Additive Graph Decompositions

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Definition

A 2-(v, k, λ) design is a pair $(V, \mathcal{B})$ such that

- V is a set of v points;
- $\mathcal{B}$ is a collection of k-subsets of V (called blocks), $|\mathcal{B}| = b$;
- each 2-subset of V is contained exactly in λ blocks.

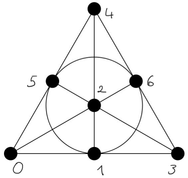


Figure: the Fano plane is a 2-$(7, 3, 1)$ design

Graph decompositions - (K_v, Γ)-designs

- a (K_v, Γ)-design is a decomposition of (the edges of) the graph K_v into copies of the graph Γ
- the graphs $\mathcal{B} = \{\Gamma_1, \ldots, \Gamma_b\}$ are "blocks"
- so a (K_v, K_k)-design is a 2-$(v, k, 1)$ design
- and a $(\lambda K_v, K_k)$-design is a 2-(v, k, λ) design

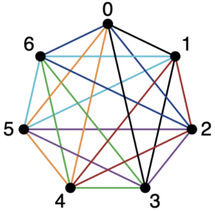
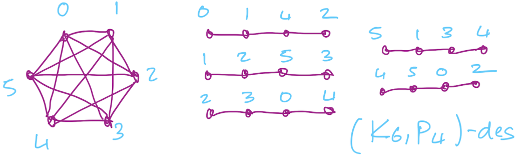


Figure: a (K_7, K_3) design

Graph decompositions - (K_v, Γ)-designs

- the most studied cases are $\Gamma = C_k$ a k-cycle (k-cycle system) and $\Gamma = P_k$ the path on k vertices (k-path system)
- the existence problem is often easier wrt to "ordinary" designs
- e.g. is completely solved for cycle and path systems
- these exist if the obvious necessary conditions are satisfied



- A $2-(v,k,\lambda)_q$ design is a $(\frac{q^v-1}{q-1},\frac{q^k-1}{q-1},\lambda)$ design $(V,\mathcal{B})$ where $V=$ points of $PG(v-1,q)$ and each block B is a subspace
- when $\lambda=1$ only non trivial known is for parameters $2-(13,3,1)_2$ (Braun, Etzion, Ostergard, Vardy, Wassermann, 2016)
- Buratti, Nakic and Wassermann (2021) introduced $(K_v,\Gamma)_q$, the q-analogs of graph decompositions
- obtaining many examples for Γ a cycle, path, generalized Petersen and other graphs
- no infinite family, but many results applicable also to the 'classical' case ($\Gamma=K_v$)

- Let $\mathcal{D} = (V, \mathcal{B})$ be a 2-(v, k, λ) design
- $\mathcal{D}$ is called additive if $V \subseteq G$, G an abelian group and $\sum_{b \in B} b = 0$ for all $B \in \mathcal{B}$ (Caggegi, Falcone, Pavone 2017)
- $\mathcal{D}$ is strictly additive if $V = G$
- $\mathcal{D}$ is almost strictly additive if $V = G \setminus \{0\}$

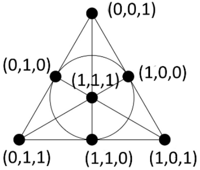
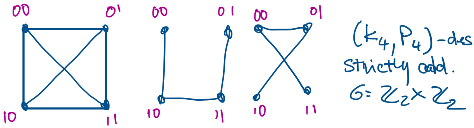


Figure: the Fano plane is almost strictly additive, $G = \mathbb{Z}_2^3$

Examples of additive designs

- first examples come from geometry; many examples with "big" λ, less for $\lambda = 1$
- see Caggegi, Falcone, Pavone 2017 & 2021, C 23, FP 21, Buratti and Nakić 2023, 2024+
- when $\lambda = 1$ we have
  - the point-line design of $AG(n, q)$ is a strictly additive $(q^n, q, 1)$-design, $G = (\mathbb{F}_q^n, +)$
  - the point-line 2-$(\frac{q^{n+1}-1}{q-1}, q+1, 1)$ design of $PG(n, q)$
  - the 2-(8191,7,1) ie 2-$(13, 3, 1)_2$ design of BEOVW
  - A sporadic 2-(124,4,1) design
  - the super regular designs (more later, very large v)
- no additive $(v, k, 1)$ design is known with v "reasonable" or k not a prime power or a prime power+1

- A (K_v, Γ)-design $\{\Gamma_1, \Gamma_2, \dots, \Gamma_b\}$ is G-additive if vertex set $V(K_v)$ is a subset of an abelian group G and the sets $V(\Gamma_1), V(\Gamma_2), \dots V(\Gamma_b)$ are zero-sum in G
- it is strictly additive if $V(K_v) = G$, almost strictly additive if $V(K_v) = G \setminus \{0\}$



- an additive $(v, k, 1)$ design + (K_k, Γ) design, $|V(\Gamma)| = k$ = additive (K_v, Γ) design

G-super regular designs

- a a (K_v, Γ)-design is regular if there is an automorphism group G of K_v
  - acting sharply transitively on the vertices of K_v (so we identify $V(K_v)$ with G)
  - permuting the blocks of $\mathcal{B}$
  - to describe a G-regular design it is enough to provide a set of base blocks, (representatives of the orbits of G on blocks)
- if a design is both G-regular and strictly G-additive
- we call it a G-super regular design (Buratti and Nakić, 2023)
- note that it is not enough to assume only that the base blocks are zero-sum

- using constructions from Buratti and Nakić
- and finding suitable base cycles, we can prove

Theorem

let $k > 3$ be odd, and let m s.t. each prime divisor of m divides k, there exists a G-additive (K_{km}, C_k)-design, where $G = \mathbb{Z}_k \times \mathbb{F}_m$

notation: $m = p_1^{e_1} \cdot \ldots p_s^{e_s}$, with $\mathbb{F}_m$ we denote $\mathbb{F}_{p_1^{e_1}} \times \cdots \times \mathbb{F}_{p_s^{e_s}}$

- note that in the 'classical' design case, the smallest nr of points for an additive $(v, 15, 1)$-design is $v{=}3 \cdot 5^{31} \simeq 10^{20}$!
- we can build an additive (K_v, C_{15}) design with $v = 45$

an additive (K_{45}, C_{15}) design

- we build a G-additive (K_{45}, C_{15}) design with $G = \mathbb{Z}_{15} \times \mathbb{Z}_3$
- 45 cycles are the G-translates of the base cycle

$$B = ((8,0),(1,1),(-1,2),(2,1),(-2,2),(3,1),(-3,2),\\ (-4,1),(-4,2),(-3,1),(3,2),(-2,1),(2,2),(-1,1),(1,2))$$

- this decomposes $K_{3\times 15}$ (equipartite graph, 3 parts of size 15)
- if $\{C_1, \ldots, C_7\}$ is a hamiltonian cycle decomposition for K_{15}, vertices labelled by $\mathbb{Z}_{15}$, with $C_i = (c_{0,i}, c_{1,i}, \ldots, c_{6,i})$
- take the 21 zero-sum cycles $C_{i,j} = ((c_{0,i}, j), (c_{1,i}, j) \ldots, (c_{6,i}, j))$ for $i = 1, \ldots, 7$, $j \in \mathbb{Z}_3$

- Some graph decompositions already in the literature are additive (and super regular)
- Benini and Pasotti (2009) study G-regular cycle systems, G elementary abelian
- amongst their results, we have
- for p an odd prime, $p^n \equiv 1 \pmod 4$ there exist a $\mathbb{Z}_p^n$-regular C_{2p}-cycle system (a $\mathbb{Z}_p^n$-regular (p^n, C_{2p})-design)
- these CS are also strictly $\mathbb{Z}_p^n$-additive, so $\mathbb{Z}_p^n$-super regular
- also, there exist $\mathbb{Z}_p^n$-super regular (p^n, C_{3p})-designs for $p^n \equiv 1 \pmod 6$ and (p^n, C_{4p})-designs for $p^n \equiv 1 \pmod 8$

- Some graph decompositions already in the literature are additive (and super regular)
- Bonisoli, Buratti, Rinaldi (2007) studied decompositions of K_v into generalized Petersen graphs
- some of their constructions give graphs strictly $\mathbb{Z}_p^n$-regular and $\mathbb{Z}_p^n$-additive designs (ie super regular)
- for instance, there exist a super regular decomposition of K_v into Petersen graphs for $v = 5^{2n}$, $G = \mathbb{Z}_5^{2n}$

Fact

A non-trivial subgroup of $\mathbb{F}_q^*$ and all its cosets are zero-sum.

- from a cyclic (K_v, Γ)-design with base blocks $\Gamma_1, \dots, \Gamma_s$ s.t. $V(\Gamma_i), 1 \leq i \leq s$ is the union of cosets of subgroups of $\mathbb{Z}_v$
- or a 1-rotational (K_{v+1}, Γ)-design, b.b. $\Gamma_1, \dots, \Gamma_s, \Gamma_\infty$
- take a prime power $q \equiv 1 \pmod{v}$, say $q = vt + 1$, and let G be the group of order v and index t in $\mathbb{F}_q^*$
- if $\mathbb{F}_q^* = \langle g \rangle$, then $G = \langle g^t = r \rangle$
- we can obtain an additive (K_v, Γ)-design where the points are the elements of G
- resp. an additive (K_{v+1}, Γ)-design where the points are the elements of $G \cup \{0\}$
- relabeling the points and base blocks with the map $\varphi : \mathbb{Z}_v \to G$ s.t. $\varphi(x) = r^x$
- resp. $\varphi : \mathbb{Z}_v \cup \{\infty\} \to G \cup \{0\}$ s.t. $\varphi(x) = r^x, \varphi(\infty) = 0$
- and developing multiplicatively

- a theorem of Buratti (2003) guarantees the existence of a 1-rotational $(v+1, C_k)$-design for all admissible $(v+1, k)$, with k odd, not a prime power, and $v+1 < 3k$
- this result gives a different proof of existence of k-cycle systems, k odd
- easy to note that in such a CS all cycles have non trivial stabilizer, so that each base cycle the set of vertices is the union of cosets of a subgroup of $\mathbb{Z}_v$

Theorem

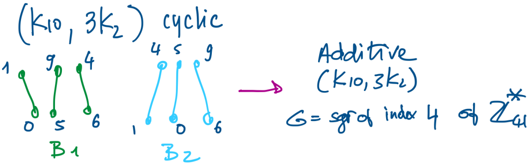
There is an additive $(v+1, C_k)$-design for all admissible $(v+1, k)$, with k odd, not a prime power, and $v+1 < 3k$

- Ex: take $k = 15$ and $v + 1 = 21$; two base cycles developed (mod 20), with ∞ a fixed point
- $B_1 = (0, 1, 19, 0 + 4, 1 + 4, 19 + 4, 0 + 8 \ldots) = (0, 1, 19, 4, 5, 3, 8, 9, 7, 12, 13, 11, 16, 17, 15)$
- $B_\infty = (\infty, 0, 3, 19, 5, 18, 6, 17, 7, 16, 8, 15, 9, 13, 10)$

- from the 1-rotational $(21, C_{15})$-design we obtain an additive $(21, C_{15})$-design
- additive under $\mathbb{Z}_{41}$
- let C^2 is the subgroup of order $20 = v$ in $\mathbb{Z}_{41}^*$, $C^2 = \langle g \rangle$ ($C^2 =$ squares of $\mathbb{Z}_{41}$, $C^2 = \langle 2 \rangle$)
- label the vertices of $K_{v=21}$ with the elements of $C^2 \cup \{0\}$
- map the vertices of the two base cycles into $C^2 \cup \{0\}$
- $\varphi : \mathbb{Z}_{20} \cup \{\infty\} \to C^2 \cup \{0\}$
  $\varphi(\infty) = 0, \quad \varphi(x) = g^x (= 2^x \pmod{41})$
- the two base cycles of the 1-rotational $(21, C_{15})$-design map to $(2^0, 2^1, 2^{19}, 2^4, \ldots, 2^{15})$ and $(0, 2^0, 2^3, \ldots, 2^{10})$
- and the full set of cycles is obtained as
  $2^i \cdot (1, 2, 21, 16, 32, 8, 10, 20, 5, 37, 33, 39, 18, 36, 9), \quad i = 0, 1, 2, 3$
  $2^i \cdot (0, 1, 8, 21, 32, 31, 23, 36, 5, 18, 10, 9, 20, 33, 40), \quad i = 0, 1, \ldots, 9.$

other examples

- the same, or similar ideas can be applied to obtain additive path decompositions
- many sporadic examples
- a (somewhat special) infinite class obtained recursively: an additive (K_v, P_4) design with $v = 7^n$
- infinite classes for other graph decompositions, eg
- a general construction using Skolem sequences for all admissible even v and $\Gamma = 3K_2$
- by exhibiting a cyclic system with vertices of base blocks are union of cosets of subgroups of $\mathbb{Z}_v$



an example with $\Gamma = P_5$

- an example of a $\mathbb{Z}_2^4$-additive (K_{16}, P_5)-decomposition
- identify $\mathbb{F}_{16}$ with $\mathbb{Z}_2[x]/\langle x^4 + x + 1\rangle$
- take x root of the primitive polynomial $x^4 + x + 1$
- the paths are
  $[0, x, x^5, x^8, x^{15}] \cdot x^i$ for $0 \leq i \leq 15$; $[x, x^6, x^8, x^9, x^{15}] \cdot x^i$ for $0 \leq i \leq 15$.
- in the additive group of $\mathbb{F}_{16} \simeq \mathbb{Z}_2^4$, $[0, x, x^5, x^8, x^{15}]$ is the zero sum set $[0000, 0100, 0110, 1010, 1000]$ and $[x, x^6, x^8, x^9, x^{15}]$ is $[0100, 0011, 1010, 0101, 1000]$
- interesting because strictly G-additive but not G-regular, $G = \mathbb{Z}_2^4$
- and no Steiner 2-design with this property is known (open question in Buratti and Nakić 2023)

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