Some results on Graphic Topology defined on Tournaments

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Index

Overview

- Properties that determine if a tournament with few vertices is decomposable.
- Properties of the graphic topology defined on finite indecomposable tournaments.
- Non-isomorphic indecomposable tournaments with homeomorphic graphic topologies.

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Finite topological spaces

Definition

A topological space $(X, \mathcal{T})$ is an **Alexandroff** space if every arbitrary intersection of open sets is an open set.
Finite topological space $\Longrightarrow$ Alexandroff space.

Definition

A topological space $(X, \mathcal{T})$ is $\mathbf{T_0}$ if for any two points of X, there exists an open set that includes one of them but not the other one.

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For any $x \in X$, the **minimal open set of x**, U_x, is the intersection of all open sets that contain x.

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We define a **relation** $\leq$ over the set X as

$$x \leq y \text{ iff } x \in U_y \equiv U_x \subseteq U_y.$$

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The family of open sets $\{U_x\}_{x \in X}$ is the unique minimal basis of the finite space $(X, \mathcal{T})$.

Proposition

There exists a biyective correspondence between T_0 finite spaces and finite partially ordered sets (finite posets).

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A **tournament** is an oriented complete graph, i. e., a directed graph or digraph in which any pair of vertices is connected by only one directed edge.

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Let $T = (V, A)$ be a tournament.
x dominates y, $x \to y$, if there is a directed edge from x to y, i. e. $(x, y) \in A$.

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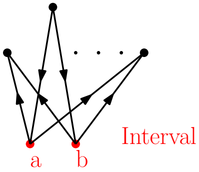
Let $T = (V, A)$ be a tournament. A subset $I \subseteq V$ is an **interval** if for any $a, b \in I$ it is true that for all $x \in V \setminus I$, $(a, x) \in A$ if and only if $(b, x) \in A$.

Definition

The **trivial intervals** of a tournament are the subsets $\emptyset$, V and $\{x\}, \forall x \in V$.



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A tournament is **indecomposable** if all its intervals are trivial ones. Otherwise, the tournament is called **decomposable**.

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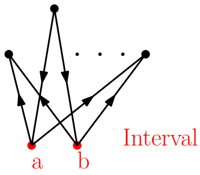

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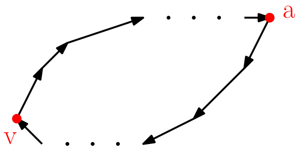
A tournament is **indecomposable** if all its intervals are trivial ones. Otherwise, the tournament is called **decomposable**.

Tournaments

Definition

A finite tournament $T = (V, A)$ of three or more vertices is **strongly connected** if $\exists\, a \in V$ so that $\forall\, v \in V$, $v \neq a$, there exist paths in T from a to v and from v to a.



Definition

A **path in T** is a sequence $x = z_0, z_1, \ldots, z_n = y$ so that $\forall i \in \{0, \ldots, n-1\}$, $z_i \to z_{i+1}$.

Tournaments

Proposition

A tournament of three or more vertices is strongly connected if and only if it is indecomposable.

Definition

Let $T = (V, A)$ be a tournament.

- **Outset:** $N_x^+ = \{y \in V : x \to y\}$
- **Inset:** $N_x^- = \{y \in V : y \to x\}$
- **Out-degree:** $d^+(x) = |N_x^+|$
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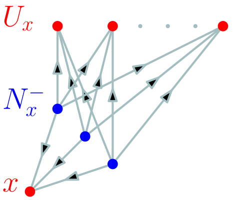
Lemma

$\mathbf{S_T} = \{N_x^+ : x \in V\}$ *is a subbasis for a certain topology* $\mathcal{T}_T$ *on* V*.*

Lemma

For all $x \in V$*,*
$U_x = \bigcap_{y \in N_x^-} N_y^+$.



Definition

The topology $\mathcal{T}_T$ on V generated by the subbasis S_T is named the **graphic topology** of T.

Proposition

Let $T = (V, A)$ *be a finite indecomposable tournament,* $|V| \geq 3$*. Then* $(V, \mathcal{T}_T)$ *is a* T_0 *finite space.*

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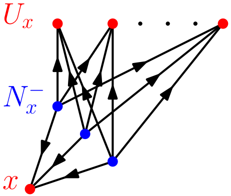

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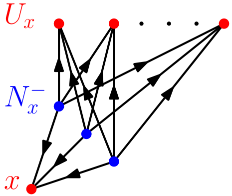

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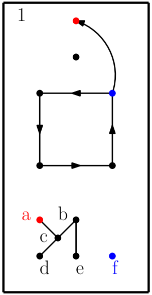
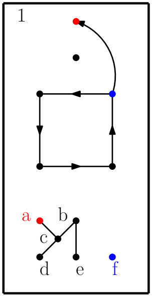
Notation (tournaments via diagrams)



- No edge ≡ ↓
- Alphabetic order: ↓ ⟳

Notation (graphic topology)



- First level: vertices with unitary minimal open sets.
- Nth level: vertices with minimal open sets that contain at least one vertex from level $n-1$.
- Vertices in the same level are placed in a horizontal line.
- "$x \leq y$ and $\nexists\, z \neq x, y$ with $x \leq z \leq y$" iff the vertex y is in a level immediately superior to the level of x and there is a line binding them.

Notation (graphic topology)

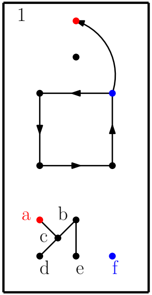


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Table 1

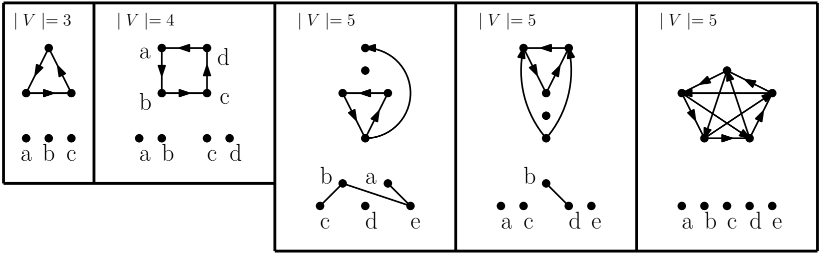


Table 2

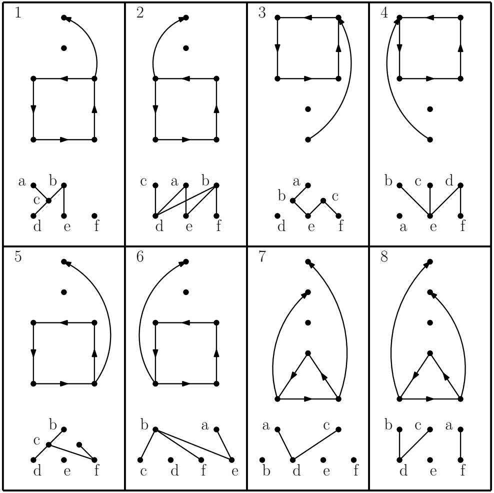

1
a b c d e f
2
c a b d e f
3
a b c d e f
4
b c d a e f
5
b c d e f
6
b a c d f e
7
a c b d e f
8
b c a d e f

Table 3

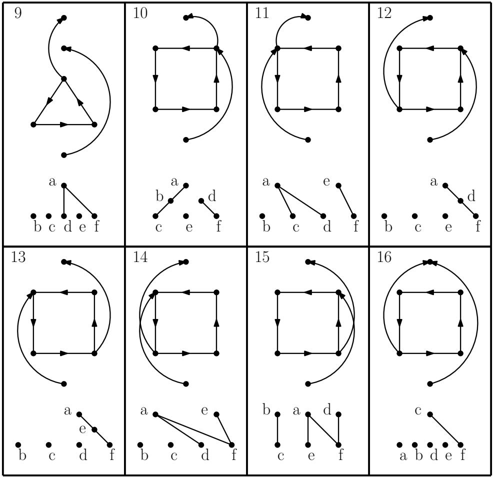

9
a
b c d e f
10
a
b d
c e f
11
a e
b c d f
12
a
d
b c e f
13
a
e
b c d f
14
a e
b c d f
15
b a d
c e f
16
c
a b d e f

Table 4

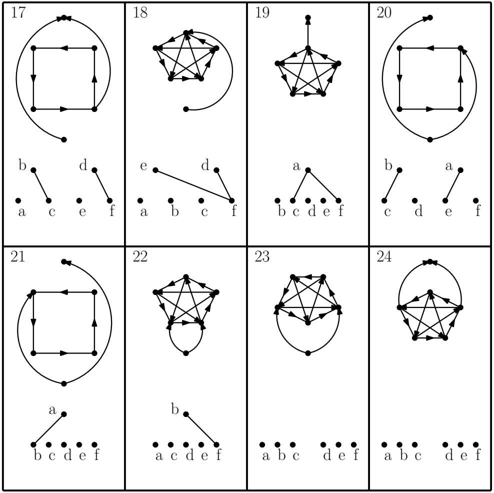
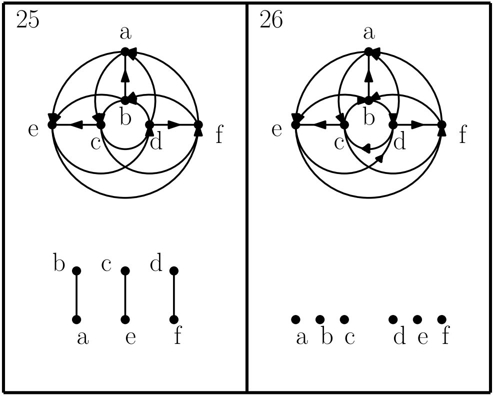
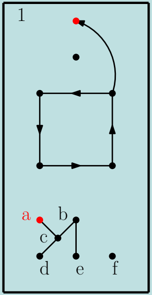

17
18
19
20
b d a c e f
e d a b c f
a b c d e f
b a c d e f
21
22
23
24
a b c d e f
b a c d e f
a b c d e f
a b c d e f

Table 5



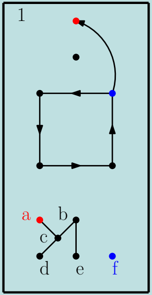
Example (Indecomposable tournament)



$\forall v \in V \ (v \neq a)$, *there exist paths from a to v and from v to a.*

$a \longrightarrow b \longrightarrow c \longrightarrow d \longrightarrow e \longrightarrow f \longrightarrow a.$

Example (Graphic topology)



$N_a^- = \{f\}$, $N_a^+ = \{b, c, d, e\}$
$N_b^- = \{a\}$, $N_b^+ = \{c, d, e, f\}$
$N_c^- = \{a, b, f\}$, $N_c^+ = \{d, e\}$
$N_d^- = \{a, b, c, f\}$, $N_d^+ = \{e\}$
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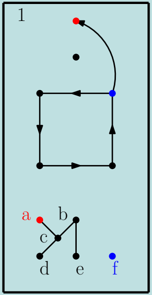

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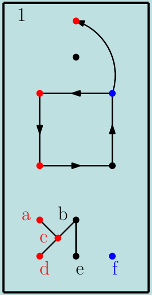

$$U_x = \cap_{y \in N_x^-} N_y^+, \; \forall x \in V.$$

$$U_a = N_f^+ = \{a, c, d\}$$
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$$U_c = N_a^+ \cap N_b^+ \cap N_f^+ = \{c, d\}$$
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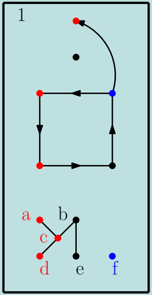

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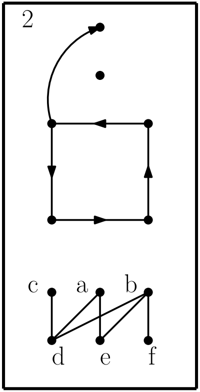
Index

Connectivity

There is only one indecomposable tournament of six vertices with a connected graphic topology.



Proposition

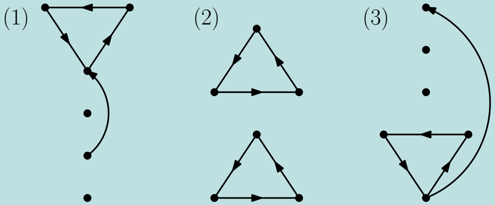
A tournament of n vertices, with $3 \leq n \leq 6$, is decomposable iff its representation (following the chosen notation) verifies one of these properties:

(1) *There is a vertex v over or bellow the rest of the representation, without incident edges.*

(2) *There is a directed cycle represented, with fewer than n vertices, that does not have any exterior incident edges.*

(3) *There are two points, one bellow the other one, without exterior incident edges.*

Example (Tournaments that verify the previous proposition)



Proposition

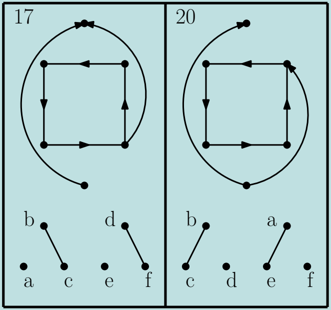
There exist T_0 finite spaces of six vertices that can't be obtained from the graphic topology of an indecomposable tournament.

Theorem

The graphic topologies of the tournaments of each one of the following families are homeomorphic:

- **Family 1** *7 and 18.*
- **Family 2** *9 and 19.*
- **Family 3** *12 and 13.*
- **Family 4** *16, 21 and 22.*
- **Family 5** *17 and 20.*
- **Family 6** *23, 24 and 26.*

Example (Homeomorphism between topologies of Family 5)



$$h := \begin{cases} a \longmapsto d \\ b \longmapsto b \\ c \longmapsto c \\ d \longmapsto a \\ e \longmapsto f \\ f \longmapsto e \end{cases}$$

Conclusions and future work

- We have **characterized indecomposable tournaments** from three to six vertices and **obtained the graphic topologies** of the non-isomorphic ones.
- We have proven that the **family of graphic topologies** of non-isomorphic indecomposable tournaments of six vertices is **strictly included** in the **family of T_0 finite spaces of six vertices**.
- We have proven that there exist **non-isomorphic indecomposable tournaments** of six vertices with **homeomorphic graphic topologies**.

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References

J. Dammak, R. Salem, Graphic topology on tournaments, *Adv. Pure Appl. Math.* **9**(4) (2018), 279-–285. https://doi.org/10.1515/apam-2018-0024

S. M. Jafarian Amiri, A. Jafarzadeh, H. Khatibzadehan, Alexandroff topology on graphs, *Bull. Iranian Math. Soc.* **39**(4) (2013), 647-–662.

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