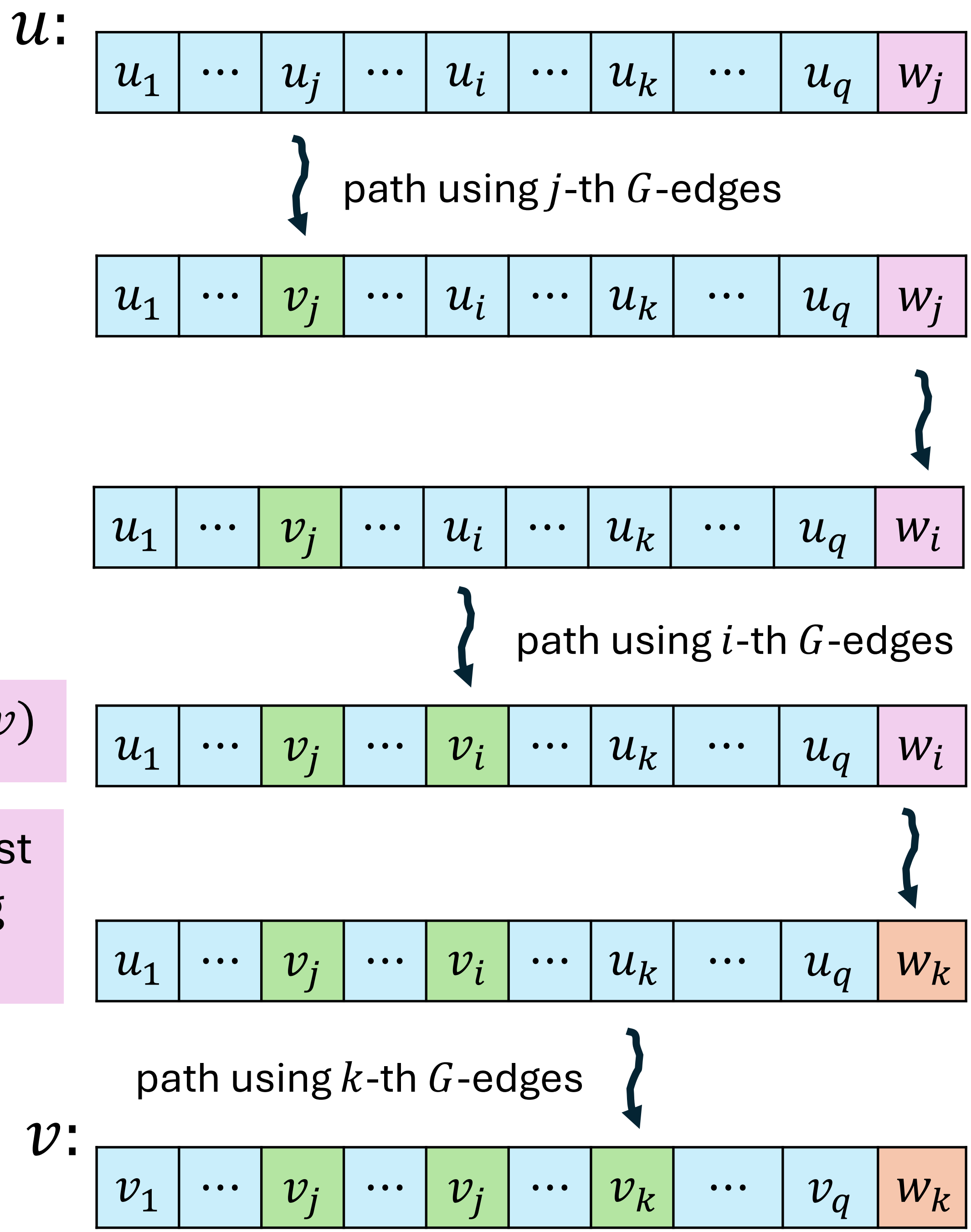
$$\mathrm{diam}(G) = \max_{u,v \in V(G)} \mathrm{dist}_G(u,v)$$

$\mathrm{dist}_G(u,v)$: length of shortest path from u to v

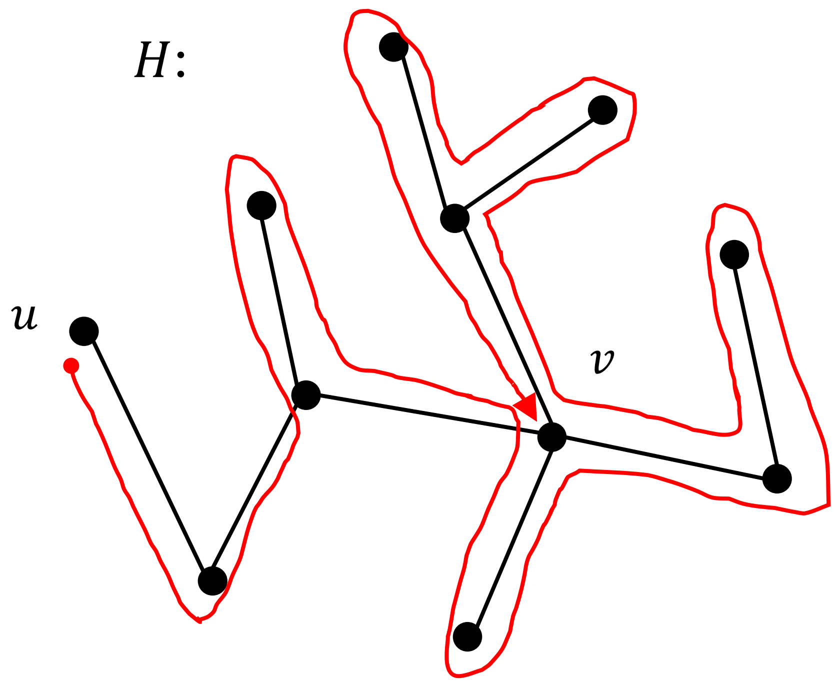
$$\mathrm{diam}^*(H) = \max_{u,v \in V(H)} \mathrm{dist}^*_H(u,v)$$

$\mathrm{dist}^*_H(u,v)$: length of shortest walks from u to v containing every vertex in H.

$\ell(P) \leq \mathrm{diam}^*(H)$



Based on a spanning tree in H, we can construct a walk between any two vertices which goes through every vertex in H.



$$\mathrm{diam}^*(H) = \max_{u,v \in V(H)} \mathrm{dist}^*_H(u,v)$$

$\mathrm{dist}^*_H(u,v)$: length of shortest walks from u to v containing every vertex in H.

$$\mathrm{diam}^*(H) \leq 2(|V(H)| - 1) = 2q - 2$$

Diameter

$$\mathrm{diam}^*(H) \le 2q - 2$$

$$\begin{aligned}\mathrm{diam}(G^H) &= \mathrm{diam}(G)q + \mathrm{diam}^*(H)\\ &\le (\mathrm{diam}(G) + 2)q - 2\end{aligned}$$

- $|V(G^H)| = p^q q$
- $\log|V(G^H)| = q\log p + \log q$

$$\mathrm{diam}(G) = O(\log p) \Rightarrow \mathrm{diam}(G^H) = O(q\log p)$$

□ G has logarithmic diameter $\Rightarrow$ G^H has logarithmic diameter

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3. Hamiltonicity
4. Applications
   - Multiexponential bounded-degree networks with logarithmic diameter

Connectivity

Theorem 2.1

If G and H are connected, then G^H is maximally connected, i.e., $\kappa(G^H) = \delta(G^H) = \delta(G) + \delta(H)$.

$\kappa(G^H)$: connectivity of G^H

$\delta(G^H)$: minimum degree of G^H

G, H: graphs

Cartesian product $G \times H$:

$$V(G \times H) = V(G) \times V(H)$$

$(u_1, u_2)(v_1, v_2) \in E(G \times H)$

$\Longleftrightarrow$

$u_1 v_1 \in E(G)$ and $u_2 = v_2$

or

$u_1 = v_1$ and $u_2 v_2 \in E(H)$

G, H: graphs

Cartesian product $G \times H$:

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$\Leftrightarrow$

$u_1 v_1 \in E(G)$ and $u_2 = v_2$

or

$u_1 = v_1$ and $u_2 v_2 \in E(H)$

Cartesian product of q copies of G:

$$G^q = \underbrace{G \times G \times \cdots \times G}_{q}$$

$$V(G^q) = \{(u_0, u_1, \ldots, u_{q-1}) \mid u_i \in V(G), 0 \leq i < q\}$$

$$(u_0, u_1, \ldots, u_{q-1})(v_0, v_1, \ldots, v_{q-1}) \in E(G^q)$$

$\Leftrightarrow$ $u_j v_j \in E(G)$, and $u_i = v_i$ for $i \neq j$

Cartesian product of q copies of G:

G: graph of order p

H: graph of order q with $V(H) = \{w_0, w_1, \dots, w_{q-1}\}$

$$G^q = \underbrace{G \times G \times \cdots \times G}_{q}$$

$$V(G^q) = \{(u_0, u_1, \dots, u_{q-1}) \mid u_i \in V(G), 0 \le i < q\}$$

$$(u_0, u_1, \dots, u_{q-1})(v_0, v_1, \dots, v_{q-1}) \in E(G^q)$$

$\Leftrightarrow$ $u_j v_j \in E(G)$, and $u_i = v_i$ for $i \neq j$

Definition of G^H:

$$V(G^H) = \{(u_0, u_1, \dots, u_{q-1}; w_j) \mid u_i \in V(G), 0 \le i, j < q\}$$

$$(u_0, u_1, \dots, u_{q-1}; w_j)(v_0, v_1, \dots, v_{q-1}; w_k) \in E(G^H)$$

$\Leftrightarrow$

$j = k, u_j v_j \in E(G)$, and $u_i = v_i$ for $i \neq j$

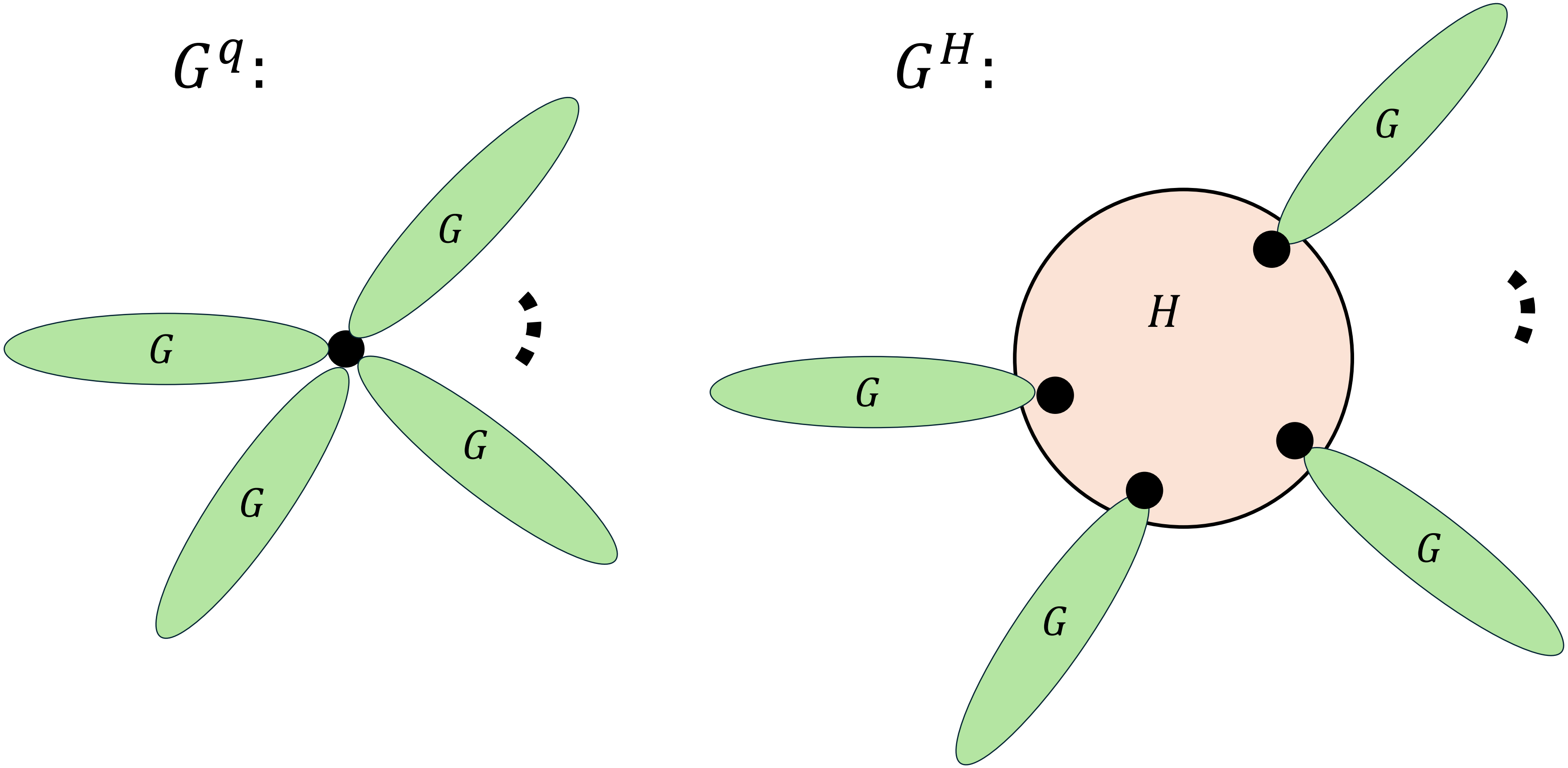
or

$j \neq k, w_j w_k \in E(H)$, and $u_i = v_i$ for $0 \le i < q$

If the vertices in $\{(u_1, u_2, \dots, u_q; w_j) \mid w_j \in V(H)\}$ are identified to a single vertex $(u_1, u_2, \dots, u_q)$, then we have G^q.

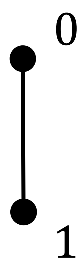
$\{(u_1, u_2, \dots, u_q; w_j) \mid w_j \in V(H)\}$ induces a copy of H.

The exponential graph G^H can be obtained from G^q by replacing each vertex with a copy of H so that each vertex w_i in the copy of H is incident to the i-th G-edges.

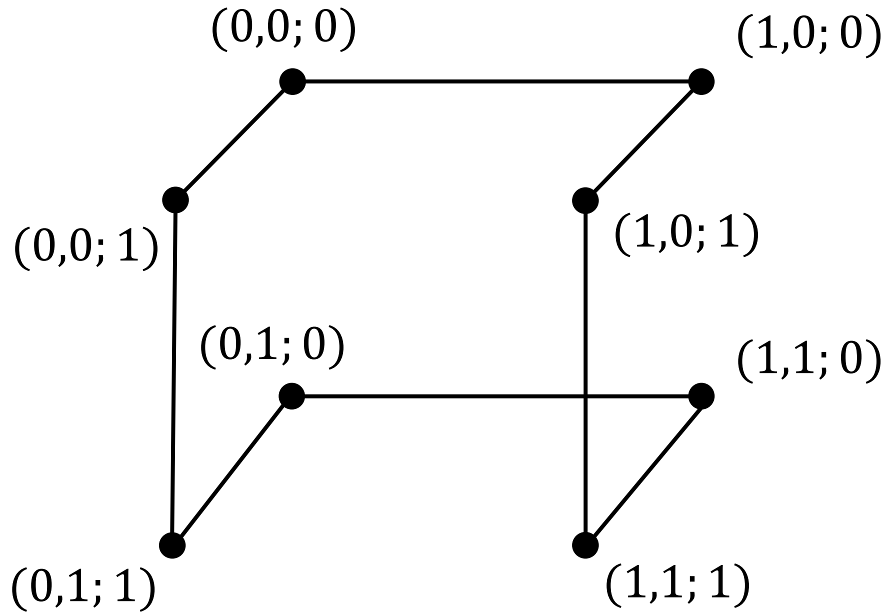


Example

K_2:

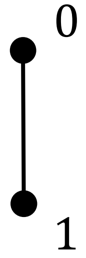


$K_2^{K_2}$:

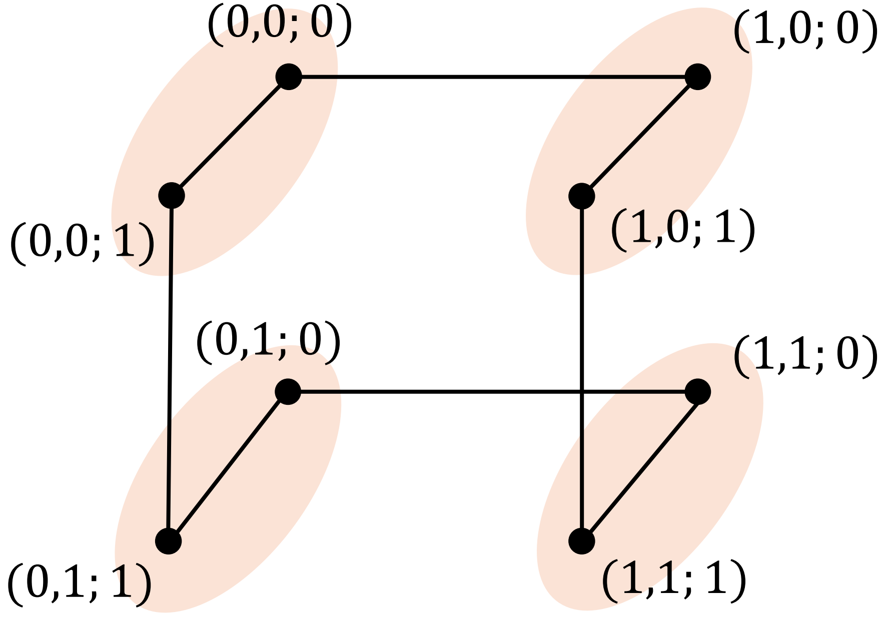


Example

K_2:

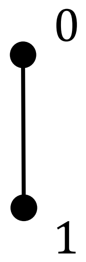


$K_2^{K_2}$:

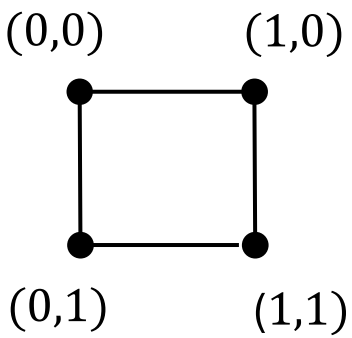


Example

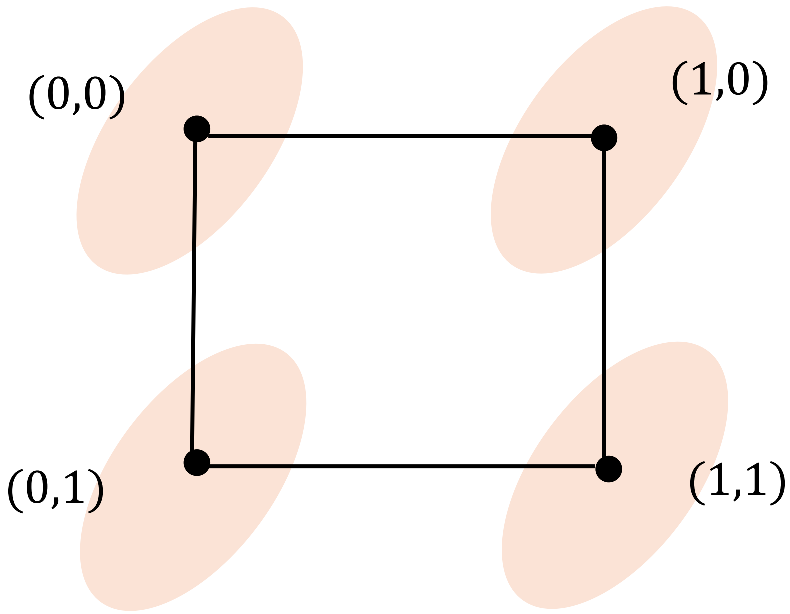
K_2:



$K_2^2 = K_2 \times K_2$:

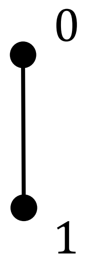


K_2^2:

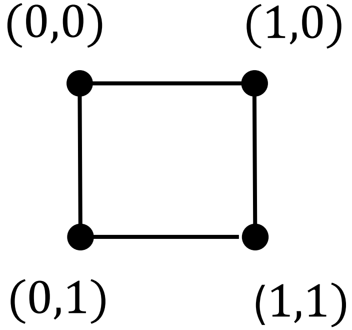


Example

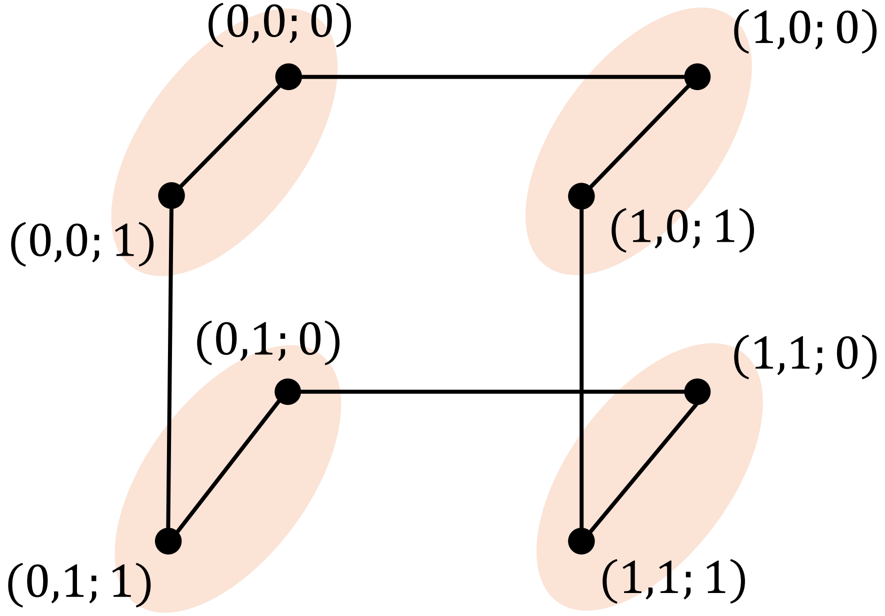
K_2:



$K_2^2 = K_2 \times K_2$:



$K_2^{K_2}$:



Connectivity of Cartesian product graphs

Theorem 2.2 (Špacapan 2008)

$$\kappa(G \times H) = \min\{\kappa(G) \cdot |V(H)|, \kappa(H) \cdot |V(G)|, \delta(G) + \delta(H)\}$$

Corollary 2.3

$$\kappa(G^q) = q \cdot \delta(G)$$

Corollary 2.3

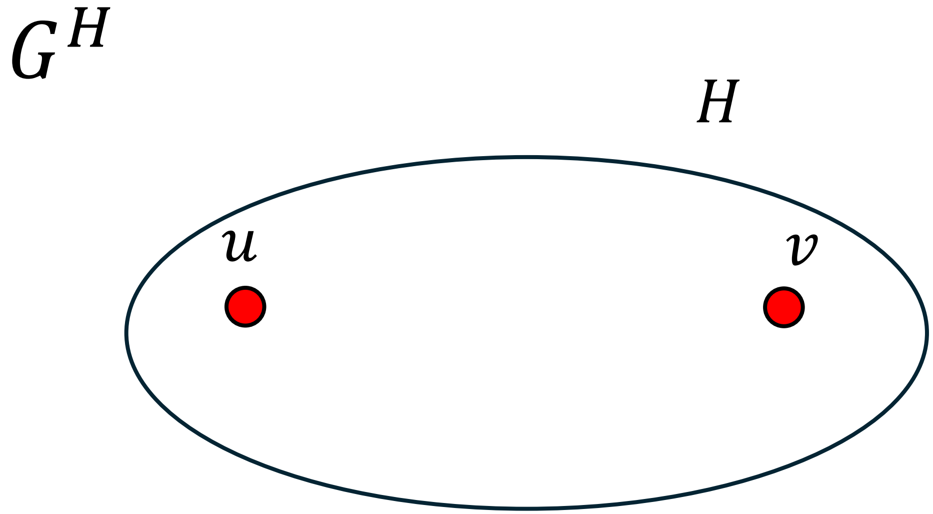
$$\kappa(G^q) = q \cdot \delta(G)$$

Theorem 2.1

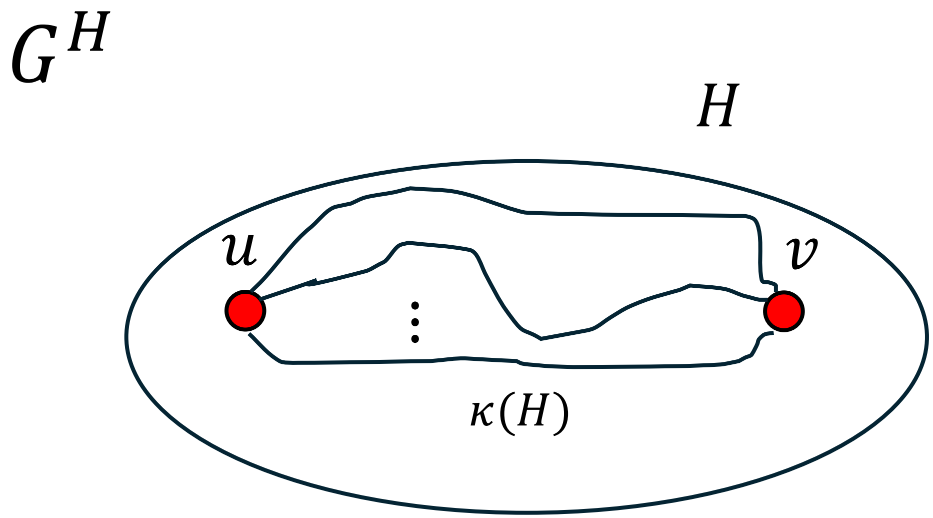
If G and H are connected, then G^H is maximally connected, i.e., $\kappa(G^H) = \delta(G^H) = \delta(G) + \delta(H)$.

We show that there are $\delta(G) + \delta(H)$ internally disjoint paths between any two vertices u and v in G^H.

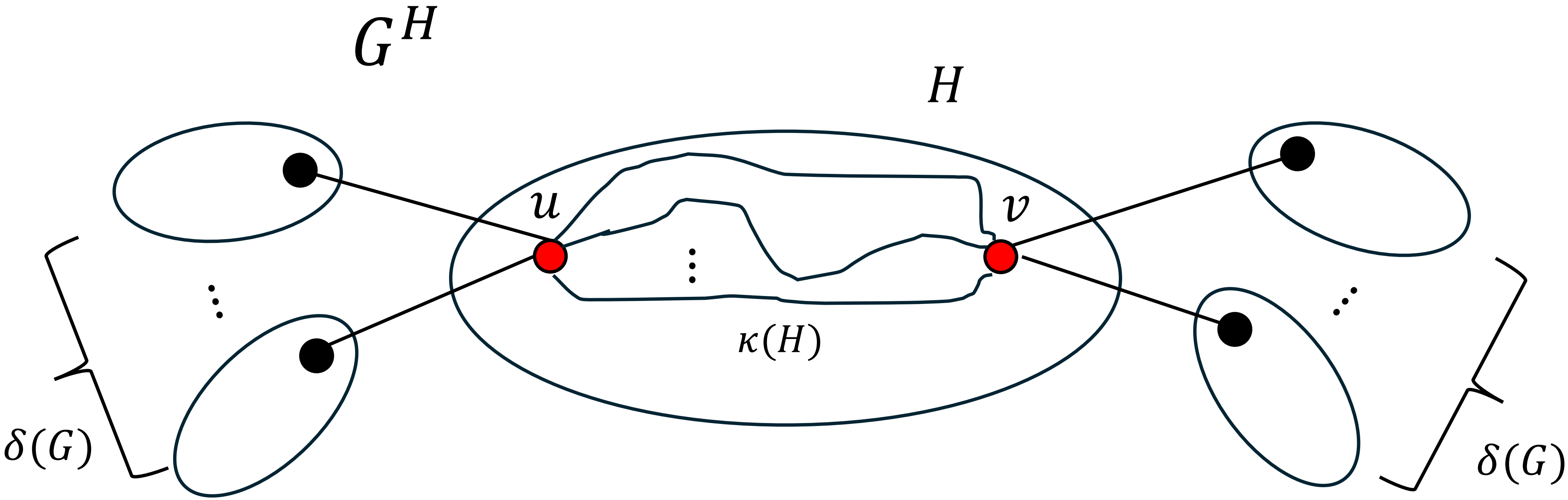
Case 1: u and v are in the same copy of H.



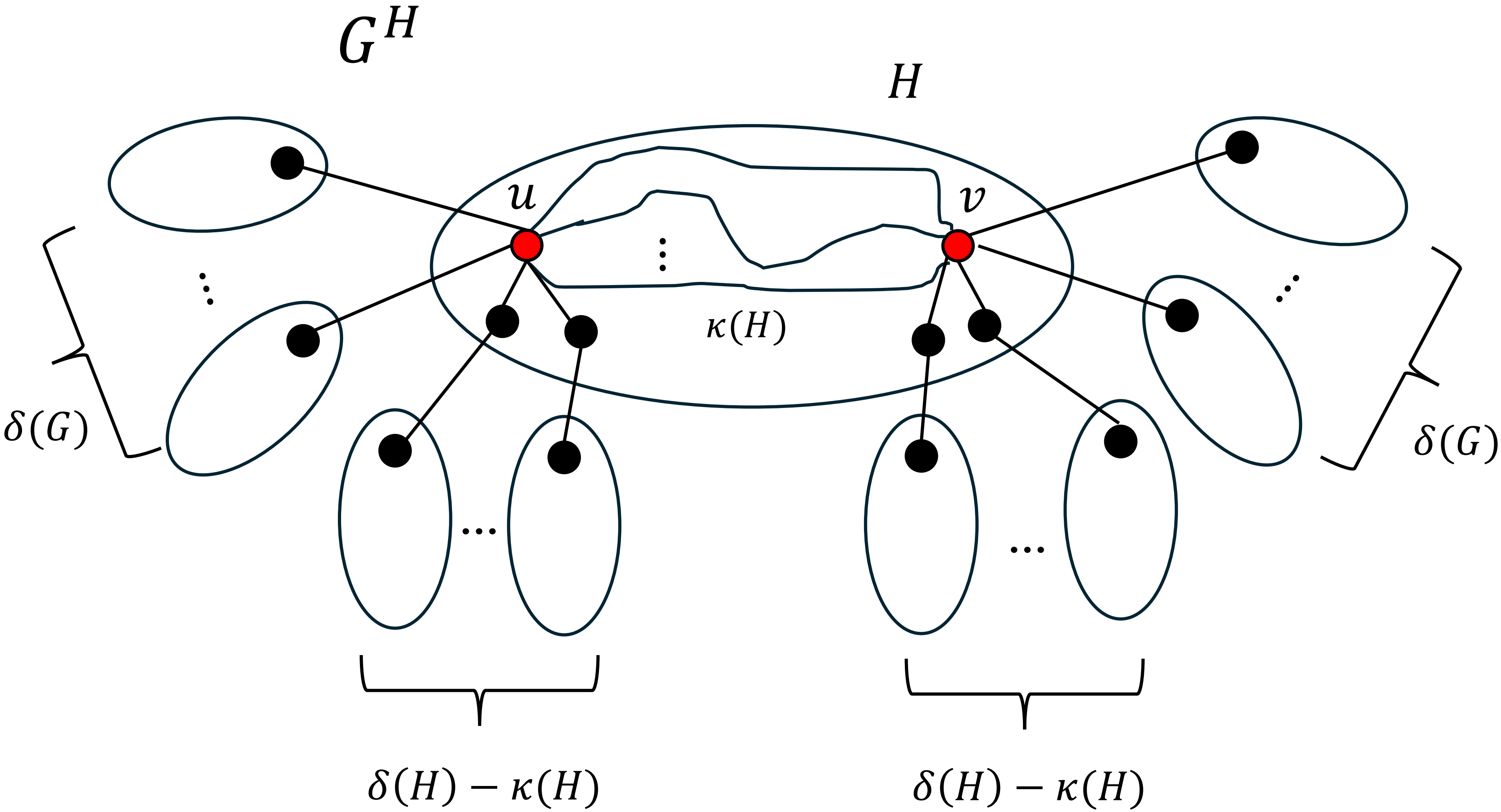
Case 1: u and v are in the same copy of H.

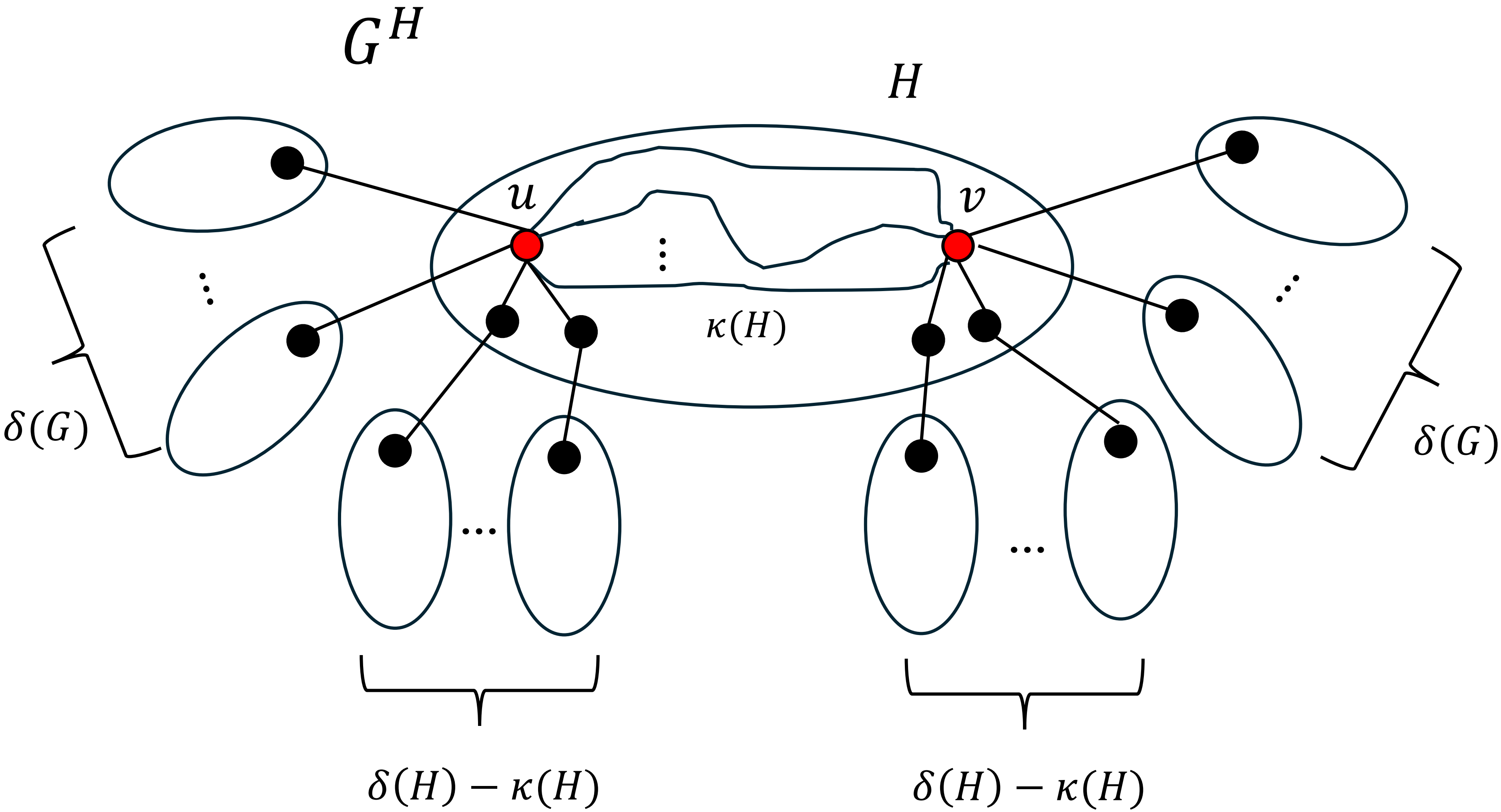


Case 1: u and v are in the same copy of H.

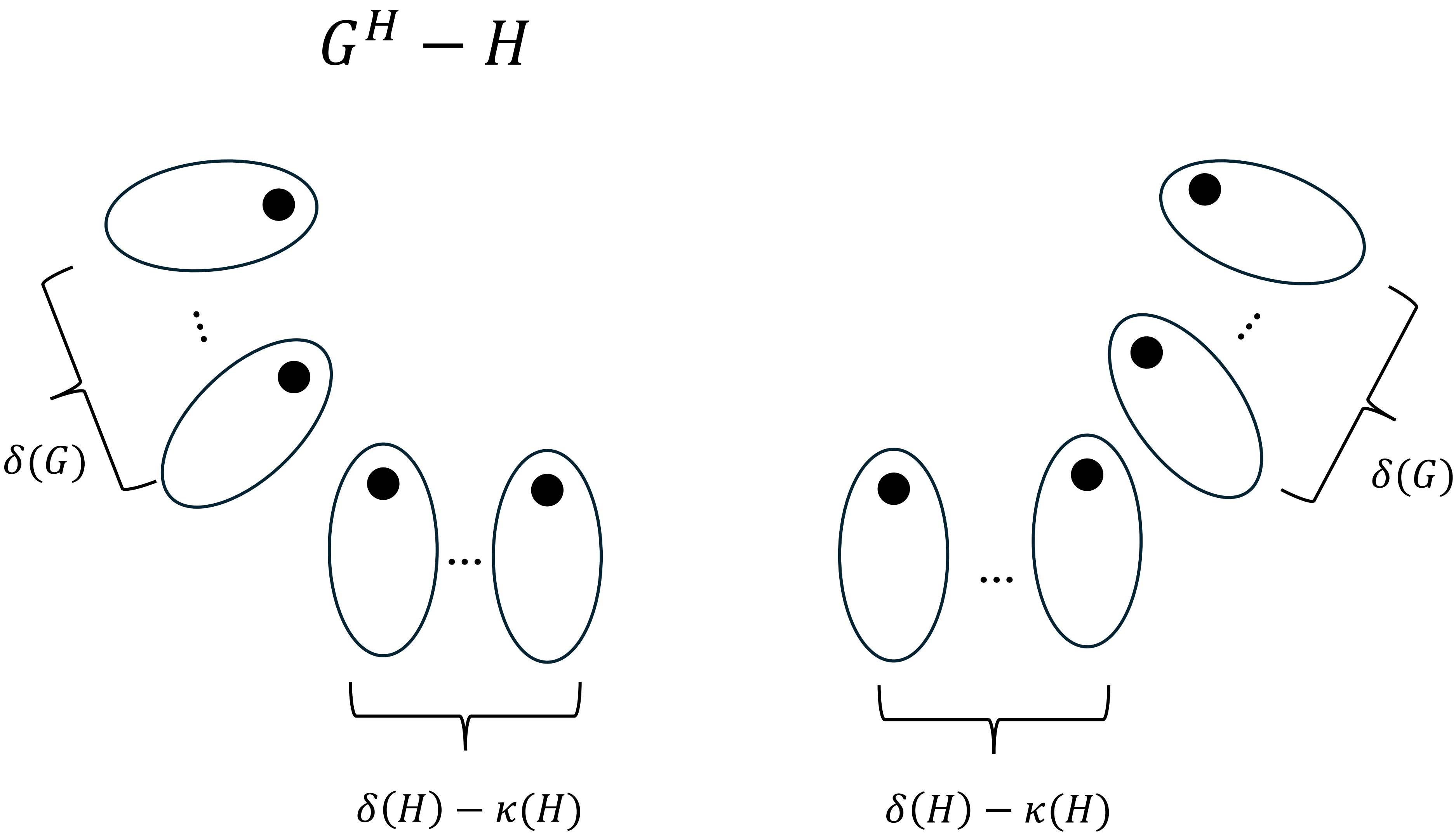


Case 1: u and v are in the same copy of H.

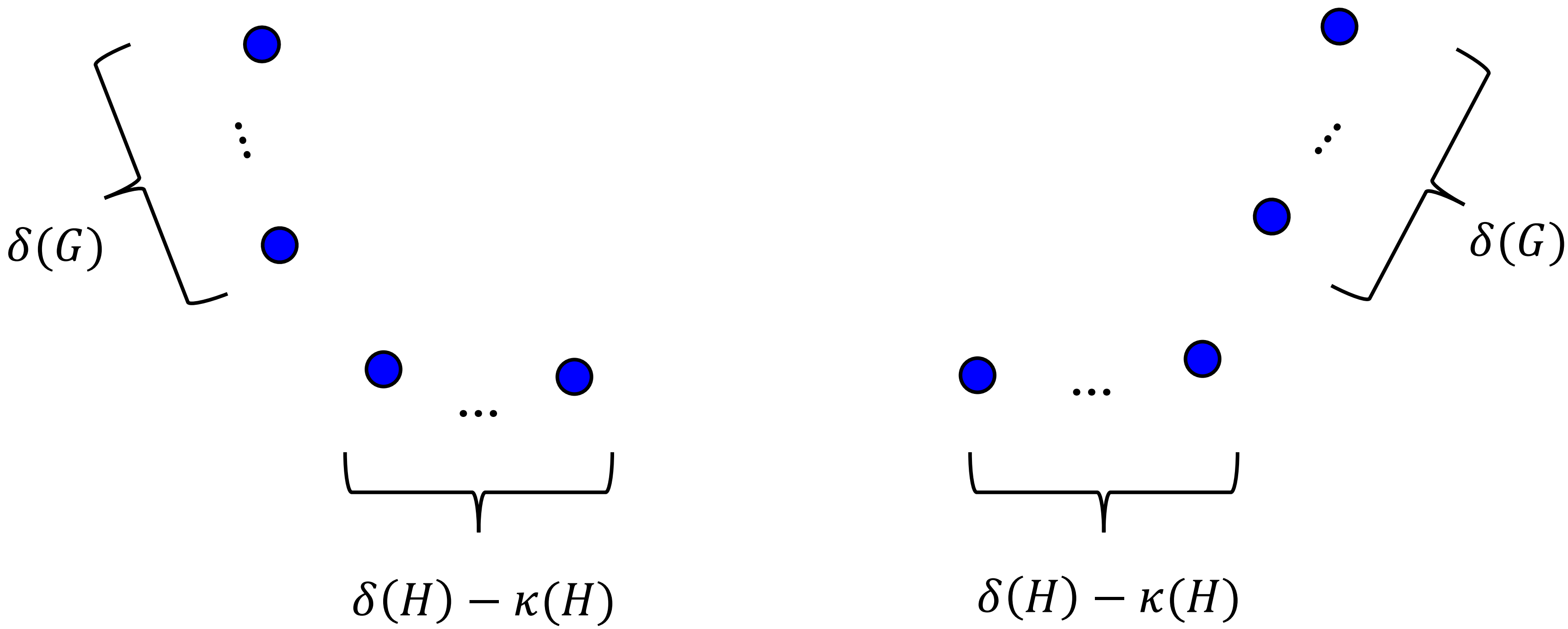




G^H
H
u
v
$\kappa(H)$
$\delta(G)$
$\delta(G)$
$\delta(H) - \kappa(H)$
$\delta(H) - \kappa(H)$

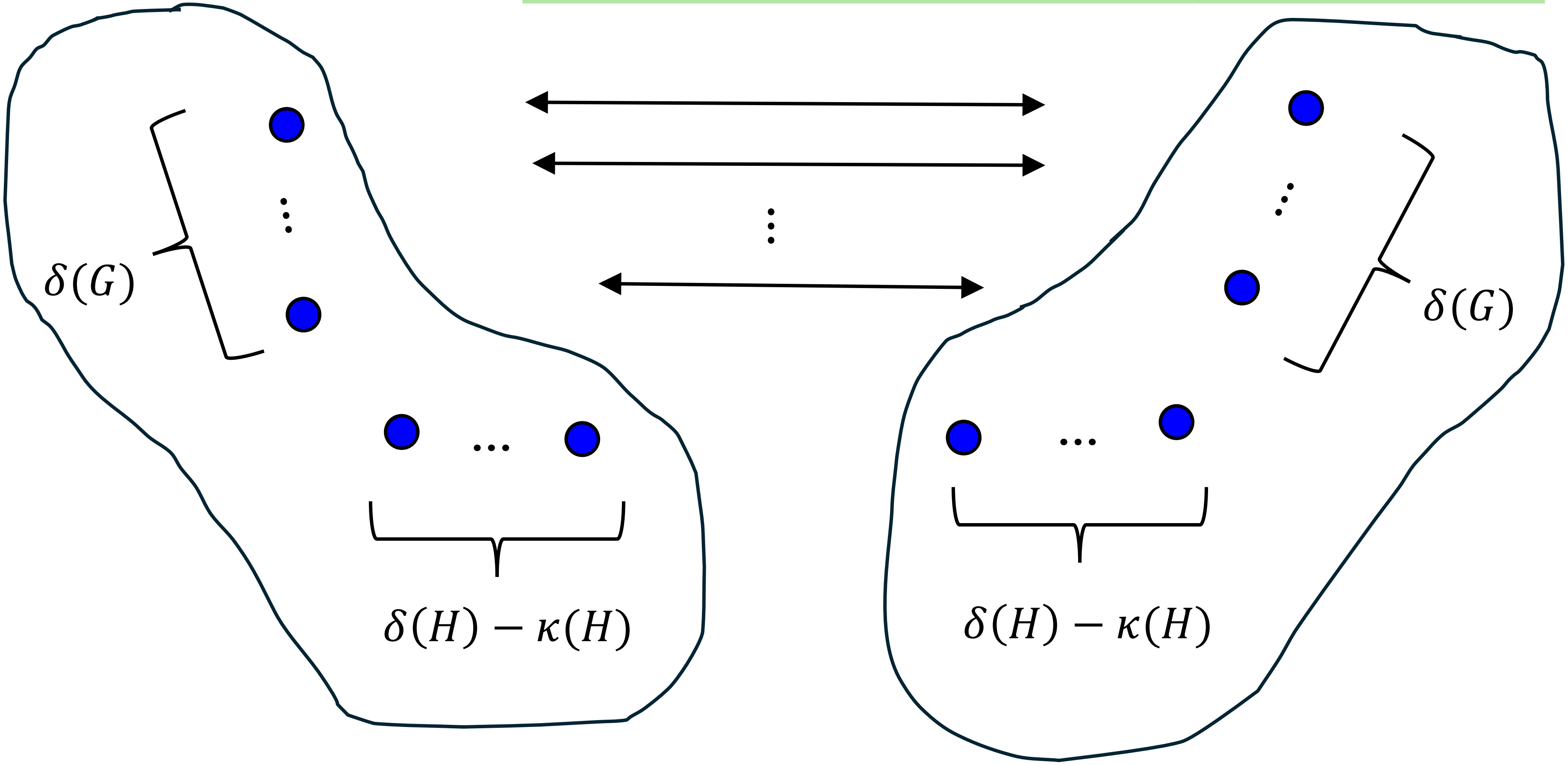

$G^H - H$
$\delta(G)$
$\delta(G)$
$\delta(H) - \kappa(H)$
$\delta(H) - \kappa(H)$

$G^q - v_H$



$G^q - v_H$

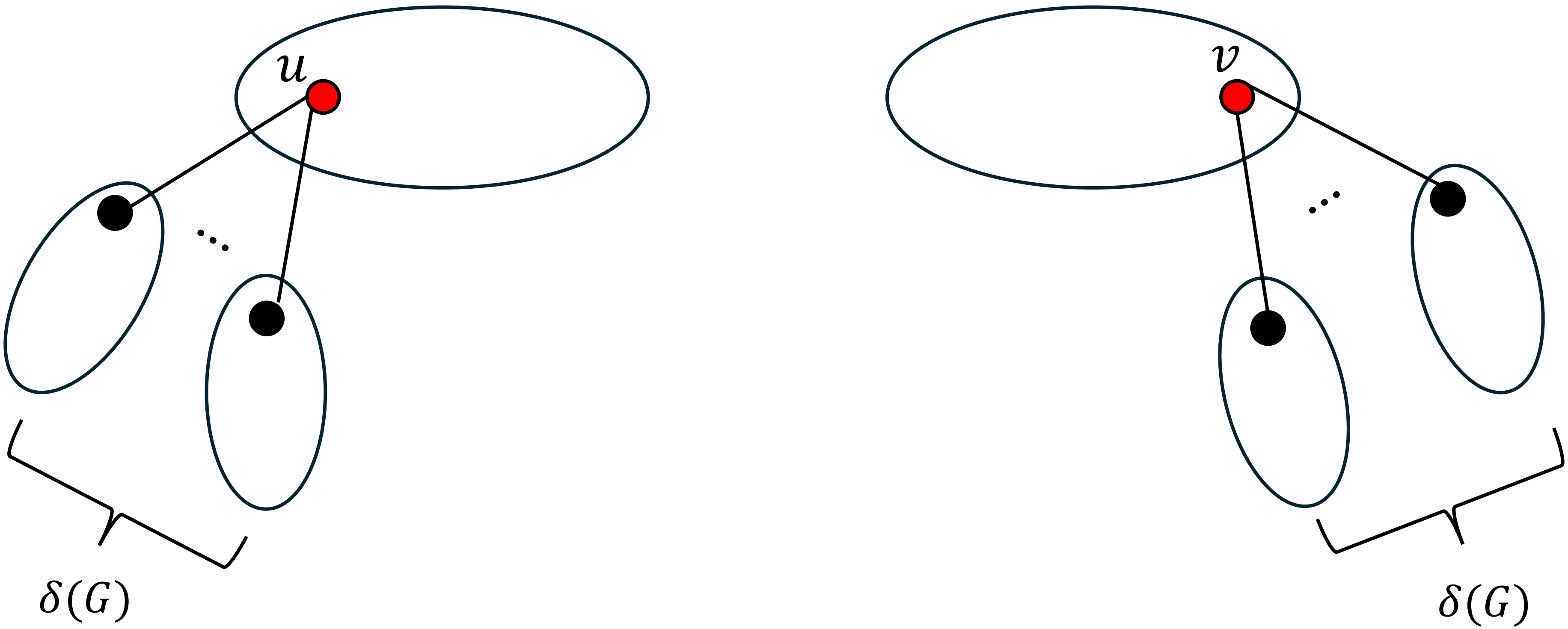
$\delta(G) + \delta(H) - \kappa(H)$ internally disjoint paths exist if
$q \cdot \delta(G) - 1 \geq \delta(G) + \delta(H) - \kappa(H)$

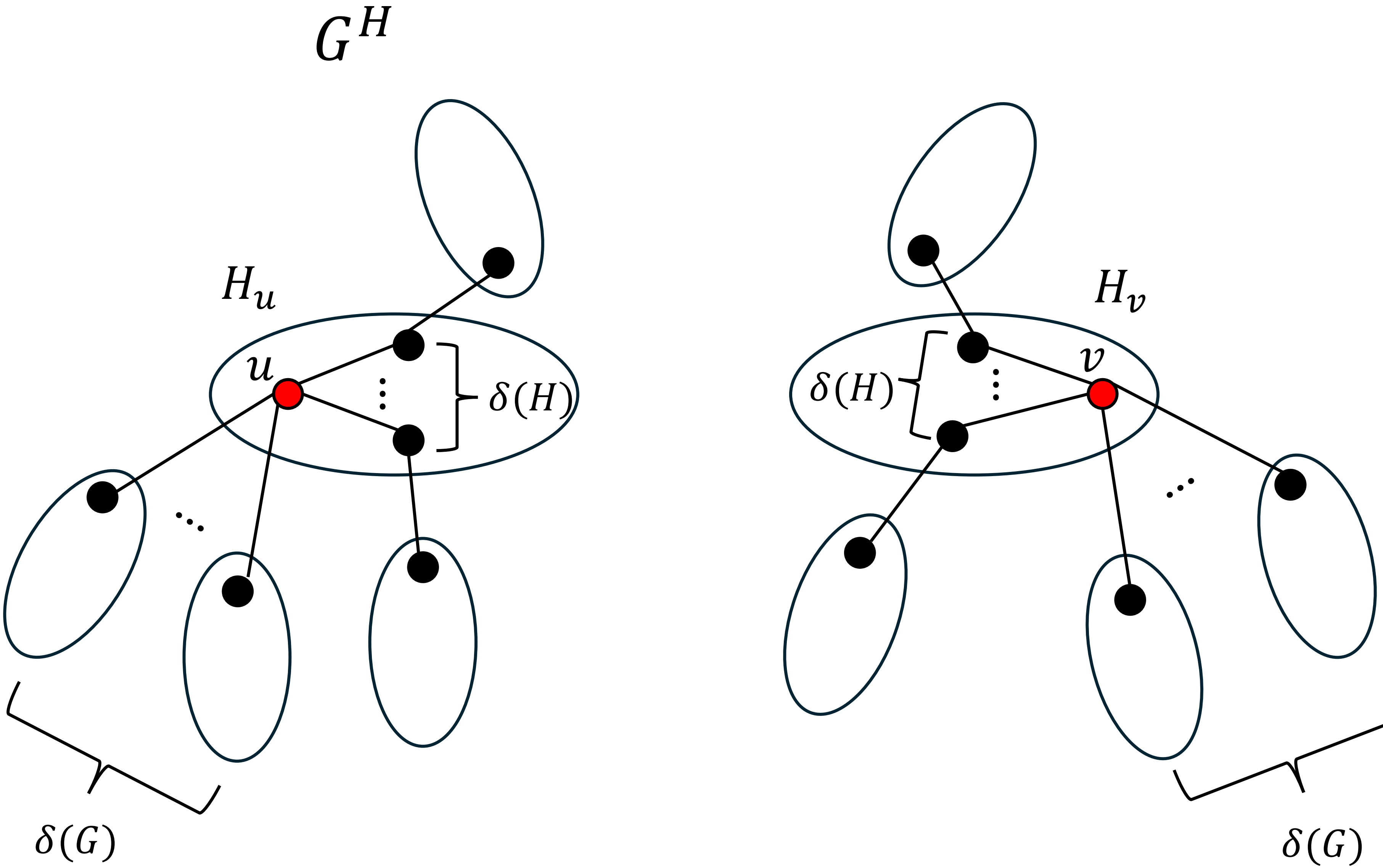


This inequality holds,
since $(q-1)\delta(G) - 1 \geq \delta(H)\delta(G) - 1 \geq \delta(H) - 1 \geq \delta(H) - \kappa(H)$

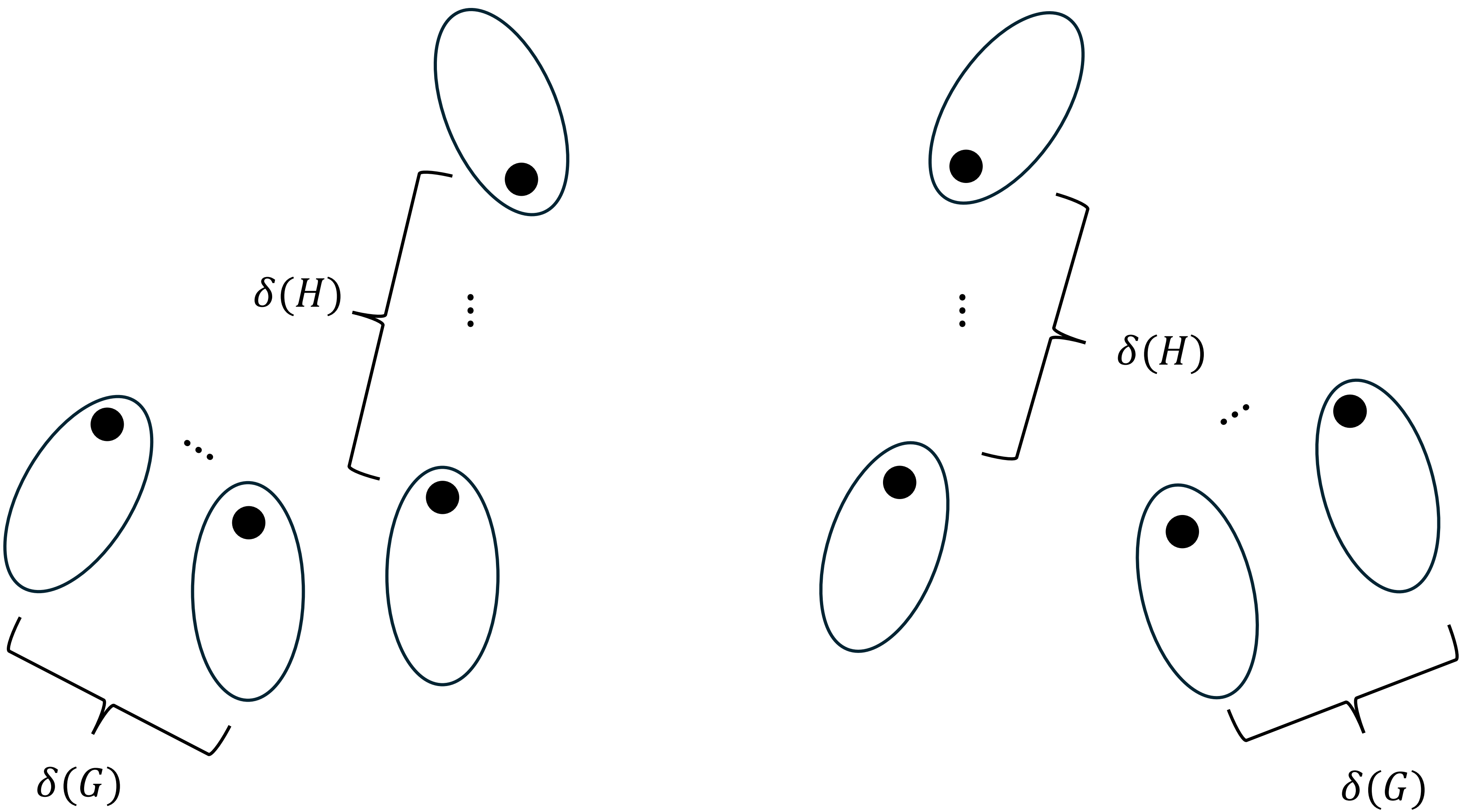
Case 2: u and v are in the distinct copies of H.

Case 2: u and v are in the distinct copies of H.

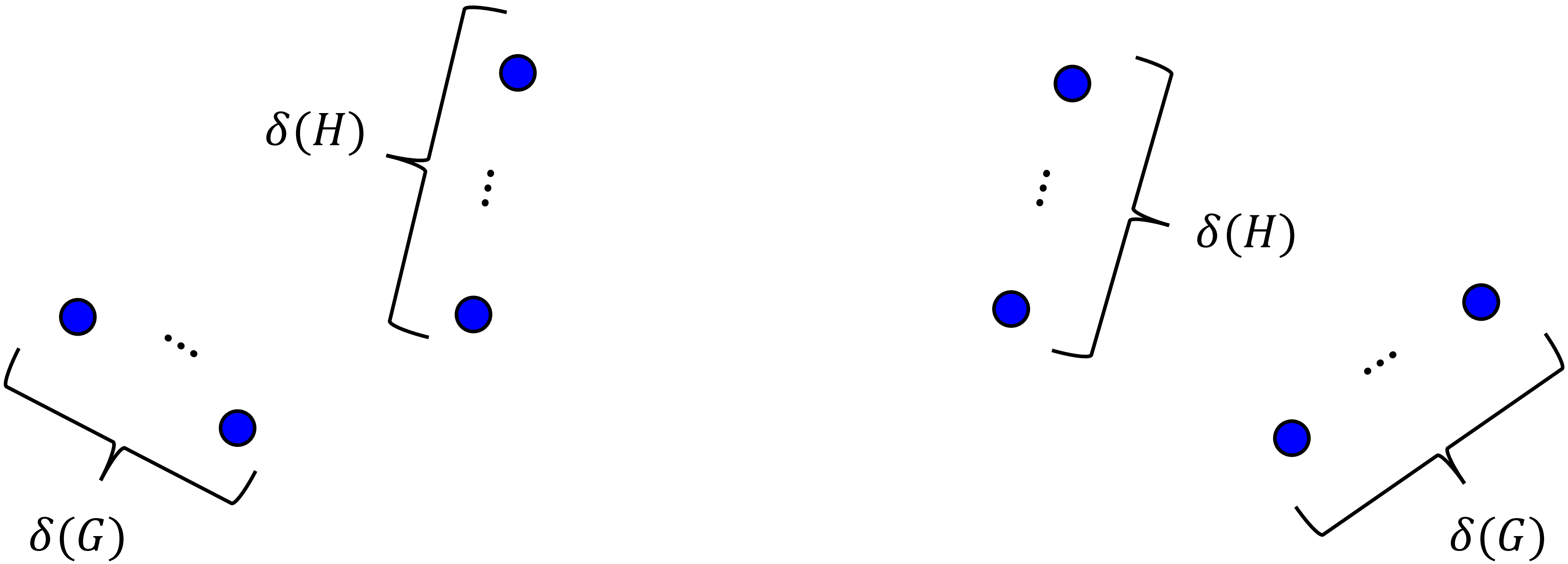



G^H
H_u
u
$\delta(H)$
$\delta(G)$
H_v
v
$\delta(H)$
$\delta(G)$

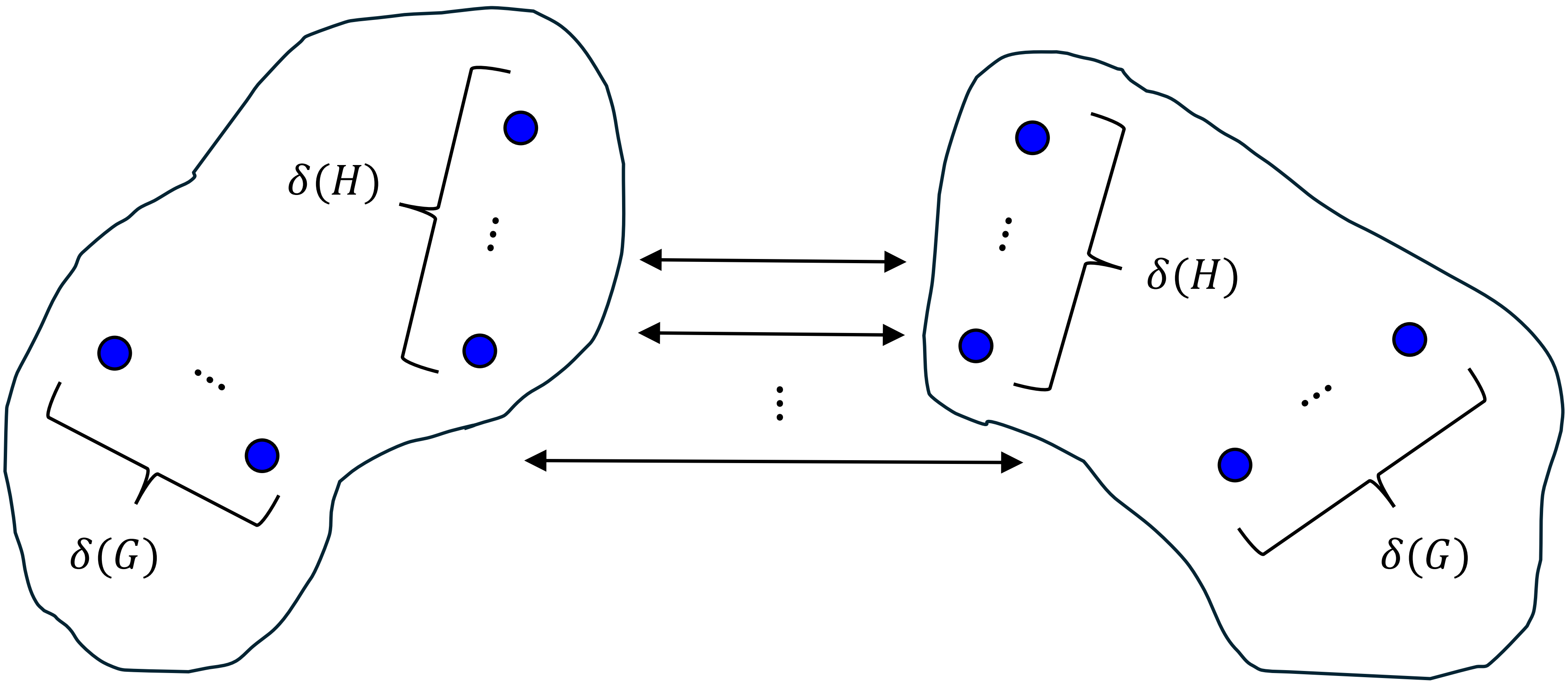
$G^H - H_u - H_v$



$$G^q - v_{H_u} - v_{H_v}$$



$G^q - x - y$



$\delta(G) + \delta(H)$internally disjoint paths paths exist if
$q \cdot \delta(G) - 2 \geq \delta(G) + \delta(H)$

$\delta(G) + \delta(H)$ internally disjoint paths paths exist if

$$q \cdot \delta(G) - 2 \geq \delta(G) + \delta(H)$$

$$\Updownarrow$$

$$(q-1)\delta(G) \geq \delta(H) + 2$$

Since $q = |V(H)|$, if $\delta(G) \geq 3$, then we have the desired result.

Case 1: $\delta(G) = 2.$ $\qquad 2(q-1) \geq \delta(H) + 2$

It is sufficient to consider the case that $q = 2$, i.e., $H \cong K_2$.

Case 2: $\delta(G) = 1.$ $\qquad q - 3 \geq \delta(H)$

It is sufficient to consider the case that $\delta(H) \geq q - 2$.

- In each special cases, we can directly construct desired paths.

Outline

1. Exponentiation of graphs
   - Definition
   - Order, Degree and Diameter
2. Connectivity
3. Hamiltonicity
4. Applications
   - Multiexponential bounded-degree networks with logarithmic diameter

Hamiltonicity

Theorem 3.1

If G is Hamiltonian, then G^{K_2} is Hamiltonian.

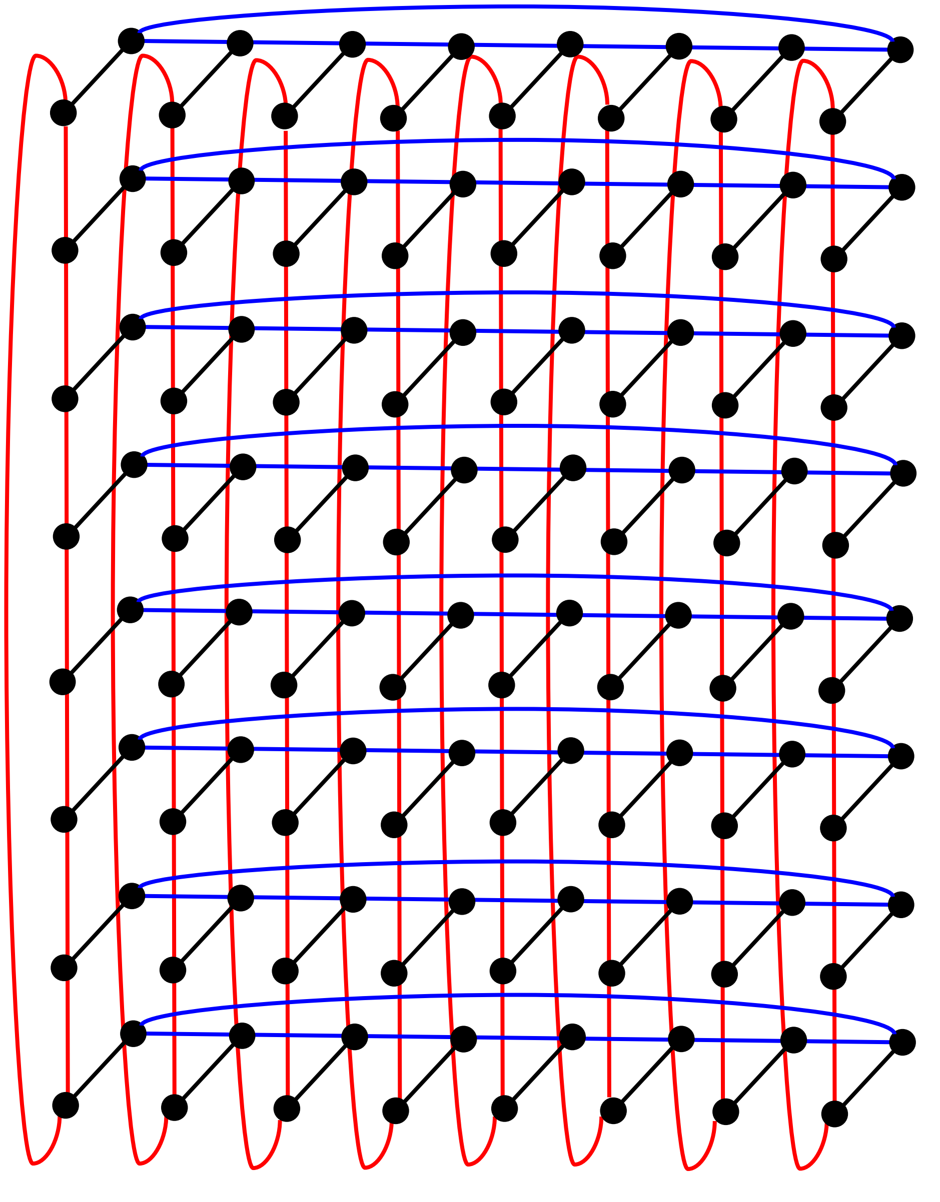
Theorem 3.2

If G is Hamiltonian such that $|V(G)|$ is even and H is Hamiltonian connected, then G^H is Hamiltonian.

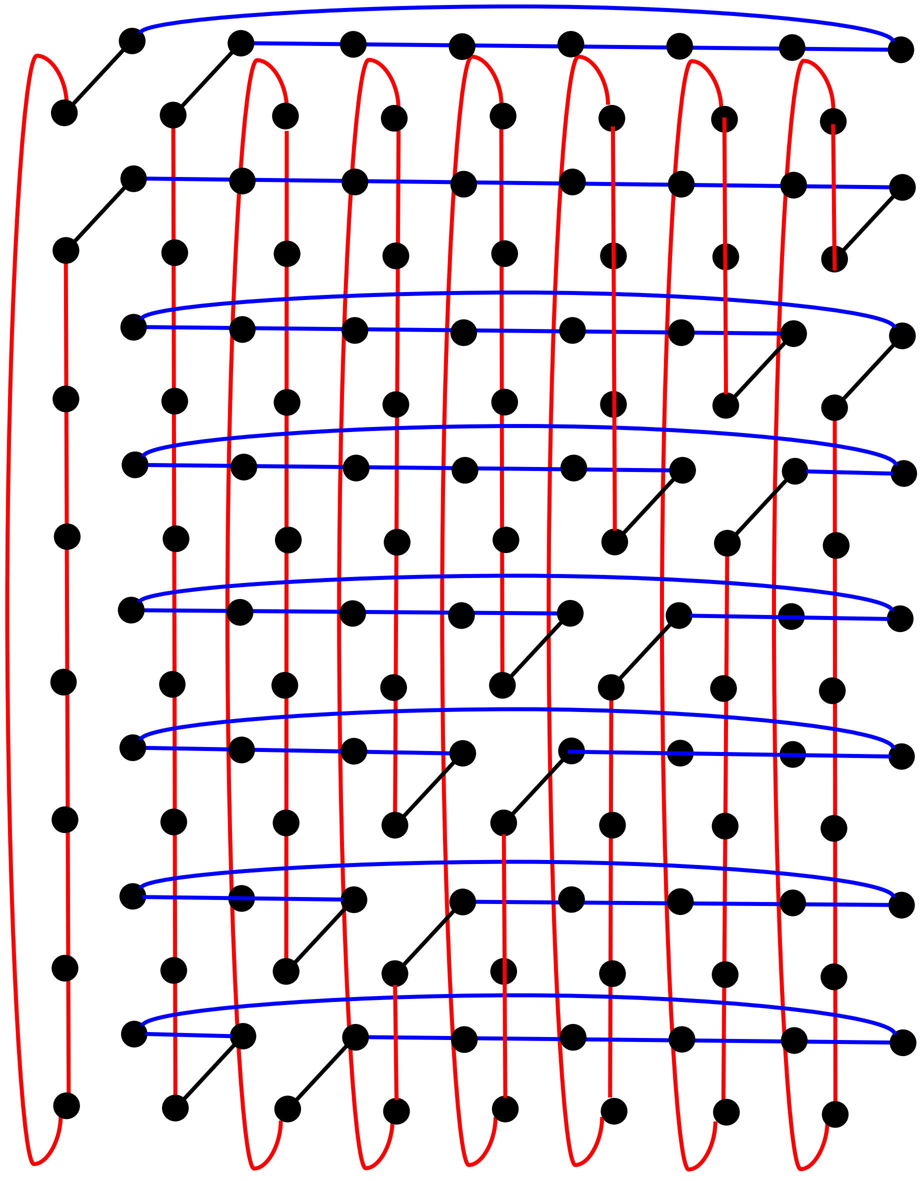
$C_p^{K_2}$:

G : Hamiltonian

C_p: Hamiltonian cycle in G



$C_p^{K_2}$:



G : Hamiltonian

C_p: Hamiltonian cycle in G

Theorem 3.1

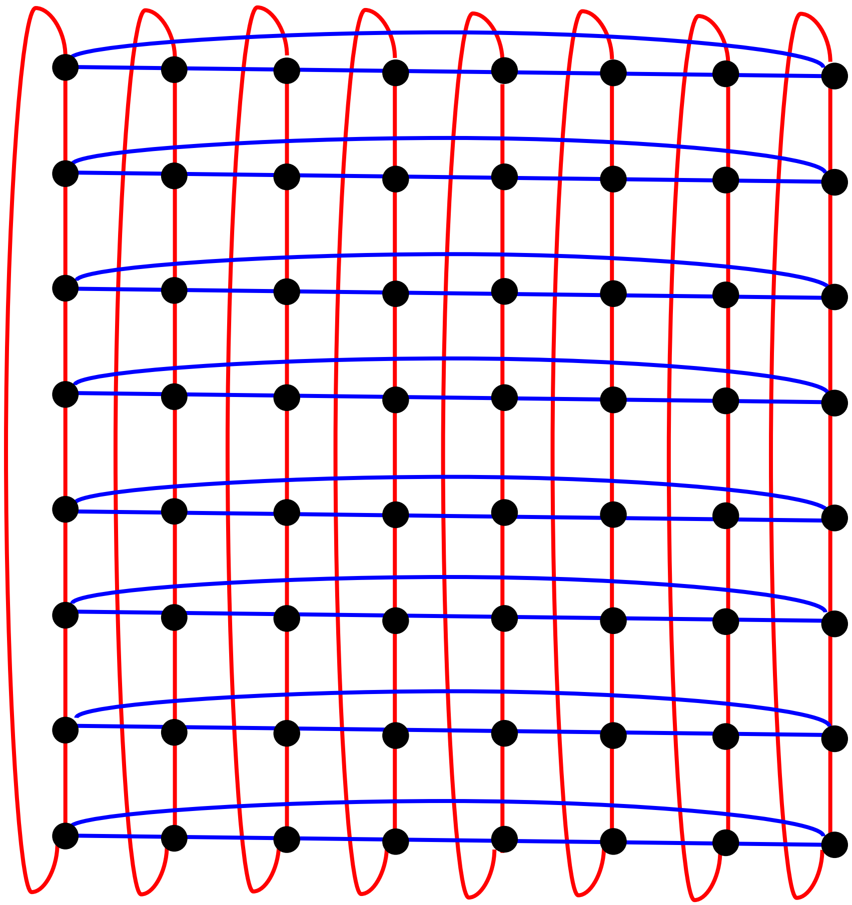
If G is Hamiltonian, then G^{K_2} is Hamiltonian.

Lemma 3.3

If G is Hamiltonian such that $|V(G)|$ is even, then G^q has a Hamiltonian cycle in which no two adjacent edges have the same dimension.

C_p^2:

C_p: Hamiltonian cycle in G

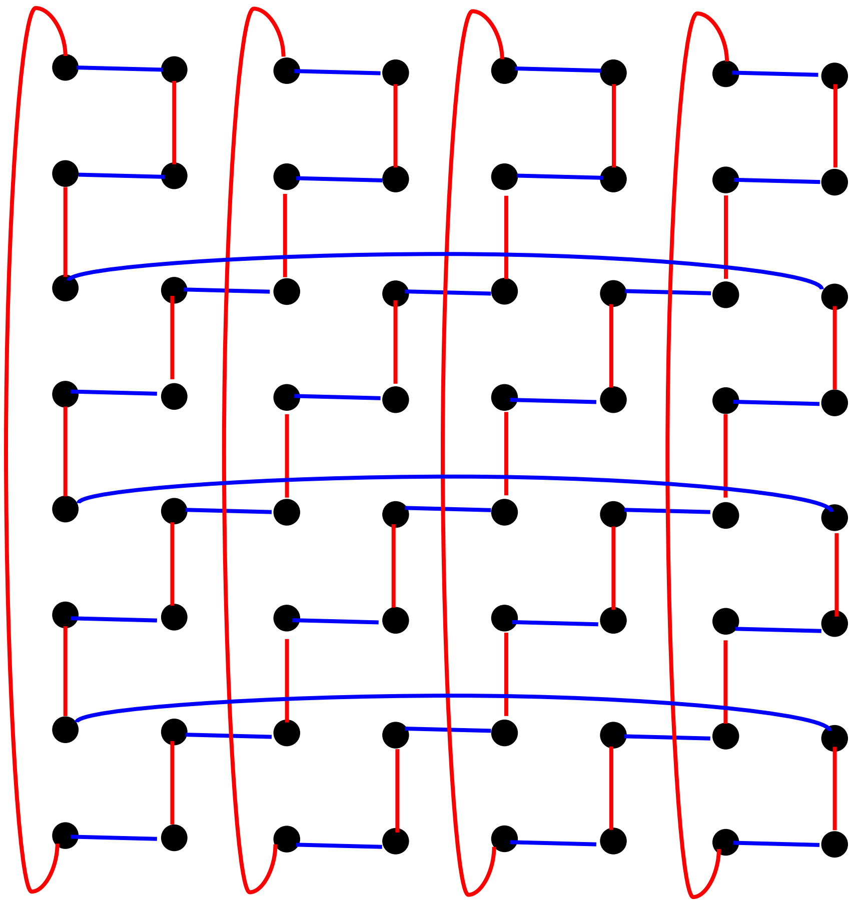


Lemma 3.3

If G is Hamiltonian such that $|V(G)|$ is even, then G^q has a Hamiltonian cycle in which no two adjacent edges have the same dimension.

C_p^2:

C_p: Hamiltonian cycle in G

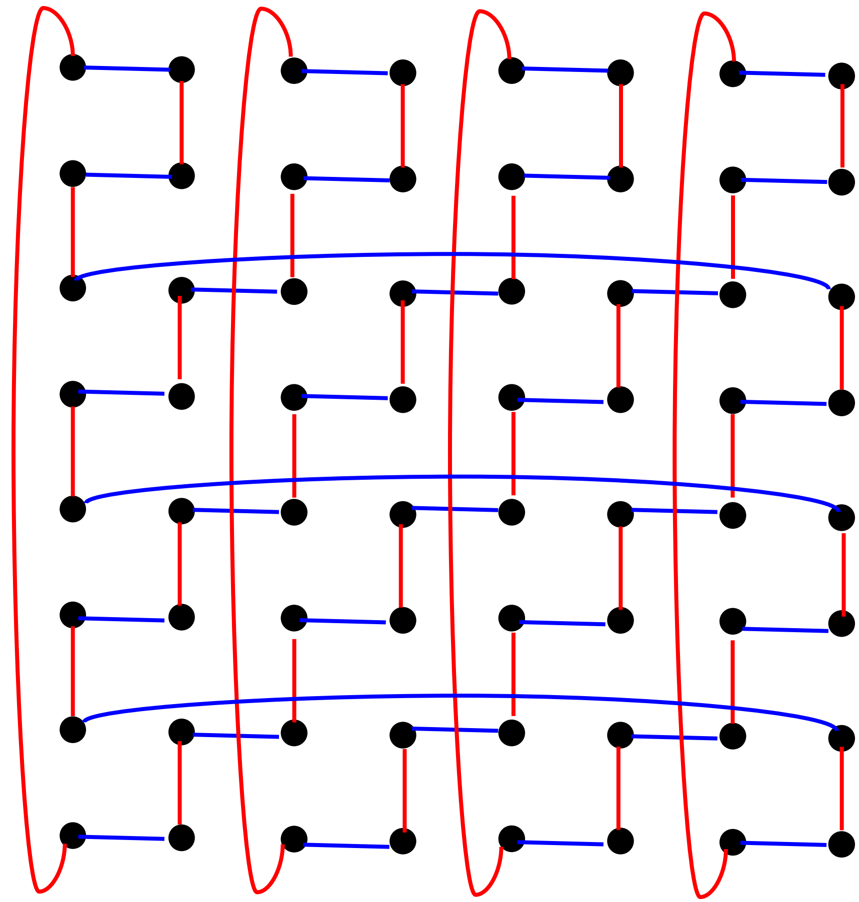


Lemma 3.3

If G is Hamiltonian such that $|V(G)|$ is even, then G^q has a Hamiltonian cycle in which no two adjacent edges have the same dimension.

Based on this lemma and the assumption that H is Hamiltonian-connected, we can construct a Hamiltonian cycle in G^H.

C_p^2:



Theorem 3.2

If G is Hamiltonian such that $|V(G)|$ is even and H is Hamiltonian connected, then G^H is Hamiltonian.

Outline

1. Exponentiation of graphs
   - Definition
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Applications

DCell $D_{k,n}$ is well-known as a doubly exponential network.

	$D_{k,n}$
Order	$\geq \left(n + \frac{1}{2}\right)^{2^k}$
Degree (regular)	$n + k - 1$
Diameter	$\leq 2^{k+1} - 1$
Connectivity	$n + k - 1$
Hamiltonicity	Yes

Bounded-degree logarithmic diameter networks

De Bruijn network $B^*(d,k)$: the underlying graph of the de Bruijn digraph $B(d,k)$

Kautz network $K^*(d,k)$: the underlying graph of the de Bruijn digraph $K(d,k)$

	$B^*(d,k)$	$K^*(d,k)$
Order	d^k	$d^k + d^{k-1}$
Maximum Degree	$2d$	$2d$
Minimum Degree	$2d-2$	$2d-2$
Diameter	k	k
Connectivity	$2d-2$	$2d-2$
Hamiltonicity	Yes	Yes

Exponential graphs using K_n, $B^*(d,k)$ and $K^*(d,k)$

$K_n^{B^*(d,k)}$ $K_n^{K^*(d,k)}$

Comparison

	$D_{k,n}$	$K_n^{B^*(d,k)}$	$K_n^{K^*(d,k)}$
Order	$\geq \left(n+\frac{1}{2}\right)^{2^k}$	$n^{d^k} d^k$	$n^{d^k+d^{k-1}}(d^k + d^{k-1})$
Maximum Degree	$n+k-1$	$n+2d-1$	$n+2d-1$
Minimum Degree	$n+k-1$	$n+2d-3$	$n+2d-3$
Diameter	$\leq 2^{k+1}-1$	$\leq 2d^k + k$	$\leq 2(d^k+d^{k-1})+k$
Connectivity	$n+k-1$	$n+2d-3$	$n+2d-3$
Hamiltonicity	Yes	-	-

- $K_n^{B^*(d,k)}$ and $K_n^{K^*(d,k)}$ have an advantage that the order can be increased while fixing the maximum degree.

Multi-exponential bounded-degree networks

$$K_n^{\left(K_n^{B^*(d,k)}\right)} \qquad K_n^{\left(K_n^{K^*(d,k)}\right)}$$

	$K_n^{\left(K_n^{B^*(d,k)}\right)}$	$K_n^{\left(K_n^{K^*(d,k)}\right)}$
Order	$n^{n^{d^k}d^k}n^{d^k}d^k$	$n^{n^{d^k+d^{k-1}}(d^k+d^{k-1})}\, n^{d^k+d^{k-1}}(d^k+d^{k-1})$
Maximum Degree	$2(n+d-1)$	$2(n+d-1)$
Minimum Degree	$2(n+d-2)$	$2(n+d-2)$
Diameter	$\leq 2n^{d^k}d^k+2d^k+k$	$\leq 2n^{d^k+d^{k-1}}\left(d^k+d^{k-1}\right)+2(d^k+d^{k-1})+k$
Connectivity	$2(n+d-2)$	$2(n+d-2)$
Hamiltonicity	-	-

Iterated exponential graphs of a graph G

$$\Omega(G,k) \quad \begin{cases} \Omega(G,1) = G, \\ \Omega(G,k) = \Omega(G,k-1)^G \quad \text{for } k \geq 2. \end{cases}$$

$$\Psi(G,k) \quad \begin{cases} \Psi(G,1) = G, \\ \Psi(G,k) = G^{\Psi(G,k-1)} \quad \text{for } k \geq 2. \end{cases}$$

$\Omega(G,k)$ and $\Psi(G,k)$

	$\Omega(K_n,k)$	$\Psi(K_n,k)$
Order	$n^{\frac{n^k-1}{n-1}}$	$n^{n^{\cdot^{\cdot^{\cdot^{n^{k+1}}}}}}+\cdots+n^{k+1}+k+1$
Maximum Degree	$k\cdot\Delta(G)$	$k\cdot\Delta(G)$
Minimum Degree	$k\cdot\delta(G)$	$k\cdot\delta(G)$
Diameter	$n^{k-1}\left(\mathrm{diam}(G)+\frac{2}{n-1}\right)-\frac{2}{n-1}$	$\leq\left((n^{\cdot^{\cdot^{\cdot^{n^{k+1}}}}}+\cdots+n^{k+1}+k\right)(\mathrm{diam}(G)+2)-2$
Connectivity	$k\cdot\delta(G)$	$k\cdot\delta(G)$
Hamiltonicity	n: even	-

$\Omega(K_2, k)$ and $\Psi(K_2, k)$

	$\Omega(K_2, k)$	$\Psi(K_2, k)$
Order	$2^{2^k - 1}$	$2^{2^{\cdot^{\cdot^{\cdot^{2^{k+1}}}}}} + \cdots + 2^{k+1} + k + 1$
Degree (regular)	k	k
Diameter	$3 \cdot 2^{k-1} - 2$	$3 \cdot \left((2^{\cdot^{\cdot^{\cdot^{2^{k+1}}}}} + \cdots + 2^{k+1} + k \right) - 2$
Connectivity	k	k
Hamiltonicity	Yes	-

- We name $\Omega(K_2, k)$ the *exponential cube* with notation Ω_k , although it has no structure concerning a cube, which is simply from an analogous for the definition of the hypercube $Q_k = K_2^k$.

Comparison of the DCell and the exponential cube

	$D_{k,n}$	Ω_k
Order	$\geq \left(n + \frac{1}{2}\right)^{2^k}$	$2^{2^k - 1}$
Degree (regular)	$n + k - 1$	k
Diameter	$\leq 2^{k+1} - 1$	$3 \cdot 2^{k-1} - 2$
Connectivity	$n + k - 1$	k
Hamiltonicity	Yes	Yes

- The exponential cube Ω_k might be another candidate for large-scale communication networks.

Conclusion

- We have newly introduced exponentiation of graphs.

- We have checked fundamental properties of the exponential graph G^H and shown that
  - G^H is maximally connected if G and H are connected,
  - G^H is Hamiltonian if G is Hamiltonian, $|V(G)|$ is even and H is Hamiltonian-connected.

- As an application to very large-scale networks, we have considered special exponential graphs with logarithmic diameter.

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  - G^H is Hamiltonian if G is Hamiltonian, $|V(G)|$ is even and H is Hamiltonian-connected.

- As an application to very large-scale networks, we have considered special exponential graphs with logarithmic diameter.

Thank you very much for your kind attention.