130+ Years of the Hadamard Conjecture

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17 YEARS AGO ... SEVILLA 2007 HADAMARD CONF.

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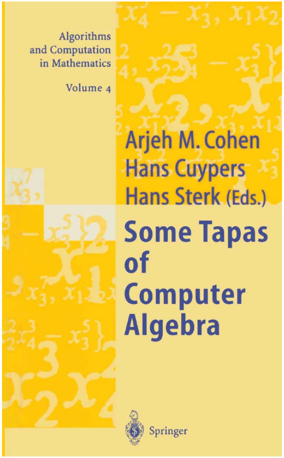
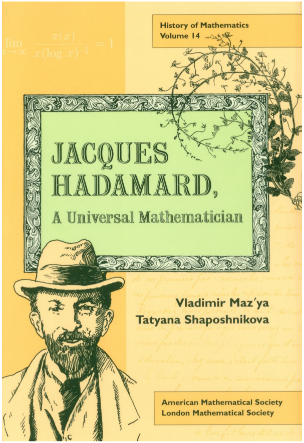

Algorithms
and Computation
in Mathematics
Volume 4
Arjeh M. Cohen
Hans Cuypers
Hans Sterk (Eds.)
Some Tapas
of
Computer
Algebra
Springer



Jacques Salomon Hadamard

8 December 1865 (Versailles) – 17 October 1963 (Paris)

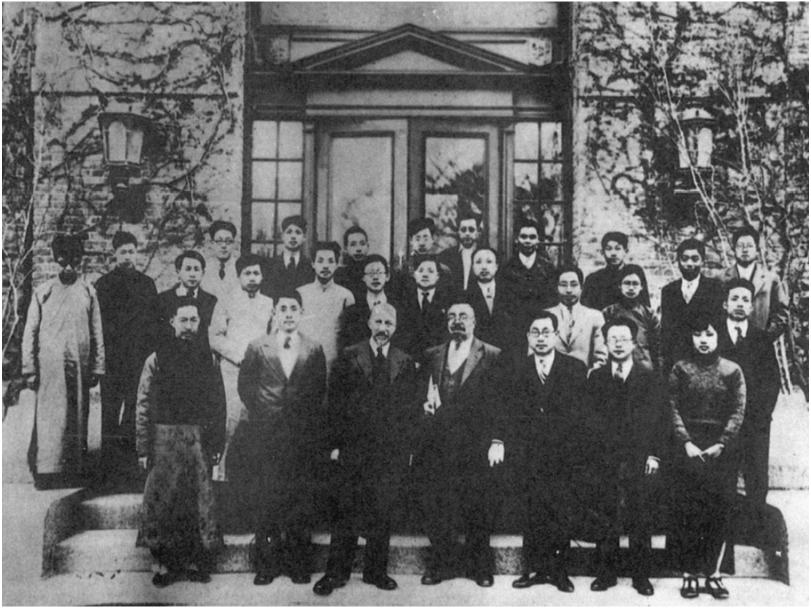
https://www.fondation-hadamard.fr/

A foundation promoting scientific excellence in mathematics

Biographical highlights

https://www.britannica.com/biography/Jacques-Salomon-Hadamard

- 1869: The Hadamard family moves to Paris
- 1884: 1st place in entrance examinations: École Polytechnique & ENS
- 1892: doctorate: ENS
- 1892: Grand Prix des Sciences Mathématiques: "Determination of the Number of Primes Less than a Given Number"
- 1893–1896: University of Bordeaux
- 1896: proof of the prime number theorem
- 1897: Alfred Dreyfus affair
- 1897-1935: professorship: Collège de France, Paris
- 1912–1935: professorship: École Polytechnique, Paris
- Spent WWII in the United States and the United Kingdom
- 1944: Hardy: LMS intro: "mathematics living legend"
- 1945: PUP: "The Psychology of Invention in the Mathematical Field"



Genesis of the Hadamard conjecture

1893

Sur le module maximum que puisse atteindre un déterminant, C. R. Acad. Sci. Paris **116**, 1500.

Résolution d'une question relative aux déterminants, Bull. Sci. Math. (2) **17**, 240–246.

Hadamard matrices are $n \times n$ (square) matrices H with ± 1 elements s.t.

$$H \cdot H^t = nI_n \longrightarrow HM(n)$$

- $\frac{1}{\sqrt{n}} H$ is an orthogonal matrix, $|\det(H)| = n^{n/2}$
- trivial cases: $n = 1, [1]$ and $n = 2$, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- well-known necessary condition: $n \equiv 0 \pmod 4$
- the sufficiency of this condition is the celebrated Hadamard conjecture
- "There exists a Hadamard matrix of order n, for every $n \equiv 0 \pmod 4$" (1893)

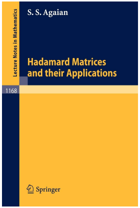
Computational state-of-the-art

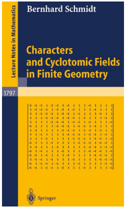
- smallest unresolved order until 1985: 268, K. Sawade, Graphs Combin.
- smallest unresolved order until 2005: 428, H. Kharaghani+BTR, JCD
- $HM(764)$, D. Z. Djokovic, Combinatorica 2008
- **smallest unresolved order until 2024: $\mathbf{668}$**
- 3 unresolved cases < 1000: $668, 716, 892$
- 10 unresolved cases < 2000:
  $1004, 1132, 1244, 1388, 1436, 1676, 1772, 1916, 1948, 1964$
- 2012: list of integers $v < 500$ for which no Hadamard matrices of order $4v$ are known consisted of 13 integers, all of them primes $\equiv 3 \pmod 4)$

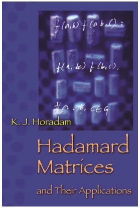
$$\mathbf{167}, \mathbf{179}, \mathbf{223}, \cancel{251}, 283, 311, 347, 359, 419, 443, 479, 487, 491$$

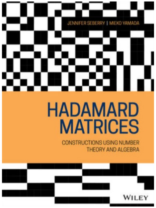
- $HM(4 \cdot 251)$ Djokovic, Golubitsky, Kotsireas, JCD, 2012
  unions of orbits approach + 4 zero-PAF sequences of lengths 251 + GS
- Magma and Sage contain databases of HMs
- N.J.A. Sloane on-line database http://neilsloane.com/hadamard/

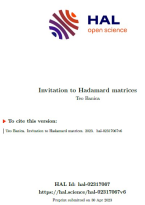
Hadamard Matrices Books & Chapters

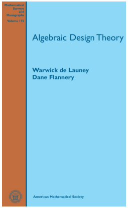


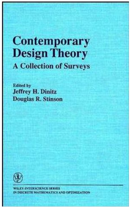


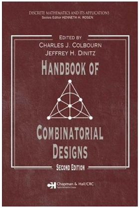


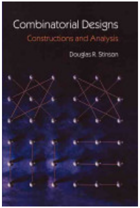


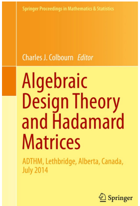


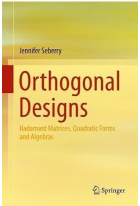












Hadamard Matrices Existence Results

- C. Lam, S. Lam, V. D. Tonchev, Bounds on the number of affine, symmetric, and Hadamard designs and matrices. JCTA 92 (2000), no. 2, 186–196.
- C. Lam, S. Lam, V. D. Tonchev, Bounds on the number of Hadamard designs of even order. JCD 9 (2001), no. 5, 363–378.

The number of inequivalent Hadamard matrices of order 40 is at least 3.66×10^{11}.

- E. Merchant, Exponentially many Hadamard designs. DCC 38 (2006), no. 2, 297–308.

If a Hadamard design of order n exists, then the number of inequivalent Hadamard matrices of size $8n$ is at least $2^{8n-16-11\log n}$.

- https://images.math.cnrs.fr/
- LA CONJECTURE DE HADAMARD (I), par Shalom Eliahou
- LA CONJECTURE DE HADAMARD (II), par Shalom Eliahou

Hadamard Matrices Asymptotic Existence Results

- J. Seberry-Wallis, JCTA 1976, "On the existence of Hadamard matrices"
- R. Craigen, JCTA 1995, "Signed groups, sequences, and the asymptotic existence of Hadamard matrices"
- W. de Launey, JCTA 2009, "Note On the asymptotic existence of Hadamard matrices"

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It is conjectured that Hadamard matrices exist for all orders $4t$ ($t > 0$). However, despite a sustained effort over more than five decades, the strongest overall existence results are asymptotic results of the form: for all odd natural numbers k, there is a Hadamard matrix of order $k2^{[a+b\log_2 k]}$, where a and b are fixed non-negative constants. To prove the Hadamard Conjecture, it is sufficient to show that we may take $a = 2$ and $b = 0$. Since Seberry's ground-breaking result, which showed that we may take $a = 0$ and $b = 2$, there have been several improvements where b has been by stages reduced to 3/8. In this paper, we show that for all $\epsilon > 0$, the set of odd numbers k for which there is a Hadamard matrix of order $k2^{2+[\epsilon\log_2 k]}$ has positive density in the set of natural numbers. The proof adapts a number-theoretic argument of Erdos and Odlyzko to show that there are enough Paley Hadamard matrices to give the result.

Important topics

- cocyclic approach:
  K. Horadam, D. Flannery, P. O'Cathain. R. Egan, The Sevilla Group
- cohomology approach: A. Goldberger, G. Dula
- Eliahou Theory, Coding Theory reformulation of the (structured) Hadamard conjecture
- large body of work on complexity/randomness
  A. Winterhof, C. Mauduit, A. Sarkozy, K. Gyarmati etc
- skew HMs symmetris HMs, Butson-Hadamard, complex HMs,

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  Obama HMs, ~~Trump HMs~~, ~~Biden HMs~~ replaced by N. Haley & K. Harris.

Formulae for the number of Hadamard matrices of order n

1. Shalom Eliahou
   Enumerative combinatorics and coding theory
   Enseign. Math. (2), 40(1-2):171–185, 1994.
2. Warwick de Launey and Daniel A. Levin
   A Fourier-analytic approach to counting partial Hadamard matrices
   Cryptogr. Commun., 2(2):307–334, 2010.

- The Eliahou formula uses Coding Theory: binary linear code + weight enumerator
- The de Launey-Levin formula uses multidimensional integrals associated with lattice walks
- Both formulas require a certain amount of technical definitions before they can be stated in a self-contained manner and are very difficult (computationally) to evaluate

It is far from evident why these two formulae (should) agree for all $n \equiv 0 \pmod 4$

Constructions for Hadamard matrices

1. Sylvester Kronecker product construction: $HM(n) \otimes HM(m) \longrightarrow HM(nm)$
2. U. Scarpis, 1898: $HM(n)$ $s.t.$ $(n-1)$ is prime $\longrightarrow HM((n-1)n)$
3. R.E.A.C. Paley, 1933: $n \equiv 0 \pmod 4$ s.t. $n-1$ or $\frac{n}{2}-1$ is p^k $\longrightarrow HM(n)$
4. J. Seberry Wallis, 1976: $\forall\, q$ odd, $\exists\, HM(2^t \cdot q)$, for t large enough.
5. Gruner's theorem: p and $p+2$ are twin primes, $\longrightarrow HM(p(p+2)+1)$
6. Circulant HMs: **Ryser conjecture (1963): CHM(4) is the only one ...**
7. Williamson method: uses the Williamson array $\begin{pmatrix} -A & B & C & D \\ B & A & D & -C \\ C & -D & A & B \\ D & C & -B & A \end{pmatrix}$

   where A, B, C, D are $n \times n$ circulant matrices satisfying certain properties.
8. Miyamoto JCTA 1991:

   requires 32 $(0,-1,1)$-matrices U_{ij}, V_{ij} s.t. $U_{ij} \pm V_{ij}$ are $(-1,1)$-matrices

There are literally $100s$ of HM constructions ...

They all suffer from two kinds of disadvantages:

- they produce a sparse set of orders (e.g twin primes)
- they fail for specific parameter values (e.g. Williamson for $n = 35$)

Opinion: The Hadamard conjecture is too general

The introduction of the **circulant** structure, furnishes two promising candidates and their variants for a proof of the (general) Hadamard conjecture $\rightsquigarrow$

1. Turyn/Golay quadruples conjecture: 4 sequences with **aperiodic autoc.** 0
2. Legendre pairs conjecture: 2 sequences with **periodic autocorrelation** -2

These two conjectures:

- introduce some **structure** into the more general Hadamard Conjecture. This structure is described in terms of four/two circulant matrices whose first rows (seen as binary sequences) have constant aperiodic/periodic autocorrelation $0/-2$
- **do not fail** for any value of the parameter, i.e. cover the full range of multiples of 4

1. I. S. Kotsireas et al. Hadamard ideals and Hadamard matrices with two circulant cores *European J. Combin.*, 27(5):658–668, 2006.
2. I. S. Kotsireas, Structured Hadamard Conjecture, 2013
in: Number Theory and Related Areas Eds: J. M. Borwein, I. Shparlinski, and W. Zudilin

Turyn/Golay quadruples & Turyn-type sequences

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- Turyn/Golay quadruples:
  Four $\{-1,+1\}$-sequences X, Y, Z, W of lengths n, n, n, n s.t.
  $NPAF(X,s) + NPAF(Y,s) + NPAF(Z,s) + NPAF(W,s) = 0, s = 1, \ldots n-1$

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- Turyn-type sequences:
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"similar" combinatorial objects can have very different characteristics

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▷ 5 known constructions for Turyn/Golay quadruples, smallest open case?

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▷ no known constructions for Turyn-type sequences, algorithmic ad-hoc methods.

n	$T(n,n,n,n)$	n	$T(n,n,n,n)$	n	$T(n,n,n,n)$	n	$T(n,n,n,n)$	n	$T(n,n,n,n)$
1		51	🔴	101	$g+g$	151		201	🟢$BS(101,100)$
2	$g+g$	52	$g+g$	102	$g+g$	152	$g+g$	202	$101\otimes 2$
3	$g+g$	53	$g+g$	103		153		203	
4	$g+g$	54	$g+g$	104	$g+g$	154	$g+g$ $77\otimes 2$	204	$102\otimes 2$
5	$g+g$	55	🟢	105	🟢$BS(53,52)$ $g+g$	155		205	
6	$g+g$	56	$g+g$	106	$g+g$	156	$g+g$	206	
7	🔴	57	🟢	107	🟣	157		207	
8	$g+g$	58	$g+g$	108	$g+g$	158	$79\otimes 2$	208	$104\otimes 2$
9	$g+g$	59	🔴🟢	109		159		209	🟢$BS(105,104)$
10	$g+g$ 🔵	60	$g+g$	110	$g+g$	160	$g+g$	210	$105\otimes 2$
11	$g+g$ 🔴🟢	61	🟢	111		161	🟢$BS(81,80)$ $g+g$	211	
12	$g+g$	62	$g+g$	112	$g+g$	162	$g+g$	212	$106\otimes 2$
13	🟢	63	🟢	113	🟣	163		213	
14	$g+g$ 🔵	64	$g+g$	114	$g+g$	164	$g+g$	214	
15	🔴🟢	65	$g+g$ 🟢	115		165		215	
16	$g+g$	66	$g+g$	116	$g+g$	166		216	$108\otimes 2$
17	$g+g$	67	🟢	117		167		217	
18	$g+g$ 🔵	68	$g+g$	118	$59\otimes 2$	168	$g+g$	218	$10+2\cdot 104DS$
19	🔴	69	🟢	119	🟣	169		219	
20	$g+g$	70	$35\otimes 2$	120	$g+g$	170	$g+g$	220	$110\otimes 2$
21	$g+g$	71	🟢	121		171		221	
22	$g+g$	72	$g+g$	122	$61\otimes 2$	172	$86\otimes 2$	222	
23	🔴	73	🟢$BS(37,36)$	123		173		223	
24	$g+g$	74	$g+g$	124	$g+g$	174		224	$112\otimes 2$
25	🟢	75	🟢 $BS(38,37)$	125	🟣	175		225	
26	$g+g$	76	$38\otimes 2$	126	$g+g$	176	$g+g$	226	
27	$g+g$ 🔴	77	🟢 $BS(39,38)$	127		177		227	
28	$g+g$	78	$g+g$	128	$g+g$	178		228	
29	🟢	79	🟢 $BS(40,39)$	129	🟢$BS(65,64)$ $g+g$	179		229	
30	$g+g$	80	$g+g$	130	$g+g$	180	$g+g$	230	
31	🔴	81	🟢 $BS(41,40)$ $g+g$	131		181		231	
32	$g+g$	82	$g+g$	132	$g+g$	182		232	
33	$g+g$	83	🟣 $BS(55,28)$	133		183		233	
34	$g+g$	84	$g+g$	134	$67\otimes 2$	184	$g+g$	234	
35	🟢	85	🟢?$BS(43,42)$	135		185		235	
36	$g+g$	86	$43\otimes 2$	136	$g+g$	186	$g+g$	236	
37	🟢	87	🟢?$BS(44,43)$	137		187		237	
38	$19\otimes 2$	88	$g+g$	138	$g+g$	188	$94\otimes 2$	238	
39	🟢	89	🟣 $BS(59,30)$	139		189		239	
40	$g+g$	90	$g+g$	140	$g+g$	190		240	
41	$g+g$ 🟢	91	🟢?$BS(46,45)$	141		191		241	
42	$g+g$	92	$g+g$	142	$71\otimes 2$	192	$g+g$	242	
43	🟢	93	🟢?$BS(47,46)$	143		193		243	
44	$g+g$	94	$47\otimes 2$	144	$g+g$	194		244	
45	🟢	95	🟣$BS(63,32)$	145		195		245	
46	$g+g$	96	$g+g$	146	$73\otimes 2$	196	$98\otimes 2$	246	
47	🟢	97	🟢?$BS(49,48)$	147		197		247	
48	$g+g$	98	$49\otimes 2$	148	$g+g$	198		248	
49	🟢	99	🟢?$BS(50,49)$	149		199		249	
50	$g+g$	100	$g+g$	150	$75\otimes 2$	200	$g+g$	250	

State-of-the-art result on Turyn-type sequences $TT(n)$

Theorem

Turyn-type sequences $TT(n)$ *exist for every even* $n = 2, \ldots, 40$.

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```

$HM(668), 167 = 3 \cdot 56 - 1$

Autocorrelation (periodic and aperiodic)

- The **periodic autocorrelation function** associated to a finite sequence $A = [a_0, \dots, a_{n-1}]$ of length n is defined as

$$P_A(s) = \sum_{k=0}^{n-1} a_k a_{k+s}, \; s = 0, \dots, n-1,$$

  where $k + s$ is taken modulo n, when $k + s > n$.
- The **aperiodic autocorrelation function** associated to a finite sequence $A = [a_0, \dots, a_{n-1}]$ of length n is defined as

$$N_A(s) = \sum_{k=0}^{n-1-s} a_k a_{k+s}, \; s = 0, \dots, n-1,$$

We are mostly concerned with binary $\{-1, +1\}$, ternary $\{-1, 0, +1\}$ and quaternary $\{\pm 1, \pm i\}$ sequences.
Note that for sequences with complex elements, a_{k+s} is replaced by $\overline{a_{k+s}}$.

Example: $n = 7$, $A = [a_1, \ldots, a_7]$

$$
\begin{aligned}
P_A(0) &= {a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2 + {a_5}^2 + {a_6}^2 + {a_7}^2 \\
P_A(1) &= a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_7 + a_7a_1 \\
P_A(2) &= a_1a_3 + a_2a_4 + a_3a_5 + a_4a_6 + a_5a_7 + a_6a_1 + a_7a_2 \\
P_A(3) &= a_1a_4 + a_2a_5 + a_3a_6 + a_4a_7 + a_5a_1 + a_6a_2 + a_7a_3 \\
P_A(4) &= a_1a_4 + a_2a_5 + a_3a_6 + a_4a_7 + a_5a_1 + a_6a_2 + a_7a_3 \\
P_A(5) &= a_1a_3 + a_2a_4 + a_3a_5 + a_4a_6 + a_5a_7 + a_6a_1 + a_7a_2 \\
P_A(6) &= a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_7 + a_7a_1
\end{aligned}
$$

$$
\begin{aligned}
N_A(0) &= {a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2 + {a_5}^2 + {a_6}^2 + {a_7}^2 \\
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N_A(2) &= a_1a_3 + a_2a_4 + a_3a_5 + a_4a_6 + a_5a_7 \\
N_A(3) &= a_1a_4 + a_2a_5 + a_3a_6 + a_4a_7 \\
N_A(4) &= a_1a_5 + a_2a_6 + a_3a_7 \\
N_A(5) &= a_1a_6 + a_2a_7 \\
N_A(6) &= a_1a_7
\end{aligned}
$$

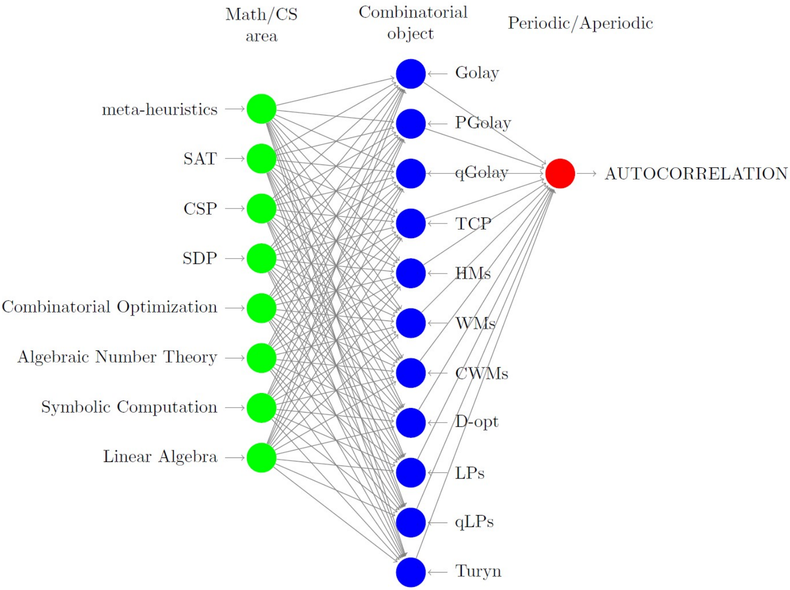
Circulant matrices

A $n \times n$ matrix $C(A)$ is called **circulant** if every row (except the first) is obtained by the previous row by a right cyclic shift by one.

$$C(A) = \begin{bmatrix} a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_2 & a_3 & \dots & a_0 & a_1 \\ a_1 & a_2 & \dots & a_{n-1} & a_0 \end{bmatrix}$$

- Consider a finite sequence $A = [a_0, \dots, a_{n-1}]$ of length n and the circulant matrix $C(A)$ whose first row is equal to A. Then $P_A(i)$ is the inner product of the first row of $C(A)$ and the $i+1$ row of $C(A)$.
- **symmetry property** $\rightsquigarrow$ $P_A(s) = P_A(n-s), s = 1, \dots, n-1.$
- 2^{nd} **ESF property** $\rightsquigarrow$ $P_A(1) + P_A(2) + \dots + P_A(n-1) = 2e_2(a_0, \dots, a_{n-1})$
- $\rightsquigarrow$ $N_A(s) + N_A(n-s) = P_A(s), s = 1, \dots, n-1.$

Unified description of combinatorial objects



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- General AI formalism to solve search problems, using V, D, C
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```
#
# 2cc Hadamard matrix Legendre pairs CSP, {V,D,C}, for ell = 5
# Created by: Ilias S. Kotsireas, ikotsire@gmail.com, Date Created: December 15, 2020
# 10 variables, 10 {-1,+1} domains, 4 constraints, (2 linear and 2 quadratic)
#

V :=
a1, a2, a3, a4, a5
b1, b2, b3, b4, b5

D :=
Da1 = ... = Da5 = {-1,+1}
Db1 = ... = Db5 = {-1,+1}

C :=
c1 := a1*a2+a2*a3+a3*a4+a4*a5+a5*a1+b1*b2+b2*b3+b3*b4+b4*b5+b5*b1 = -2

c2 := a1*a3+a2*a4+a3*a5+a4*a1+a5*a2+b1*b3+b2*b4+b3*b5+b4*b1+b5*b2 = -2

c3 := a1+a2+a3+a4+a5 = 1

c4 := b1+b2+b3+b4+b5 = 1
```

Legendre Pairs (Seberry, 2001)

Definition

$\forall$ odd n, two sequences $A = [a_0, \ldots, a_{n-1}]$ and $B = [b_0, \ldots, b_{n-1}]$, with $\{-1, +1\}$ elements, form a **Legendre Pair LP(n)** of order/length n if:

$$PAF(A, s) + PAF(B, s) = -2, s = 1, \ldots, \frac{n-1}{2}$$

- Normalization: $a_0 + \cdots + a_{n-1} = 1$, $b_0 + \cdots + b_{n-1} = 1$

  $\rightsquigarrow$ upper bound on potential A,B seqs: $\binom{n}{\frac{n+1}{2}}$
- Consequence/Property: the PSD constant Wiener–Khinchin

$$PSD(A, s) + PSD(B, s) = 2n + 2, s = 1, \ldots, \frac{n-1}{2}$$

- LPs characterized by: constancy of PAF & PSD invariants

Examples of Legendre pairs

Example (1)

$n = 11, \quad LP(11), \quad PSD = 2 \cdot 11 + 2 = 24$

A = [1, 1, -1, 1, 1, 1, -1, -1, -1, 1, -1]
B = [1, -1, 1, -1, -1, -1, 1, 1, 1, -1, 1]
first 5 PAF values for A: -1, -1, -1, -1, -1
first 5 PAF values for B: -1, -1, -1, -1, -1

Example (2)

$n = 13, \quad LP(13), \quad PSD = 2 \cdot 13 + 2 = 28$

A = [-1, -1, -1, 1, -1, 1, -1, -1, 1, 1, 1, 1, 1],
B = [-1, -1, 1, -1, 1, -1, -1, 1, 1, -1, 1, 1, 1]
first 6 PAF values for A: 1, 1, -3, -3, -3, 1
first 6 PAF values for B: -3, -3, 1, 1, 1, -3

Example (3)

$n = 37, \quad LP(37), \quad PSD = 2 \cdot 37 + 2 = 76$

A = [-1,-1,-1,-1,1,-1,1,1,-1,-1,1,-1,1,-1,1,-1,-1,1,1,-1,1,-1,-1,-1,-1,1,1,1,-1,1,1,1,1,-1,1,1,1]
B = [-1,-1,-1,-1,1,-1,-1,-1,1,1,1,1,-1,1,-1,-1,1,1,-1,1,1,1,-1,1,1,-1,-1,1,-1,1,1,1,1,-1,-1,-1,1]

first 18 PAF values for A: -3, 1, -3, -3, 1, 1, -3, 1, -3, -3, -3, -3, 1, 1, 1, -3, 1, 1
first 18 PAF values for B: 1, -3, 1, 1, -3, -3, 1, -3, 1, 1, 1, 1, -3, -3, -3, 1, -3, -3

1, 25.83447494, 50.16552507, 76
2, 50.16552503, 25.83447496, 76
3, 25.83447496, 50.16552506, 76
4, 25.83447493, 50.16552505, 76
5, 50.16552507, 25.83447493, 76
6, 50.16552504, 25.83447494, 76
7, 25.83447496, 50.16552505, 76
8, 50.16552503, 25.83447494, 76
9, 25.83447496, 50.16552504, 76
10, 25.83447495, 50.6165525, 76
11, 25.83447494, 50.1655250, 76
12, 25.83447491, 50.1655250, 76
13, 50.16552505, 25.8344749, 76
14, 50.16552507, 25.8344749, 76
15, 50.16552504, 25.8344749, 76
16, 25.83447493, 50.1655250, 76
17, 50.16552507, 25.8344749, 76
18, 50.16552505, 25.8344749, 76

Example (4)

$n = 55, \quad LP(55), \quad PSD = 2 \cdot 55 + 2 = 112$

1. $A := [1, -1, -1, 1, 1, 1, -1, -1, -1, -1, -1, -1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1,$
   $-1, 1, 1, 1, -1, 1, 1, -1, 1, 1, -1, -1, -1, 1, 1, 1, 1, -1, 1, -1, 1, -1, -1, -1, 1, -1, 1, -1];$
2. $B := [1, -1, -1, -1, 1, 1, 1, 1, -1, 1, 1, 1, 1, -1, 1, 1, -1, -1, 1, -1, -1, -1, 1, 1, 1, -1, 1, -1,$
   $-1, -1, 1, 1, 1, 1, -1, 1, 1, -1, -1, 1, -1, -1, -1, 1, 1, 1, 1, -1, -1, -1, 1, -1, -1, -1, -1];$
3. first 27 PAF values for A:

   $-9, 3, 3, -5, -1, 3, 3, 3, -5, 3, -1, -1, 3, -1, -1, 3, -1, 3, -1, -9, -9, -1, -1, -5, 3, -5, -1$
4. first 27 PAF values for B:

   $7, -5, -5, 3, -1, -5, -5, -5, 3, -5, -1, -1, -5, -1, -1, -5, -1, -5, -1, 7, 7, -1, -1, 3, -5, 3, -1$

Exhaustive searches for Legendre Pairs

ℓ	order of $H_{2\ell+2}$	total number of $LP(\ell)$
3	8	$9 = 1 \times 3^2$
5	12	$50 = 2 \times 5^2$
7	16	$196 = 4 \times 7^2$
9	20	$972 = 12 \times 9^2$
11	24	$2,904 = 24 \times 11^2$
13	28	$7,098 = 42 \times 13^2$
15	32	$38,700 = 172 \times 15^2$
17	36	$93,058 = 322 \times 17^2$
19	40	$161,728 = 448 \times 19^2$
21	44	$433,944 = 984 \times 21^2$
23	48	$1,235,744 = 2,336 \times 23^2$
25	52	$2,075,000 = 3,320 \times 25^2$
27	56	$5,353,776 = 7,344 \times 27^2$
29	60	$12,401,386 = 14,746 \times 29^2$
31	64	$22,472,024 = 23,384 \times 31^2$

} exhaustive searches for $LP(\ell)$

LPs of prime lengths: Legendre symbol construction

For every odd prime $p, \exists\ LP(p)$, via the `Legendre symbol`.

Maple code:

```
with(NumberTheory):
L:=[seq(LegendreSymbol(i,p),i=1..p-1)];
A:=[1,op(L)];
B:=[1,-op(L)];
```

(A, B) is a Legendre pair of length p, for $p = 3, 5, 7, \ldots$

An interesting behavior occurs, according to the parity of $p \mod 4$:

the mod 4 dichotomy

- $p \equiv 3 \,(mod\ 4)$
  all the PAF values of (a,b) are equal to -1
  all the PSD values are equal to $p+1$
  (so we get the PAF const -2 and the PSD const $2p+2$)

- $p \equiv 1 \,(mod\ 4)$
  all the PAF values of (a,b) belong to $\{1, -3\}$
  there are only two different PSD values
  Gauss sum interpretation, Arne Winterhof
  (so we get the PAF const -2 and the PSD const $2p+2$)

LPs twin primes construction

For twin primes $p, p+2$, $\exists\ LP(p \cdot (p+2))$

TWO CAVEATS:

1. Twin prime conjecture $\rightsquigarrow$ infinite classes of LPs & HMs
2. the twin primes must have a common primitive root
   turns out this is an open problem in Number Theory
   (for which there is no known counter-example)

CONSTRUCTION DETAILS:

1. $g =$ common primitive root of p and $p+2$,
2. $n = p \cdot (p+2), ub = (p^2 - 3)/2$
3. Positions of the $-1's$ are encoded by:
   $[g^i \mod n, i = 0 \ldots ub, i(p+2) \mod n, i = 0 \ldots p-1)]$

LPs $\rightsquigarrow$ HMs, Two circulant cores (2cc) construction

From an $LP(n), (A, B)$, form the two circulants $C(A), C(B)$.
Then a 2cc Hadamard matrix $HM(2n+2)$ is given by:

$$H_{2n+2} = \left[\begin{array}{cc|ccc ccc} - & - & + & \cdots & + & + & \cdots & + \\ - & + & + & \cdots & + & - & \cdots & - \\ \hline + & + & & & & & & \\ \vdots & \vdots & & C(A) & & & C(B) & \\ + & + & & & & & & \\ \hline + & - & & & & & & \\ \vdots & \vdots & & C(B)^t & & & -C(A)^t & \\ + & - & & & & & & \end{array}\right]$$

$LP(p) \rightsquigarrow HM(2p+2)$

$LP(p(p+2)) \rightsquigarrow$
$HM(2 \cdot p \cdot (p+2) + 2)$

Legendre pairs $\rightsquigarrow$ "structured" version of the Hadamard conjecture

Djokovic-Kotsireas Compression of Legendre pairs (1)

Definition (Djokovic-Kotsireas, DCC 2015)

Let $A = [a_0, a_1, \ldots, a_{v-1}]$ be a sequence of length $v = d \cdot m$.
Set $a_j^{(d)} = a_j + a_{j+d} + \ldots + a_{j+(m-1)d}$, for $j = 0, \ldots, d-1$.
The sequence $A^{(d)} = [a_0^{(d)}, a_1^{(d)}, \ldots, a_{d-1}^{(d)}]$ of length d is the m-compression of A.

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- m-compression of $LP(v), (A, B)$, by the same m, yields two sequences $(A^{(d)}, B^{(d)})$ of length d each, $\{-m, \ldots + m\}$.
- $PAF(A^{(d)}, s) + PAF(B^{(d)}, s) = (-2) \cdot m, \quad \forall s = 1, \ldots, \frac{d-1}{2}$
- $PSD(A^{(d)}, s) + PSD(B^{(d)}, s) = 2 \cdot v + 2, \quad \forall s = 1, \ldots, \frac{d-1}{2}$
- m-compression $\rightsquigarrow$ PAF scales linearly, PSD remains invariant

Djokovic-Kotsireas Compression of Legendre pairs (2)

Example

$LP(15), \quad n = 15 = 3 \cdot 5 = 5 \cdot 3$

3-compression $\rightsquigarrow$

$$A^{(5)} = [a_0 + a_5 + a_{10}, a_1 + a_6 + a_{11}, a_2 + a_7 + a_{12}, a_3 + a_8 + a_{13}, a_4 + a_9 + a_{14}]$$

$$B^{(5)} = [b_0 + b_5 + b_{10}, b_1 + b_6 + b_{11}, b_2 + b_7 + b_{12}, b_3 + b_8 + b_{13}, b_4 + b_9 + b_{14}]$$

5-compression $\rightsquigarrow$

$$A^{(3)} = [a_0 + a_3 + a_6 + a_9 + a_{12}, a_1 + a_4 + a_7 + a_{10} + a_{13}, a_2 + a_5 + a_8 + a_{11} + a_{14}]$$

$$B^{(3)} = [b_0 + b_3 + b_6 + b_9 + b_{12}, b_1 + b_4 + b_7 + b_{10} + b_{13}, b_2 + b_5 + b_8 + b_{11} + b_{14}]$$

Djokovic-Kotsireas Compression of Legendre pairs (3)

Example

$LP(133) = 133 = 7 \cdot 19$ compute its 19-compression:

1. $A^{(7)} = [1, 1, -3, 1, -3, -3, 7], \quad B^{(7)} = [-5, -5, 5, -5, 5, 5, 1]$
2. $PAF(A^{(7)}, s) + PAF(B^{(7)}, s) = (-2) \cdot 19 = -38, \quad s = 1, 2, 3$
3. $PSD(A^{(7)}, s) + PSD(B^{(7)}, s) = 2 \cdot 133 + 2 = 268, \quad s = 1, 2, 3$

1. Conversely, given $A^{(7)}, B^{(7)}$, recover $LP(133)$ decompression
2. Compression is not an one-to-one mapping, regrettably:

$\tilde{A^{(7)}}, \tilde{B^{(7)}}$ with PAF $= -38$ and PSD $= 268$, it is not guaranteed that decompression will yield $LP(133)$

Legendre pair of length 77, DCC, 2021 Turner/Kotsireas/Bulutoglu/Geyer

- Exploit the idea of simultaneous decompressions for LPs of composite length based on generating binary matrices with fixed row and column sums.
- PhD thesis of Jonathan Turner, AFIT, Ohio
- First construction of $LP(77)$, $77 = 7 \times 11$ & the only known example of $LP(77)$, open problem since 2001, i.e. 20+ years

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- First construction of $LP(77)$, $77 = 7 \times 11$ & the only known example of $LP(77)$, open problem since 2001, i.e. 20+ years
- 11-compression of $LP(77)$ reveals PAF constancy property:

$$A^{(7)} = [-3, 5, -3, -3, -5, 5, 5] \qquad B^{(7)} = [1, -1, 1, -1, -1, 1, 1]$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$-21, -21, -21 \qquad\qquad -1, -1, -1$$

Legendre pairs of lengths $\ell \equiv 0 \pmod 3$, JCD, 2021, Kotsireas/Koutschan

- elaboration of the PAF constancy property for an arbitrary divisor m of ℓ

Theorem (Kotsireas/Koutschan, 2021)

If the m-compression of (A, B), $(\mathcal{A}, \mathcal{B})$ is made up from two constant-PAF sequences of length n:
$\mathrm{PAF}(\mathcal{A}, 1) = \cdots = \mathrm{PAF}(\mathcal{A}, \frac{n-1}{2}), \mathrm{PAF}(\mathcal{B}, 1) = \cdots = \mathrm{PAF}(\mathcal{B}, \frac{n-1}{2})$
then the PSD values at integer multiples of m of A and B are integers, with the explicit evaluations

$$\mathrm{PSD}(A, m \cdot s) = p_2(\mathcal{A}) - \mathrm{PAF}(\mathcal{A}, 1), \quad s = 1, 2, \ldots, \frac{n-1}{2}$$

$$\mathrm{PSD}(B, m \cdot s) = p_2(\mathcal{B}) - \mathrm{PAF}(\mathcal{B}, 1), \quad s = 1, 2, \ldots, \frac{n-1}{2}$$

- Determination of the complete spectrum of the $\ell/3$-rd value of the DFT/PSD for $LP(\ell)$ s.t. $\ell \equiv 0 \pmod 3$.

  sample result: for LPs of length $\ell = 117 = 3 \cdot 39$:

$$\big[PSD(A, 39), PSD(B, 39)\big] \in \big\{[28, 208], [64, 172], [112, 124]\big\},$$

- state-of-the-art list of twelve integers in the range < 200 for which the question of existence of Legendre pairs remains unresolved.

$$85, 87, 115, 145, 159, 161, 169, 175, 177, 185, 187, 195.$$

Legendre pairs of lengths $\ell \equiv 0 \pmod 5$, SPMA 2023, Kotsireas/Koutschan/Bulutoglu/Arquette/Turner/Ryan

- Exploit a conjecture regarding the value of $PSD(\cdot, \frac{\ell}{5})$
  For every $\ell \equiv 0 \pmod 5$, there exist Legendre pairs (A, B) of length ℓ s.t. for some $x \geq 0$ we have:

$$PSD(A, \frac{\ell}{5}) = \ell + 1 + \frac{\sqrt{5}}{2} \cdot x$$

$$PSD(B, \frac{\ell}{5}) = \ell + 1 - \frac{\sqrt{5}}{2} \cdot x$$

- state-of-the-art list of ten integers (< 200) for which the question of existence of Legendre pairs remains unresolved.

$$115, 145, 159, 161, 169, 175, 177, 185, 187, 195.$$

(half of them are multiples of 5)

Multiplication Theorems

Turyn's multiplication of Golay pairs

1. Hall polynomial: (discrete GF) $[a_0, \ldots, a_{n-1}] \leftrightarrow a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}$
2. (A, B) Golay pair of length g, (C, D) Golay pair of length v
3. The product pair (E, F) is a Golay pair of length gv
4. $E(z) = \frac{1}{2}[A(z) + B(z)]C(z^g) + \frac{1}{2}[A(z) - B(z)]D(z^{-g})z^{gv-g}$
5. $F(z) = \frac{1}{2}[B(z) - A(z)]C(z^{-g})z^{gv-g} + \frac{1}{2}[A(z) + B(z)]D(z^g)$

Multiplication of Golay & periodic Golay pairs (Djokovic-Kotsireas)

Theorem

If (A, B) is a Golay pair of length g and (C, D) is a periodic Golay pair of length v, then Turyn's pair (E, F) is a periodic Golay pair of length gv.

Open Problem: find multiplication theorems for Legendre pairs

Quaternary Legendre pairs, 2023-2024, Kotsireas/Koutschan/Winterhof

1. "Quaternary Legendre pairs", in New Advances in Designs, Codes and Cryptography, Stinson66, Toronto, Canada, June 13-17, 2022, Eds: Charles J. Colbourn, Jeffrey H. Dinitz, Fields Institute Communications, volume 86
2. "Quaternary Legendre pairs II", submitted

Definition

Two sequences $A = [a_0, \ldots, a_{\ell-1}]$ and $B = [b_0, \ldots, b_{\ell-1}]$, of the same length ℓ, with $\{-1, -i, +1, +i\}$ elements, form a **quaternary Legendre Pair** if:

1. $PAF(A, s) + PAF(B, s) = -2$, for $s = 1, \ldots, \frac{\ell-1}{2}$

- Pay attention to use complex conjugate in the definiton of PAF.
- Note that the parity restriction on the length has been removed
- Algebraic Number Theory provides new restrictions/constraints

Quaternary Legendre pairs are balanced

Lemma

Let $A = [a_0, a_1, \ldots, a_{\ell-1}]$, $B = [b_0, b_1, \ldots, b_{\ell-1}]$ *be a quaternary Legendre pair of length* ℓ. *Put*

$$\alpha = \sum_{j=0}^{\ell-1} a_j \quad \text{and} \quad \beta = \sum_{j=0}^{\ell-1} b_j.$$

Then we have $|\alpha|^2 + |\beta|^2 = 2$,

$$\alpha, \beta \in \{-1, 1, -i, i\} \quad \text{if } \ell \text{ is odd}$$

and

$$\{\alpha, \beta\} \in \{\{0, 1+i\}, \{0, 1-i\}, \{0, -1+i\}, \{0, -1-i\}\} \quad \text{if } \ell \text{ is even.}$$

Lemma

Let (A,B) be a quaternary Legendre pair of odd length ℓ with $\alpha = \beta = 1$. Then

$$\left(\begin{array}{cc|cccccc}
-1 & -1 & 1 & \dots & 1 & 1 & \dots & 1 \\
-1 & 1 & 1 & \dots & 1 & -1 & \dots & -1 \\
\hline
1 & 1 & & & & & & \\
\vdots & \vdots & & C(A) & & & C(B) & \\
1 & 1 & & & & & & \\
\hline
1 & -1 & & & & & & \\
\vdots & \vdots & & C(\overline{B})^T & & & -C(\overline{A})^T & \\
1 & -1 & & & & & &
\end{array}\right)$$

is a quaternary complex Hadamard matrix of order $2(\ell+1)$

Lemma

Let (A, B) *be a quaternary Legendre pair of even length* ℓ *with* $\alpha = 0$ *and* $\beta = 1 + i$. *Then*

$$\left(\begin{array}{cc|cccccc}
-1 & i & 1 & \dots & 1 & 1 & \dots & 1 \\
-i & 1 & 1 & \dots & 1 & -1 & \dots & -1 \\
\hline
1 & 1 & & & & & & \\
\vdots & \vdots & & C(A) & & & C(B) & \\
1 & 1 & & & & & & \\
\hline
1 & -1 & & & & & & \\
\vdots & \vdots & & C(\overline{B})^T & & & -C(\overline{A})^T & \\
1 & -1 & & & & & &
\end{array}\right)$$

is a quaternary complex Hadamard matrix of order $2(\ell + 1)$

qLPs toy examples

1. even $n = 2$, $A = (1, -1)$, $B = (1, i)$

$$PAF(A, 1) = -2, \quad PAF(B, 1) = 0$$

2. odd $n = 15$,

$$A = (1, 1, 1, -1, 1, 1, i, -i, -1, 1, -i, -1, -1, i, -1)$$

$$B = (1, 1, i, 1, i, -i, i, -1, -i, i, 1, -i, -1, -1, -i)$$

7 PAF values for A:	$-1+2I$	-1	$-1+4I$	1	-1	-3	$-1+2I$
7 PAF values for B:	$-1-2I$	-1	$-1-4I$	-3	-1	1	$-1-2I$
	-2	-2	-2	-2	-2	-2	-2

Seed Sequences

Definition

For a prime $p > 2$ let two "seed" sequences $A_p = (a_j^{(p)})$ and $B_p = (b_j^{(p)})$ be defined by:

$$a_j^{(p)} = \begin{cases} 0, & j \equiv 0 \bmod p, \\ 2\left(\frac{j}{p}\right), & j \not\equiv 0 \bmod p, \end{cases} \qquad j = 0, 1, \ldots$$

and

$$b_j^{(p)} = \begin{cases} 1 + i, & j \equiv 0 \bmod p, \\ 0, & j \not\equiv 0 \bmod p, \end{cases} \qquad j = 0, 1, \ldots$$

Theorem

The construction of the previous Definition satisfies:

$$PAF(A_p,0) = 4(p-1), \qquad PAF(B_p,0) = 2,$$

$$PAF(A_p,s) = -4, \quad PAF(B_p,s) = 0, \quad s = 1,2,\dots,p-1,$$

$$DFT(A_p,s) = \begin{cases} 2\left(\frac{s}{p}\right)p^{1/2}, & p \equiv 1 \bmod 4, \\ 2i\left(\frac{s}{p}\right)p^{1/2}, & p \equiv 3 \bmod 4, \end{cases}$$

$$DFT(B_p,s) = 1+i, \quad s = 1,2,\dots,p-1,$$

$$PSD(A_p,s) = 4p, \quad PSD(B_p,s) = 2, \quad s = 1,2,\dots,p-1,$$

$$PSD(A_p,s) + PSD(B_p,s) = 2(2p+1), \quad s = 1,2,\dots,p-1.$$

seed sequences A_p, B_p provide promising 2-compressions of qLPs