Schur rings and multiplicity one subgroups

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We also let $X^{(-1)} = \{x^{-1} : x \in X\}$.

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(2) if $0 \leq i, j \leq m$, then

$$\overline{C_i}\,\overline{C_j} = \sum_{k=0}^{r} \lambda_{ijk} \overline{C_k},$$

where $\lambda_{ijk} \in \mathbb{Z}^{\geq 0}$ for all i, j, k.

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The C_i are called the *principal sets* of the Schur ring.

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3. Direct products of Schur rings.

Some examples

4. Relative difference sets sometimes give Schur rings: suppose $1 < H \lhd G$ where there is $D \subset G \setminus H$ with
(i) $\overline{G} = \overline{D} + \overline{D^{(-1)}} + \overline{H}$;
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Then
(a) $H = \langle t \rangle \cong \mathcal{C}_2$,
(b) $|G| \equiv 0 \mod 8$,
(c) G is not abelian,
(d) a Sylow 2-subgroup is a generalized quaternion group and
(e) $\{1\}, \{t\}, D, D^{(-1)}$ is a Schur ring.

Basic facts

Each Schur ring is an association scheme (not necessarily commutative) with Bose-Mesner matrices $\mathcal{P}_i$ given by:

$$(\mathcal{P}_i)_{x,y} = 1 \text{ if and only if } yx^{-1} \in C_i.$$

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ORIGINAL MOTIVATION (Schur, Wielandt): Character-free method to study permutation groups.

One use of Schur rings

Random walks on G: Take probabilities $x_g \geq 0, \sum_{g \in G} x_g = 1$, that are constant on the principal sets of some commutative Schur ring.

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EXAMPLE: $G = S_3, H = \langle (1,2) \rangle$, with $\mathfrak{S}(G,H)$: $C_0 = \{Id(S_3)\}, C_1 = \{(1,2)\}, C_2 = \{(2,3),(1,3)\}, C_3 = \{(1,3,2),(1,2,3)\}$ Let

$$\mathcal{P} = \sum_{i=0}^{3} x_i \mathcal{P}_i = \begin{bmatrix} x_0 & x_3 & x_2 & x_3 & x_2 & x_1 \\ x_3 & x_0 & x_1 & x_3 & x_2 & x_2 \\ x_2 & x_1 & x_0 & x_2 & x_3 & x_3 \\ x_3 & x_3 & x_2 & x_0 & x_1 & x_2 \\ x_2 & x_2 & x_3 & x_1 & x_0 & x_3 \\ x_1 & x_2 & x_3 & x_2 & x_3 & x_0 \end{bmatrix}$$

Then commutativity and semisimplicity says that this matrix is similar to:

$$\mathrm{diag}(x_0 + x_1 + 2x_2 + 2x_3, x_0 - x_1 - 2x_2 + 2x_3, x_0 - x_1 + x_2 - x_3,$$

$$x_0 - x_1 + x_2 - x_3, x_0 + x_1 - x_2 - x_3, x_0 + x_1 - x_2 - x_3)$$

Gelfand pairs

QUESTION: How do you construct commutative Schur rings over a group?
Let G be a group and $H \leq G$. Then (G, H) is a *Gelfand pair* if the double cosets $HgH, g \in G$, commute.

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Usual definition: (G, H) is a *Gelfand pair* if the induced character $1_H \uparrow G$ has the multiplicity one property.

Definition: A character χ of G has the *multiplicity one property* if $\chi = \sum_i c_i \psi_i, c_i \leq 1$, where $\hat{G} = \{\psi_1, \psi_2, \cdots, \psi_r\}$.

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CLASSIC EXAMPLES: (S_n, S_{n-1}); $(S_{2n}, Cox(B_n))$.

Gelfand's Lemma

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STANDARD EXAMPLE: $X = S^n \subset \mathbb{R}^{n+1}$. Then $G = SO_{n+1}$ acts 2-point transitively on S^n and $H = SO_n$ so that (SO_{n+1}, SO_n) is a Gelfand pair.

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ONE COMBINATORIAL APPLICATION: The Gelfand pair $(S_{2n}, Cox(B_n))$ was used by N. Lindzey to study perfect matchings of K_{2n}.

Strong Gelfand pairs

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For S_n (based on results of Saxl).

Theorem (Anderson, H., Nicholson)

For $G = S_n, n \geq 7$, the strong Gelfand pairs are:
(i) (S_n, S_n);
(ii) (S_n, A_n);
(iii) (S_n, S_{n-1});
(iv) $(S_n, S_{n-2} \times S_2)$.

Strong Gelfand pairs, examples ctd.

Let $G = SL(2, p^n)$.
Let B denote the upper-triangular subgroup of G.
Let $B_2 \leq B$ be the subgroup with squares on the diagonal.

Theorem (Barton, Gardiner, H.)

For $G = SL(2, p^n), p^n > 11$, the proper strong Gelfand pairs are:
(i) For $p \equiv 1 \mod 4$ only (G, B);
(ii) For $p \equiv 3 \mod 4$ there are (G, B) and (G, B_2);
(iii) For $p = 2$ they are $(G, B), (G, D_{2(2^n+1)})$ and $(G, \mathcal{C}_{2^n+1})$.

S_n again

Let (S_n, H) be a strong Gelfand pair, so that $\mathfrak{S}(S_n, H)$ is commutative. Then H acts on $(1,2)^{S_n}$, not necessarily transitively, and so $(1,2)^{S_n}$ is a union of principal sets.

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Theorem (H)

If $n \geq 6$, then in the above situation $(1,2)^{S_n}$ splits into at most three principal sets, and the corresponding partitions of $(1,2)^{S_n}$ are the same as those obtained using the above strong Gelfand subgroups $(S_n, A_n, S_{n-1}, S_2 \times S_{n-2})$.

$\mathfrak{S}(G, H)$ maximal

FACT: The maximal dimension of a commutative Schur ring over G is $s_G := \sum_{\chi \in \hat{G}} \chi(1)$.

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If (G, H) is a strong Gelfand pair and $\mathfrak{S}(G, H)$ is of maximal dimension s_G, then H is abelian.

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TEMPTING DEFINITION: Let $H \leq G$. Then (G, H) is an *extra strong Gelfand pair* if (G, H) is a strong Gelfand pair and $\mathfrak{S}(G, H)$ is of maximal dimension.

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Corollary

(i) For $n \geq 6$, S_n has no extra strong Gelfand pairs.
(ii) For $SL(2, p^n), p^n > 11$, only $SL(2, 2^n)$ has an extra strong Gelfand pair: $\mathcal{C}_{2^n+1}$

Symplectic groups

$V = \mathbb{F}_2^n$ with n even and symplectic form $\cdot$ (antisymmetric, nondegerate).

$Sp(n,2)$ - Symplectic group

Symplectic transvections: for $0 \neq a \in V$ define

$$T_a : V \to V, \quad v \mapsto v + (a \cdot v)a.$$

Let $\mathcal{T} = \{T_a : a \in V, a \neq 0\}$

Lemma

(i) $Sp(n,2)$ is generated by the symplectic transvections.
(ii) $\mathcal{T}$ is a conjugacy class in $Sp(n,2)$.
(iii) $Sp(n,2)$ is a 3-transposition group.
[(i) and (ii) and: if $T_a, T_b \in \mathcal{T}$, then $T_a T_b$ has order $1,2,3$.]

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For small n:

Theorem

(1) For $Sp(2, 2) \cong S_3$ *there are two proper (extra) strong Gelfand subgroups* $\langle(1, 2)\rangle, \langle(1, 2, 3)\rangle$.

(2) For $Sp(4, 2) \cong S_6$ *the proper strong Gelfand subgroups are:* $S_4 \times S_2, S_2 \wr S_3, S_3 \wr S_2, A_6, S_5, \langle(1, 5)(2, 3)(4, 6), (1, 3, 6, 4, 5, 2)\rangle \cong S_5$

(3) For $Sp(6, 2)$ *there is one strong Gelfand subgroup:* $SO^-(6, 2)$*; there are* 31 *Gelfand subgroups.*

(4) for $n = 8, 10, 12$ *there are no strong Gelfand subgroups [Magma].*

Symplectic groups

Problem: $\mathrm{Sp}(n,2), n \geq 14$?

Theorem (H)

Any commutative Schur ring over $Sp(n,2), n \geq 14$, contains $\mathcal{T}$ as a principal set.

Corollary

If $(Sp(n,2), H), n \geq 8$, is a proper strong Gelfand pair, then
(i) H acts transitively on $\mathcal{T}$ (by conjugation);
(ii) H contains no elements of $\mathcal{T}$.

Dye subgroups

We note that Dye [1984/7] has constructed such subgroups - they are maximal.

These only occur when $n = 2p$ with p prime, and they are the stabilizers of a set $\{U_i\}_i$ (called a spread) of subspaces where $\{U_i \setminus \{0\}\}_i$ partitions $V \setminus \{0\}$. We denote this H by Dye_n; then

Proposition

Dye_n is not a strong Gelfand subgroup.

Proof of the main theorem - first step

Lemma (Commuting condition)

For disjoint $C, D \subset \mathcal{T}$, *we have:* $\overline{C}$, $\overline{D}$ *commute if and only if for all* $T_a \in C, T_b \in D, a \cdot b = 1$, *we have* $T_{a+b} \in C \cup D$.

THE END

THANKS FOR LISTENING