Strongly regular graphs decomposable into divisible design graph and Delsarte clique

Vladislav Kabanov

Krasovskii Institute of Mathematics and Mechanics, Yekaterinburg, Russia

joint work with Alexander Gavrilyuk – Shimane University, Matsue, Japan

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Strongly regular graphs

Definition 1

A simple graph of order v is called strongly regular with parameters (v, k, λ, μ) whenever it is not complete or edgeless and

- any vertex is adjacent to k vertices,
- any two adjacent vertices have λ common neighbours,
- any two non-adjacent vertices have μ common neighbours.

We already have this definition thanks to the previous great presentation.

We use some basic results of strongly regular graphs as set out, for example, in the recent monograph

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Cliques and cocliques in strongly regular graphs

Let Γ be a strongly regular graph with parameters (v, k, λ, μ), and eigenvalues k, r, s, where $r > 0$ and $s < 0$.

The clique and coclique numbers in strongly regular graphs satisfy the well-known Delsarte-Hoffman bound.

(i) If C is a clique in Γ, then $|C| \leq 1 + \frac{k}{-s}$. If the equality holds, then C is called a Delsarte clique.

(ii) If C is a coclique in Γ, then $|C| \leq v/(1 + \frac{k}{-s})$. If the equality holds, then C is called a Hoffman coclique.

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Strongly regular decomposition

If a strongly regular graph has a Delsarte clique or a Hoffman coclique, then there is a special case of regular decomposition (or an equitable partition) and the subgraph on $V(\Gamma) \setminus C$ has restrictions on its eigenvalues.

W.H. Haemers and D.G. Higman studied this case in general and in particular, strongly regular graphs which admit a decomposition into a strongly regular graph and a Delsarte clique (or a Hoffman coclique).

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As an analogue of these results, we study strongly regular graphs that can be decomposed into a divisible design graph and a Delsarte clique (or a Hoffman coclique).

Let Γ be the (4×4)-lattice, which is a strongly regular graph with parameters $(16, 6, 2, 2)$.

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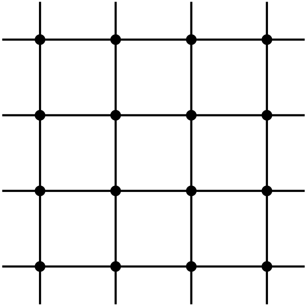
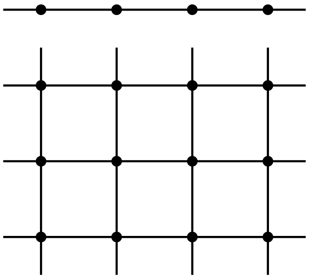


Any two vertices on the same horizontal or vertical line are adjacent.

Example

Consider the (4×4)-lattice without K_4.

Any two vertices on the same vertical line have 1 common neighbour and any two vertices on different vertical lines have 2 common neighbours.



Divisible design graphs

Definition 2

A k-regular graph Δ on v vertices is a divisible design graph with parameters $(v, k, \lambda_1, \lambda_2; m, n)$

- if its vertex set can be partitioned into m classes of size n,
- any two distinct vertices from the same class have λ_1 common neighbours,
- any two vertices from different classes have λ_2 common neighbours.

Obviously, a complete multipartite graph with equal parts is a divisible design graph, otherwise it is called a proper divisible design graph.

Divisible design graphs were first considered in
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Wallis–Fon-Der-Flaass construction

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Affine planes

For our construction, we used finite affine planes as a special case of affine designs.

A 2-dimensional affine space over the finite field of order q is a Desarguesian affine plane with q^2 points, q points on any line and $q+1$ parallel classes of lines,

Also there are many examples of finite non-Desarguesian affine planes with the same parameters.

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Construction

Let $D_1, \dots, D_{q+2}$ be affine planes with q^2 points and $q+1$ parallel classes of lines in each D_i (which are not necessarily isomorphic). Let $D_i = (V_i, L_i)$, where $i \in \{1, \dots, q+2\}$.

All parallel classes in each D_i are enumerated by integers from $\{1, \dots, q+2\} \setminus \{i\}$. Any j-th parallel class of D_i is denoted by L_i^j. For $v \in V_i$, let $\ell_i^j(v)$ denote the line in the parallel class L_i^j which contains v.

Choose one arbitrary point p_i in V_i for any $i \in \{1, \dots, q+2\}$. Let $C = \{p_1, \dots, p_{q+2}\}$. For any pair i, j, where $i \neq j$, choose an arbitrary bijection

$$\sigma_{i,j} : L_i^j \to L_j^i.$$

We require $\sigma_{i,j} = \sigma_{j,i}^{-1}$ and $\sigma_{i,j}(\ell_i^j(p_i)) = \ell_j^i(p_j)$ for any $i, j \in \{1, \dots, q+2\}$.

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Example 1.

Construct a graph Γ as follows:

The vertex set of Γ is

$$V = \bigcup_{i=1}^{q+2} V_i.$$

Any vertices x in V_i and y in V_j, where $i \neq j$, are adjacent if and only if

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Construction

If Γ is a graph from Construction, then Γ is a strongly regular graph with parameters

$$(q^2(q+2), q(q+1), q, q).$$

Moreover, C is a Delsarte clique and the induced subgraph Δ on $V(\Gamma) \setminus C$ is a divisible design graph with parameters

$$((q+2)(q^2-1), q^2+q-1, q-1, q; q+2, q^2-1).$$

For $q = 2$ we have a divisible design graph with 12 vertices. This graph was shown in one of the previous slides.

For all other q these divisible design graphs were unknown.

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Divisible design graph + clique

In this talk, we present parameters of strongly regular graphs that can be decomposed into a divisible design graph and a Delsarte clique in general.

Theorem 1. A.L. Gavrilyuk and V.V. K.

Let Γ be a primitive strongly regular graph with parameters (v, k, λ, μ) and the restricted eigenvalues r, s, where $r > 0 > s$. Suppose that Γ contains a clique C of cardinality $1 - k/s$ and the subgraph Δ induced on $V(\Gamma) \setminus C$ is a proper divisible design graph. Then either Γ has parameters $(25, 8, 3, 2)$ and $(64, 45, 32, 30)$, or there exists a natural number m such that Γ has parameters

$$v = \frac{-sm(m-2)}{(m+s-1)}, \; k = -s(m-1), \; \lambda = \mu = -s(m+s-1). \quad (1)$$

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The proof of Theorem 1 base on the following three statements.

Lemma 2.1 [HKM]

The eigenvalues of a divisible design graph Δ with parameters $(V, K, \lambda_1, \lambda_2, m, n)$ are $\{K, \pm\sqrt{K - \lambda_1}, \pm\sqrt{K^2 - \lambda_2 V}\}$.

Proposition 3.2 [HKM]

If g_1 and g_2 are the multiplicities of $\pm\sqrt{K^2 - \lambda_2 V}$, then $0 \leq K + (g_1 - g_2)\sqrt{K^2 - \lambda_2 V} \leq m(n-1)$.

Corollary from Theorem 2.4 [HH]

Let Γ be a primitive strongly regular graph on v vertices, with spectrum k^1, r^f, s^g, where $k > r > s$. If Γ contains a Delsarte clique C, then the subgraph of Γ, induced by the vertices in Γ outside C, is a regular, connected graph with spectrum $\{(k + k/s + r)^1, r^{f-c}, (r + s + 1)^{c-1}, s^{g-c+1}\}$, which has four distinct eigenvalues if $c < g + 1$.

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Let us comment on the two sporadic cases in Theorem 1.

The (5×5)-lattice graph is a unique strongly regular graph with parameters $(25, 8, 3, 2)$. It obviously contains Delsarte cliques (of order 5); for every such clique, the other part is the (5×4)-lattice which obviously is a divisible design graph.

Sporadic cases in Theorem 1

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Sporadic cases in Theorem 1

The complementary graph of a strongly regular graph with parameters $(64, 45, 32, 30)$ were determined by W.H. Haemers and E. Spence.

There are 167 strongly regular graphs with these parameters and exactly eleven of them are the incidence graphs of some systems of linked symmetric 2-$(16, 6, 2)$ designs. Thus, the respective strongly regular graphs admit a partition of the vertex set into four Delsarte cliques of order 16.

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Furthermore, it follows from a direct inspection of these graphs that every Delsarte clique is a part of such a partition. For each of these four cliques, the other part in the corresponding regular decomposition is a divisible design graph with parameters $(48, 35, 26, 25; 3, 16)$.

Finally, besides these eleven strongly regular graphs, the others 156 do not have Delsarte cliques.

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Wallis–Fon-Der-Flaass construction

From Theorem 1 we have the following parameters

$$v = \frac{-sm(m-2)}{(m+s-1)},\ k = -s(m-1),\ \lambda = \mu = -s(m+s-1).$$

Let q be a prime power.

If $-s = q$ and $m = q+2$ then we have the parameters of strongly regular graphs from our Construction

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Strongly regular graphs from Steiner 2-design

There is another series of parameters that satisfy of Theorem 1.

Recall that the block graph of a Steiner 2-design on u points with blocks of order w is the graph with the blocks as vertices, where two distinct blocks are adjacent whenever their intersection is nonempty. Such a graph is strongly regular, with parameters

$$\left(\frac{u(u-1)}{w(w-1)}, \frac{w(u-w)}{w-1}, (w-1)^2 + \frac{u-2w+1}{w-1}, w^2\right)$$

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Strongly regular graphs from Steiner 2-design

Suppose that $r = -s$, then $u = 2w^2 - w$.

Example 2.

Let Γ be a strongly regular graph with parameters $(4w^2 - 1, 2w^2, w^2, w^2)$. In this case, $r = -s = w$ and $m + s - 1 = w$, then these parameters match of Theorem 1. Moreover, w is odd.

Steiner 2-designs on $u = 2w^2 - w$ points for odd w are known to exist for $w = 3, 5, 7$. We study $w = 5, 7$ known cases using the following database:

V. Krcadinac, Steiner 2-designs. https://web.math.pmf.unizg.hr

If $w = 3$, then the parameters are $(35, 18, 9, 9)$. All strongly regular graphs with these parameters were enumerated by B.D. McKay and E. Spence: there are 3854 of those and exactly 499 of them satisfy Theorem 1.

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Let Γ be a strongly regular graph with parameters $(4w^2 - 1, 2w^2, w^2, w^2)$. In this case, $r = -s = w$ and $m + s - 1 = w$, then these parameters match of Theorem 1. Moreover, w is odd.

Steiner 2-designs on $u = 2w^2 - w$ points for odd w are known to exist for $w = 3, 5, 7$. We study $w = 5, 7$ known cases using the following database:

V. Krcadinac, Steiner 2-designs. https://web.math.pmf.unizg.hr

If $w = 3$, then the parameters are $(35, 18, 9, 9)$. All strongly regular graphs with these parameters were enumerated by B.D. McKay and E. Spence: there are 3854 of those and exactly 499 of them satisfy Theorem 1.

The smallest unknown examples

The smallest (in the number of vertices) case of feasible parameters of strongly regular graphs that satisfy Eq. (1) of Theorem 1 but do not fit in with the series from Example 1 and Example 2 is $(112, 75, 50, 50)$. The existence of strongly regular graphs with these parameters is currently unknown.

In total, the Brouwer's database of strongly regular graphs contains 14 tuples of feasible parameters of putative strongly regular graphs whose parameters are satisfing of Theorem 1.

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Divisible design graph + coclique

Strongly regular graphs that can be decomposed into a divisible design graph and a coclique have also been studied.

In particular, it was found new construction of strongly regular graphs with parameters of the complement of a symplectic graph and shown in general that when the least eigenvalue of such a strongly regular graph is a prime power, its parameters coincide with those of the complement of a symplectic graph.

[VK1] V.V. Kabanov, New versions of the Wallis–Fon-Der-Flaass construction to create divisible design graphs, Discrete Math., 345(11), (2022) 113054.
[VK2] V.V. Kabanov, A new construction of strongly regular graphs with parameters of the complement symplectic graph, Electron. J. Comb., 30(1) (2023) # P1.25
[GK] A.L. Gavrilyuk, V.V. Kabanov, Strongly regular graphs decomposable into a divisible design graph and a Hoffman coclique. Des. Codes Cryptogr. 92(5), (2024) 1379–1391.

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Thank you for your attention!