

Regular digraphs and related linear codes

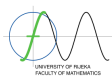
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Groups and designs

Construction of 1-designs from groups

Digraphs

Construction of k -regular digraphs from groups

Codes

Construction of codes from incidence matrix of 1-designs

Group action

A group G **acts** on a set S if there exists function $f : G \times S \mapsto S$ such that

1. $f(e, x) = x, \forall x \in S,$
2. $f(g_1, f(g_2, x)) = f(g_1 g_2, x), \forall x \in S, \forall g_1, g_2 \in G.$

Denote the described action by $xg, \forall x \in S, \forall g \in G.$

The set $G_x = \{g \in G \mid xg = x\}$ is a group called **stabilizer** of the element $x \in S.$
The set $xG = \{xg \mid g \in G\}$ is **orbit** of the element $x \in S.$ If there is only one orbit than the action is **transitive.**



Designs

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set \mathcal{P} , block set \mathcal{B} and incidence \mathcal{I} is called a t - (v, k, λ) design, if

1. \mathcal{P} contains v points,
2. each block $B \in \mathcal{B}$ is incident with k points, and
3. every t distinct points are incident with λ blocks.

The incidence matrix of a design is a $b \times v$ matrix $[m_{ij}]$ where b and v are the numbers of blocks and points respectively, such that $m_{ij} = 1$ if the point P_j and the block B_i are incident, and $m_{ij} = 0$ otherwise.

A bijective mapping of points to points and blocks to blocks which preserves incidence of a design \mathcal{D} is called an automorphism of \mathcal{D} . The set of all automorphisms of \mathcal{D} forms its full automorphism group denoted by $\text{Aut}(\mathcal{D})$.

- ▶ A design is **block design** if $t = 2$.
- ▶ A design is **symmetric** if the number of points is equal to the number of blocks.
- ▶ A design is **weakly p -self-orthogonal** (p -WSO) if all the block intersection numbers gives the same residue modulo p . A weakly p -self-orthogonal design is **p -self-orthogonal** if the block intersection numbers and the block sizes are multiples of p .
- ▶ Specially, weakly 2-self-orthogonal design is called **weakly self-orthogonal** (WSO) design, and 2-self-orthogonal design is called **self-orthogonal**.





Construction of 1-designs from groups

Theorem (D. Crnković, VMC, A. Švob)

Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n , respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \cup_{i=1}^s G_{\alpha} \cdot \delta_i$, where $\delta_1, \dots, \delta_s \in \Omega_2$ are representatives of distinct G_{α} -orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{g \cdot \Delta_2 : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, \dots, \delta_s) = (\Omega_2, \mathcal{B})$ is a $1 - (n, |\Delta_2|, \frac{|G_{\alpha}|}{|G_{\Delta_2}|} \sum_{i=1}^s |G_{\delta_i} \cdot \alpha|)$ design with $\frac{m \cdot |G_{\alpha}|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G / \cap_{x \in \Omega_2} G_x$ acts as an automorphism group on (Ω_2, \mathcal{B}) , transitively on points and blocks of the design.

-  D. Crnković, VMC, A. Švob, On some transitive combinatorial structures constructed from the unitary group $U(3, 3)$, J. Statist. Plann. Inference, 144, (2014) 19–40.
-  D. Crnković, VMC, A. Švob, New 3-designs and 2-designs having $U(3, 3)$ as an automorphism group, Discrete Math. 340 (2017), 2507-2515.
-  D. Crnković, S. Rukavina, A. Švob, New strongly regular graphs from orthogonal groups $O^+(6, 2)$ and $O^-(6, 2)$, Discrete Math. 341 (2018) 2723-2728.
-  A. E. Brouwer, D. Crnković, A. Švob, A construction of directed strongly regular graphs with parameters $(63, 11, 8, 1, 2)$, Discrete Math. 347 (2024), 114146, 3 pages.

- ▶ The incidence matrix of a symmetric 1-design is the adjacency matrix of a regular digraph.
- ▶ The incidence matrix of a symmetric 1-design with symmetric incidence matrix is the adjacency matrix of a regular graph.

Quasi-strongly regular digraphs

A **quasi-strongly regular digraph**¹ (QSRD) \mathcal{G} with parameters $(n, k, t, a; c_1, c_2, \dots, c_p)$ is a k -regular digraph on n vertices such that

- ▶ each vertex is incident with t undirected edges,
- ▶ for any two distinct vertices x, y the number of paths of length 2 from x to y is a if $x \rightarrow y$,
- ▶ for any two distinct vertices x, y the number of paths of length 2 from x to y is c_i , for $i \in \{1, \dots, p\}$, if $x \nrightarrow y$
- ▶ for each $c_i, i \in \{1, \dots, p\}$, there exist two distinct vertices $x \nrightarrow y$ such that the number of paths of length 2 from x to y is c_i .

Number p is **grade** of \mathcal{G} and $c_1 > c_2 > \dots > c_p$.

¹D. Jia, Z. Guo, G. Zhang, *Some constructions of quasi-strongly regular graphs*, *Graphs and Combinatorics*, **38** (2022)

Quasi-strongly regular digraphs

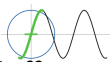
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Number p is **grade** of \mathcal{G} and $c_1 > c_2 > \dots > c_p$.

- ▶ If $p = 1$, a quasi-strongly regular digraph is **strongly regular digraph** (SRD) with parameters (n, k, a, c_1, t) .
- ▶ If $p = 1$ and $k = t$, a quasi-strongly regular digraph is **strongly regular graph**.

¹D. Jia, Z. Guo, G. Zhang, *Some constructions of quasi-strongly regular graphs*, *Graphs and Combinatorics*, **38** (2022)



Construction of k -regular digraphs from groups

Theorem (VMC, Matea Zubović Žutolija)

Let G be a finite permutation group acting transitively on the set Ω . Let $\alpha \in \Omega$ and let $\Delta = \cup_{i=1}^s \delta_i G_\alpha$ be a union of orbits of the stabilizer G_α of α , where $\delta_1, \dots, \delta_s$ are representatives of different G_α -orbits. Let $T = \{g_1, \dots, g_t\}$ be a set of representatives of left cosets in $G/G_\alpha = \{g_1 G_\alpha, \dots, g_t G_\alpha\}$. Let $\mathcal{V} = \{g_i \cdot \alpha \mid i = 1, \dots, t\}$ and let $\mathcal{E} = \{(g_i \cdot \alpha, g_j \cdot \beta) \mid i = 1, \dots, t, \beta \in \Delta\}$.

Then $\Gamma = (\mathcal{V}, \mathcal{E})$ is a directed graph with $|\Omega|$ vertices that is $|\Delta|$ -regular and such that $g_i \cdot \Delta$ is a set of out-neighbours of the vertex $g_i \cdot \alpha$, $i = 1, \dots, t$. The group G acts on the constructed graph as automorphism group, transitively on the set of vertices.

Theorem

If a group G acts transitively on a set of vertices of a directed regular graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, then there exists a set Ω such that vertices and arcs of a digraph \mathcal{G} are defined in the way described in Theorem.

Classifications

- ▶ There are, up to isomorphism, 2920 QSRD, on which a transitive automorphism group of degree n , $n \in \{1, \dots, 30\} \setminus \{22, 24, 28, 30\}$, is acting. 478 of them are SRD.
- ▶ There are, up to isomorphism, 18 QSRD on which the transitive irregular automorphism group of degree 22 acts. Two of them are SRD.
- ▶ There are, up to isomorphism, 68235 QSRD on which the transitive irregular automorphism group of degree 24 acts, of which 64 are SRD.
- ▶ There are, up to isomorphism, 469 QSRDs on which the transitive irregular automorphism group of degree 28 acts, of which 22 are SRD.
- ▶ There are, up to isomorphism, 642 QSRD on which the transitive irregular automorphism group of degree 30 acts, of which 12 are directed SRD.
- ▶ There are, up to isomorphism, 124 QSRD, on which the primitive automorphism group of degree n , $n \in \{31, \dots, 110\}$, is acting, no SRD.

More results

- ▶ There is no SRD with parameters $(22, 9, 3, 4, 6)$ such that the automorphism group G acts transitively on the set of vertices of that directed graph.
- ▶ There is no SRD with parameters $(24, 10, 3, 5, 5)$ such that the automorphism group G acts transitively on the set of vertices of that directed graph.
- ▶ There is no SRD with parameters $(28, 6, 2, 1, 3)$ such that the automorphism group G acts transitively on the set of vertices of that directed graph.
- ▶ There is no SRD with parameters $(30, 11, 2, 5, 9)$ and $(30, 12, 4, 5, 11)$ such that the automorphism group G acts transitively on the set of vertices of that directed graph.

Codes

We will talk only about **linear codes**, i.e. subspaces of the ambient vector space over a field \mathbb{F}_q of order $q = p^l$, where p is prime.

A code C of length n and dimension k over the field \mathbb{F}_q is denoted by $[n, k]_q$.

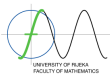
Specially, if $q = 2$, parameters of code C with minimum distance d are denoted by $[n, k, d]$.

A **generator matrix** of a $[n, k]$ code C is a $k \times n$ matrix whose rows form basis of C .

The **dual code** of a code C is code C^\perp , $C^\perp = \{v \in (\mathbb{F}_q)^n \mid (v, c) = 0, \forall c \in C\}$.

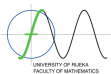
A code is **self-orthogonal** (SO) if $C \subseteq C^\perp$, **self-dual** if $C = C^\perp$, and **LCD** if $C \cap C^\perp = \{0\}$.

The **code** $C_{\mathbb{F}}(\mathcal{D})$ of a **design** \mathcal{D} over the finite field \mathbb{F} is the space spanned by the incidence vectors of the blocks over \mathbb{F} .



Two linear codes are **isomorphic** if they can be obtained from one another by permuting the coordinate positions. An **automorphism** of a code C is an isomorphism from C to C . The full automorphism group will be denoted by $\text{Aut}(C)$.

If code $C_{\mathbb{F}}(\mathcal{D})$ is a linear code of a design \mathcal{D} over a finite field \mathbb{F} , then the full automorphism group of \mathcal{D} is contained in the full automorphism group of code $C_{\mathbb{F}}(\mathcal{D})$.



SO codes from p -WSO 1-designs**Theorem (VMC, I. Traunkar)**

Let $q = p^l$ be prime power and \mathbb{F}_q a finite field of order q . Let \mathcal{D} be a weakly p -self-orthogonal design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of a design \mathcal{D} . Let M be its $b \times v$ incidence matrix.

1. If \mathcal{D} is p -self-orthogonal design, then M generates a self-orthogonal code over \mathbb{F}_q .
2. If $a = 0$ and $d \neq 0$, then the matrix $[\sqrt{d} \cdot I_b, M, \sqrt{-d} \cdot \mathbf{1}]$ generates a self-orthogonal code over \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-d$ is a square in \mathbb{F}_q , and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
3. If $a \neq 0$ and $d = 0$, then the matrix $[M, \sqrt{-a} \cdot I_b]$ generates a self-orthogonal code over \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-a$ is a square in \mathbb{F}_q , and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
4. If $a \neq 0$ and $d \neq 0$, there are two cases:
 - 4.1 if $a = d$, then the matrix $[M, \sqrt{-d} \cdot \mathbf{1}]$ generates a self-orthogonal code over \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-a$ is a square in \mathbb{F}_q , and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise, and
 - 4.2 if $a \neq d$, then the matrix $[\sqrt{d-a} \cdot I_b, M, \sqrt{-d} \cdot \mathbf{1}]$ generates a self-orthogonal code over \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-d$ is a square in \mathbb{F}_q , and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.

LCD codes from p -WSO designs**Theorem (VMC, I. Traunkar)**

Let \mathcal{D} be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design \mathcal{D} .

1. If $a = d = 0$ then

the matrix $[\mathbf{M}, x \cdot \mathbf{1}_b]$ for $x \neq 0$, and

the matrix $[\mathbf{M}, x \cdot \mathbf{1}_b, y\mathbf{1}]$ for $y \neq 0$ and $x^2 + b \cdot y^2 \neq 0$

generate an LCD code over the field \mathbb{F}_q .

2. If $a = 0$ and $d \neq 0$ then

the matrix \mathbf{M} for $(b-1) \cdot d \neq 0$ and if \mathbf{M} is of full rank,

the matrix $[\mathbf{M}, y\mathbf{1}]$ for $by^2 + (b-1) \cdot d \neq 0$ and if \mathbf{M} is of full rank,

the matrix $[\mathbf{M}, x \cdot \mathbf{1}_b]$ for $x-d \neq 0$ and $x^2 + (b-1) \cdot d \neq 0$, and

the matrix $[\mathbf{M}, x \cdot \mathbf{1}_b, y\mathbf{1}]$ for $x^2 - d \neq 0$ and $b \cdot y^2 + x^2 + (b-1) \cdot d \neq 0$

generate an LCD code over the field \mathbb{F}_q .

Theorem (VMC, I. Traunkar)

3. If $a \neq 0$ and $d = 0$ then
 the matrix \mathbf{M} if \mathbf{M} is of full rank,
 the matrix $[\mathbf{M}, \mathbf{y}\mathbf{1}]$ for $b \cdot y^2 + a \neq 0$ and if \mathbf{M} is of full rank,
 the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x^2 + a \neq 0$, and
 the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, \mathbf{y}\mathbf{1}]$ for $x^2 + a \neq 0$ and $b \cdot y^2 + x^2 + a \neq 0$
 generate an LCD code over the field \mathbb{F}_q .
4. If $a = d \neq 0$ then
 the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x \neq 0$ and $x^2 + ba \neq 0$, and
 the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, \mathbf{y}\mathbf{1}]$ for $x \neq 0$ and $b \cdot y^2 + x^2 + b \cdot d \neq 0$
 generate an LCD code over the field \mathbb{F}_q .
5. If $a \neq 0$, $d \neq 0$, $a \neq d$ then
 the matrix \mathbf{M} for $a + (b - 1) \cdot d \neq 0$ and if \mathbf{M} is of full rank,
 the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x^2 - d + a \neq 0$ and $x^2 + a + (b - 1) \cdot d \neq 0$,
 the matrix $[\mathbf{M}, \mathbf{y}\mathbf{1}]$ for $by^2 + a + (b - 1) \cdot d \neq 0$ and if \mathbf{M} is of full rank, and
 the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, \mathbf{y}\mathbf{1}]$ for $x^2 - d + a \neq 0$ and
 $b \cdot y^2 + x^2 + a + (b - 1) \cdot d \neq 0$
 generate an LCD code over \mathbb{F}_q .

Examples of SO and LCD codes over the field \mathbb{F}_q constructed by described extensions of incidence matrix and similar extensions of orbit matrices and submatrices of orbit matrices:



V. Tonchev, Self-Orthogonal Designs and Extremal Doubtly-Even Codes, *Journal of Combinatorial Theory, Series A* 52, 197-205 (1989).



D. Crnković, VMC, B. G. Rodrigues, On self-orthogonal designs and codes related to Held's simple group, *Adv. Math. Commun.* 12 (2018)



VMC, I. Traunkar, Self-orthogonal codes constructed from weakly self-orthogonal designs invariant under an action of M_{11} , *AAECC* (2023)



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Examples of SO codes from adjacency and orbit matrix of graphs and digraphs:



D. Crnković, A. Švob, Self-orthogonal codes from Deza graphs, normally regular digraphs and Deza digraphs, *Graphs Combin.* 40 (2024), article no. 35, 12 pages.

If A is adjacency matrix of a QSRD with parameters $(n, k, t, a; c_1, c_2, \dots, c_p)$ then:

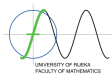
- ▶ A is square matrix and $AJ_n = J_nA = kJ$
- ▶ $A^2 = tI_n + aA + c_1C_1 + c_2C_2 + \dots + c_pC_p$ for some non-zero $(0, 1)$ -matrices C_1, C_2, \dots, C_p such that $C_1 + C_2 + \dots + C_p = J_n - I_n - A$
- ▶ $(A^T)^2 = tI_n + aA^T + c_1C_1^T + c_2C_2^T + \dots + c_pC_p^T$ for some non-zero $(0, 1)$ -matrices C_1, C_2, \dots, C_p^T such that $C_1^T + C_2^T + \dots + C_p^T = J_n - I_n - A^T$

From that one can eliminate some graphs whose adjacency matrix will generate non-interesting SO or LCD codes.

Examples: Binary SO codes constructed from SRDs and QSRDs on 12 vertices

SRD	code
$(12,4,0,2,2)$	$[12,3,4]$
$(12,5,2,2,3)$	$[12,3,6]^*$
$(12,6,5,2,2,3)$	$[12,4,4]$

QSRD or its complement	code
$(12,2,0,0;1,0)$	$[12,6,2]$
$(12,4,0,0;4,0)$	$[12,3,4]$
$(12,4,3,0;3,2,0)$	$[12,4,4]$
$(12,4,3,0;3,1)$	$[12,5,4]^*$
$(12,1,0,0;1,0)$	$[24,12,2]$
$(12,3,0,0;3,2,0)$	$[24,12,4]$
$(12,5,4,0;5,4,0)$	$[24,12,4]$
$(12,3,2,1;1,0)$	$[12,4,4]$
$(12,3,2,1;2,0)$	$[12,5,4]^*$
$(12,5,1,2;4,2)$	$[12,6,4]^*$
$(12,5,3,2;4,1)$	$[12,6,2]$
$(12,4,0,1;4,2,1)$	$[24,12,8]^*$



Thank you for your attention!