# Hadamard Partitioned Difference Families

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- AN. *A few more Hadamard Partitioned Difference Families.*
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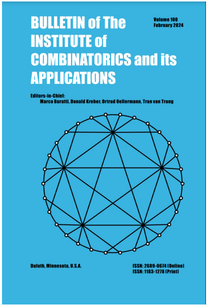


- Partitioned difference families (Ding, Yin, 2005)
- Constant-composition codes
- Application in electrical engineering: power line communication
- Hadamard partitioned difference families (Buratti, 2018)
- New sporadic examples
- $(32, [2^2, 6, 22], 16)$, $(24, [1^3, 2^2, 17], 12)$, $(36, [3, 9, 24], 18)$, $(40, [1, 3, 9, 27], 20)$

## Definition (Difference Set)

- $G$ additive group
- $k$-subset $D$ of $G$ is a $(G, k, \lambda)$ *difference set* (DS) if each non-zero element of $G$ is covered $\lambda$ times by the list of differences of $D$:

$$\Delta D = \{x - y \,:\, x \neq y, x, y \in D\} = \lambda\,(G \setminus \{0\}).$$

## Definition (Difference Family)

- $G$ additive group
- Collection of subsets $\mathcal{F} = \{D_1, \ldots, D_t\}$ of $G$ of sizes $k_1, \ldots, k_t$ is a $(G, [k_1, \ldots, k_t], \lambda)$ *difference family* (DF) if each non-zero element of $G$ is covered $\lambda$ times by the list of differences of the blocks:

$$\Delta\mathcal{F} = \uplus \Delta D_i = \lambda\,(G \setminus \{0\}).$$

## Definition (Partitioned Difference Families)

A $(G, [k_1, \ldots, k_t], \lambda)$ difference family is *partitioned difference family* (PDF) if its blocks partition $G$.

## Example

- $G \simeq \mathbb{Z}_{13}$

$$\mathbb{Z}_{13} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

- $D_1 = \{0, 3, 12\}$

$$\Delta D_1 = \{\pm 1, \pm 3, \pm 4\}$$

- $D_2 = \{5, 7, 10, 11\}$

$$\Delta D_2 = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\}$$

- $D_3 = \{1, 2, 4, 6, 8, 9\}$

$$\Delta D_3 = \{\pm 1, \pm 1, \pm 2, \pm 2, \pm 2, \pm 3, \pm 3, \pm 4, \pm 4, \pm 5, \pm 5, \pm 5 \pm 6, \pm 6, \pm 6\}$$

- $\mathcal{F} = \{D_1, D_2, D_3\}$ is a $(\mathbb{Z}_{13}, [3, 4, 6], 4)$-PDF

### Definition (Constant-Composition Code)

An $(n, M, d, [\lambda_0, \lambda_1, \dots \lambda_{q-1}])_q$ *constant-composition code* is a code $C \subset \mathbb{Z}_n^q$ with size $M$ and minimum Hamming distance $d$ such that in every codeword the element $i \in \mathbb{Z}_q$ appears exactly $\lambda_i$ times.

### Theorem (Ding, Yin, 2005)

$$(v, [\lambda_0, \lambda_1, \dots \lambda_{q-1}]; \lambda)\text{-}PDF$$
$$\Downarrow$$
$$\textit{optimal } (n, n, n-\lambda, [\lambda_0, \lambda_1, \dots, \lambda_{q-1}])_q\text{-}CCC$$

$$A_q(n, d, [w_0, w_1, \dots, w_{q-1}]) \leq \frac{nd}{nd - n^2 + (w_0^2 0 + w_1^2 + \cdots + w_{q-1}^2)}$$

### Example

- $(\mathbb{Z}_7, [3,4], 3)$-PDF $\Rightarrow$ optimal $(7, 4, [3,4])_2$-CCC of size $A_2(7, 4, [3,4]) = 7$

| 013 2456 | 124 3560 | 235 4601 | 346 5012 | 450 6123 | 561 0234 | 602 1345 |
|---|---|---|---|---|---|---|
| | ↓ | | ↓ | | ↓ | |
| 0001111 | 0110110 | 1011100 | 0111001 | 1101010 | 1100101 | 1010011 |

### Definition (Hadamard Partitioned Difference Family, Buratti, 2018)

A $(G, [k_1, \ldots, k_t], \lambda)$-PDF $\mathcal{F}$ is said to be *Hadamard* if $G$ has order $2\lambda$.

### Example (Partitioned difference families from difference sets)

- $D$ is a $(G, k, \lambda)$-DS $\quad\Rightarrow\quad \{D, \overline{D} = G \setminus D\}$ is a $(G, [k, v-k], v-2k+2\lambda)$-PDF
- The converse is also true!

### Definition (Hadamard Difference Set)

*Hadamard difference set* (HDS) is a difference set with parameters $(4u^2, 2u^2 - u, u^2 - u)$, for some $u$.

### Example

$$D \text{ is a } (4u^2, 2u^2 - u, u^2 - u)\text{-HDS in G}$$

$$\Downarrow$$

$$(D, \overline{D} = G \setminus D) \text{ is a } (4u^2, [2u^2 + u, 2u^2 - u], 2u^2)\text{-HPDF}$$

## Proposition

*A PDF with only two blocks necessarily consists of a difference set and its complement. More specifically, a HPDF with only two blocks necessarily consists of a Hadamard difference set and its complement.*

| $\lvert G\rvert$ | $[k_1, k_2]$ | $\lambda$ |
|---|---|---|
| 16 | $[10, 6]$ | 8 |
| 36 | $[20, 16]$ | 18 |
| 64 | $[34, 30]$ | 32 |
| 100 | $[52, 48]$ | 50 |
| 144 | $[74, 70]$ | 72 |
| 196 | $[100, 96]$ | 98 |
| 256 | $[130, 126]$ | 128 |
| 324 | $[164, 160]$ | 162 |
| 484 | $[244, 240]$ | 242 |
| 576 | $[290, 286]$ | 288 |
| 676 | $[340, 336]$ | 338 |
| 784 | $[394, 390]$ | 392 |
| 900 | $[452, 448]$ | 450 |
| 1024 | $[514, 510]$ | 512 |
| 1156 | $[580, 576]$ | 578 |

**Example (Buratti, 2018)**

- $G$ is a non-abelian group whose elements are all pairs of the Cartesian product $\mathbb{Z}_4 \times \mathbb{Z}_8$ and whose operation law is

$$(x_1, y_1) \rtimes (x_2, y_2) = (x_1 + x_2, 5^{x_2} y_1 + y_2)$$

- There exists $(32, [2, 2, 6, 22], 16)$-HPDF in $G$ with blocks

$$X_1 = \{(0,0), (2,0)\}, \qquad X_2 = \{(1,0), (3,4)\},$$

$$X_3 = \{(0,1), (0,3), (1,2), (1,5), (1,6), (3,3)\}, \qquad X_4 = G \setminus (X1 \cup X2 \cup X3)$$

**Are there any other sporadic examples?**

### Proposition (Necessary conditions)

- $k_1+\cdots+k_t=2\lambda=|G|$
- $|\Delta\mathcal{F}|=k_1(k_1-1)+\cdots+k_t(k_t-1)=\lambda(2\lambda-1) \quad\Rightarrow\quad k_1^2+\cdots k_t^2=\lambda(2\lambda+1)$
- $\lambda\equiv 0 \pmod 2 \quad\Rightarrow\quad |G|\equiv 0 \pmod 4$

| $v$ | $K$ | $\lambda$ |
|---|---|---|
| 20 | $[1,2,3,14]$ | 10 |
| 24 | $[1^3,2^2,17]$ | 12 |
| 28 | $[1,9,18]$ | 14 |
| 28 | $[3,6,19]$ | 14 |
| 32 | $[2^2,6,22]$ | 16 |
| 36 | $[3,9,24]$ | 18 |
| 36 | $[3,4^2,25]$ | 18 |
| 36 | $[1^5,6,25]$ | 18 |
| 40 | $[1,3,9,27]$ | 20 |
| 40 | $[3^4,28]$ | 20 |
| 40 | $[1^2,3^2,4,28]$ | 20 |
| 40 | $[1^4,4^2,28]$ | 20 |
| 40 | $[1^3,2^2,5,28]$ | 20 |

## Proposition

*In a* $(v, [k_1, k_2, k_3], \lambda)$-*HPDF we necessarily have*

$$k_{1,2} = \frac{2\lambda - k_3 \pm \sqrt{2\lambda(2k_3 + 1) - 3k_3^2}}{2}$$

## Corollary

*The existence of a* $(v, [k_1, k_2, k_3], \lambda)$-*HPDF necessarily implies that no prime divisor of* $(2k_1 + 1)(2k_2 + 1)(2k_3 + 1)$ *is congruent to* $5 \pmod{6}$.

- As a consequence, in a $(v, [k_1, k_2, k_3], \lambda)$-HPDF we cannot have, for instance, blocks of size $2, 5, 7, 8, 11, 12, 14, 16, 17, \ldots$

## Proposition

*A* $(v, [k_1, k_2, 1], \lambda)$-*HPDF cannot exist.*

### Proposition

*Let* $\mathcal{F} = \{B_1, \dots, B_t\}$ *be a* $(G, [k_1, \dots, k_t], \lambda)$*-HPDF, assume that* $G$ *has a subgroup* $H$ *of index* $2$*, and set* $|B_i \cap H| = s_i$ *for* $i = 1, \dots, t$*. Then the following identities hold:*

$$s_1 + \ \dots \ + s_t = \lambda \qquad \text{and} \qquad 2s_1(k_1 - s_1) + \ \dots \ + 2s_t(k_t - s_t) = \lambda^2$$

### Corollary

*If there exists a* $(G, [k_1, \dots, k_t], \lambda)$*-HPDF and* $G$ *has a subgroup of index* $2$*, then the diophantine system*

$$\begin{cases} x_1 & + \dots + & x_t & = & \lambda \\ 2x_1(k_1 - x_1) & + \dots + & 2x_t(k_t - x_t) & = & \lambda^2 \end{cases}$$

*has a solution* $(s_1, \dots, s_t)$ *with* $0 \leq s_i \leq k_i$ *for each* $i$*.*

As application of the above corollary one can see that none of these $K$, though admissible, can be the multiset of block-sizes of a HPDF:

$$[1, 5, 20, 50]; \quad [1, 1, 1, 2, 23, 52, ]; \quad [2, 3, 38, 73];$$

$$[3, 8, 28, 77]; \quad [3, 7, 31, 79]; \quad [1, 1, 16, 21, 81]; \quad [3, 14, 35, 104].$$

# Searching for HPDFs

- Necessary conditions
- Subgroups of index 2
- Computer search

| $v$ | $K$ | $\lambda$ |
|---|---|---|
| 20 | ~~$[1, 2, 3, 14]$~~ | 10 |
| 24 | $[1^3, 2^2, 17]$ | 12 |
| 28 | ~~$[1, 9, 18]$~~ | 14 |
| 28 | ~~$[3, 6, 19]$~~ | 14 |
| 32 | $[2^2, 6, 22]$ | 16 |
| 36 | $[3, 9, 24]$ | 18 |
| 36 | ~~$[3, 4^2, 25]$~~ | 18 |
| 36 | ~~$[1^5, 6, 25]$~~ | 18 |
| 40 | $[1, 3, 9, 27]$ | 20 |
| 40 | $[3^4, 28]$ | 20 |
| 40 | $[1^2, 3^2, 4, 28]$ | 20 |
| 40 | ~~$[1^4, 4^2, 28]$~~ | 20 |
| 40 | $[1^3, 2^2, 5, 28]$ | 20 |

# New sporadic examples of HPDFs

- $(32, [2^2, 622], 16)$

| | Group $G$ |
|---|---|
| 1. | $C_{32}$ |
| 2. | $(C_4 \times C_2) \rtimes C_4$ |
| 3. | $C_8 \times C_4$ |
| 4. | $C_8 \rtimes C_4$ |
| 5. | $(C_8 \times C_2) \rtimes C_2$ |
| 6. | $((C_4 \times C_2) \rtimes C_2) \rtimes C_2$ |
| 7. | $(C_8 \rtimes C_2) \rtimes C_2$ |
| 8. | $C_2.((C_4 \times C_2) \rtimes C_2)$ |
| 9. | $(C_8 \times C_2) \rtimes C_2$ |
| 10. | $Q_8 \rtimes C_4$ |
| 11. | $(C_4 \times C_4) \rtimes C_2$ |
| 12. | $C_4 \rtimes C_8$ |
| 13. | $C_8 \rtimes C_4$ |
| 14. | $C_8 \rtimes C_4$ |
| 15. | $C_4.D_4$ |
| 16. | $C_{16} \times C_2$ |
| 17. | $C_{16} \rtimes C_2$ |
| 18. | $D_{16}$ |
| 19. | $QD_{32}$ |
| 20. | $Q_32$ |
| 21. | $C_4 \times C_4 \times C_2$ |
| 22. | $C_2 \times ((C_4 \times C_2) : C_2)$ |
| 23. | $C_2 \times (C_4 \rtimes C_4)$ |
| 24. | $(C_4 \times C_4) \rtimes C_2$ |
| 25. | $C_4 \times D_4$ |
| 26. | $C_4 \times Q_8$ |

| | Group $G$ |
|---|---|
| 27. | $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$ |
| 28. | $(C_4 \times C_2 \times C_2) \rtimes C_2$ |
| 29. | $(C_2 \times Q_8) \rtimes C_2$ |
| 30. | $(C_4 \times C_2 \times C_2) \rtimes C_2$ |
| 31. | $(C_4 \times C_4) \rtimes C_2$ |
| 32. | $(C_2 \times C_2).(C_2 \times C_2 \times C_2)$ |
| 33. | $(C_4 \times C_4) \rtimes C_2$ |
| 34. | $(C_4 \times C_4) \rtimes C_2$ |
| 35. | $C_4 \rtimes Q_8$ |
| 36. | $C_8 \times C_2 \times C_2$ |
| 37. | $C_2 \times (C_8 \rtimes C_2)$ |
| 38. | $(C_8 \times C_2) \rtimes C_2$ |
| 39. | $C_2 \times D_8$ |
| 40. | $C_2 \times QD_{16}$ |
| 41. | $C_2 \times Q_{16}$ |
| 42. | $(C_8 \times C_2) \rtimes C_2$ |
| 43. | $(C_2 \times D_4) \rtimes C_2$ |
| 44. | $(C_2 \times Q_8) \rtimes C_2$ |
| 45. | $C_4 \times C_2 \times C_2 \times C_2$ |
| 46. | $C_2 \times C_2 \times D_4$ |
| 47. | $C_2 \times C_2 \times Q_8$ |
| 48. | $C_2 \times ((C_4 \times C_2) \rtimes C_2)$ |
| 49. | $(C_2 \times D_4) \rtimes C_2$ |
| 50. | $(C_2 \times Q_8) \rtimes C_2$ |
| 51. | $C_2 \times C_2 \times C_2 \times C_2 \times C_2$ |

- $(24, [1^3, 2^2, 17], 12)$

| | Group $G$ |
|---|---|
| 1. | $C_3 \rtimes C_8$ |
| 2. | $C_{24}$ |
| 3. | $SL(2,3)$ |
| 4. | $\mathrm{Dic}_6$ |
| 5. | $C_4 \times S_3$ |
| 6. | $D_{12}$ |
| 7. | $C_2 \times \mathrm{Dic}_3$ |
| 8. | $C_3 \rtimes D_4$ |
| 9. | $C_{12} \times C_2$ |
| 10. | $C_3 \times D_4$ |
| 11. | $C_3 \times Q_8$ |
| 12. | $S_4$ |
| 13. | $C_2 \times A_4$ |
| 14. | $C_2^2 \times S_3$ |
| 15. | $C_6 \times C_2^2$ |

- $G = C_3 \rtimes C_8$
- This is the semidirect product of $C_3$ by $C_8$ with defining relations

$$C_3 \rtimes C_8 = \langle a, b \,|\, a^8 = b^3 = 1, ab^{-1} = ba \rangle$$

- Thus the elements of $G$ are of the form $a^i b^j$ with $0 \le i \le 7$ and $0 \le j \le 2$. The difference (even though we should say "ratio" since we are in multiplicative notation) between two elements $a^{i_1} b^{j_1}$ and $a^{i_2} b^{j_2}$ is given by

$$(a^{i_1} b^{j_1})(a^{i_2} b^{j_2})^{-1} = a^{i_1 - i_2} b^{(-1)^{i_2}(j_1 - j_2)} \tag{1}$$

- Let $\mathcal{F} = \{B_1, B_2, B_3, B_4, B_5, B_6\}$ be the partition of $G$ defined as follows:

$$B_1 = \{1, a, a^2, a^3, a^4, a^6, a^7, b, ab, a^3b, a^4b, a^5b, a^6b, b^2, ab^2, a^2b^2, a^4b^2\};$$

$$B_2 = \{a^3b^2\}; \quad B_3 = \{a^5b^2\}; \quad B_4 = \{a^7b^2\};$$
$$B_5 = \{a^5,\ a^2b\}; \quad B_6 = \{a^7b,\ a^6b^2\}.$$

- Using (1) it is straightforward to check that $\mathcal{F}$ is a $(G, [1^3, 2^2, 17], 12)$-HPDF.

- $G = SL(2,3)$
- This is the 2-dimensional special linear group over $\mathbb{Z}_3$. Its elements are the $2 \times 2$ matrices with elements in $\mathbb{Z}_3$ and determinant equal to 1.
- Let $\mathcal{F} = \{B_1, B_2, B_3, B_4, B_5, B_6\}$ be the partition of $G$ defined as follows:

$$B_1 = \left\{ \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \right\}; \qquad B_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}; \qquad B_3 = \left\{ \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \right\};$$

$$B_4 = \left\{ \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right\}; \qquad B_5 = \left\{ \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \right\};$$

$$B_6 = G \setminus (B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5).$$

- It is straightforward to check that $\mathcal{F}$ is a $(G, [1^3, 2^2, 17], 12)$-HPDF.

- $G = \mathbb{Z}_3 \times D_8$

$$D_{2n} = \langle x, y \mid x^n = 1;\; y^2 = 1;\; yx^i = x^{-i}y \rangle$$

- The partition of $G$ into the blocks listed below is a $(G, [1^3, 2^2, 17], 12)$-HPDF.

$$B_1 = \{(0, x^2)\}; \quad B_2 = \{(2, xy)\}; \quad B_3 = \{(2, x^3y)\};$$
$$B_4 = \{(1, x^3),\ (2, x^3)\}; \qquad B_5 = \{(1, y),\ (2, x^2y)\};$$
$$B_6 = G \setminus (B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5).$$

- $(36, [3, 9, 24], 18)$

| | Group $G$ |
|---|---|
| 1. | $\mathbb{Z}_9 \rtimes \mathbb{Z}_4$ |
| 2. | $\mathbb{Z}_{36}$ |
| 3. | $(\mathbb{Z}_2 \times \mathbb{Z}_2) : \mathbb{Z}_9$ |
| 4. | $D_{18}$ |
| 5. | $\mathbb{Z}_{18} \times \mathbb{Z}_2$ |
| 6. | $\mathbb{Z}_6 \times \mathbb{Z}_6$ |
| 7. | $\mathbb{Z}_3 \times \mathbb{Z}_{12}$ |
| 8. | $\mathbb{Z}_3 \times \mathsf{Q}_{12}$ |
| 9. | $D_6 \times D_6$ |
| 10. | $\mathbb{Z}_6 \times D_6$ |
| 11. | $\mathbb{Z}_3 \times A_4$ |
| 12. | $\mathbb{Z}_3 \rtimes \mathsf{Q}_{12}$ |
| 13. | $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$ |
| 14. | $\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes D_6$ |

- $G = \mathbb{Z}_6 \times \mathbb{Z}_6$

  $A = \{(1,1),(1,3),(1,5)\}$;
  $B = \{\{0,2),(0,3),(1,4),(2,0),(2,5),(3,4),(4,1),(4,4),(5,4)\}$;
  $C = G \setminus (A \cup B)$.
- $G = \mathbb{Z}_3 \times \mathbb{Z}_{12}$

  $A = \{(1,1),(1,5),(1,9)\}$;
  $B = \{\{0,2),(0,3),(0,4),(1,2),(1,8),(1,11),(2,0),(2,2),(2,7)\}$;
  $C = G \setminus (A \cup B)$.
- $G = \mathbb{Z}_3 \times \mathsf{Q}_{12}$

  $A = \{(0,xy),(1,xy),(2,xy)\}$;
  $B = \{\{1,1),(0,x^3),(0,x^2),(2,y),(1,x^5),(2,x^4y),(2,x^4),(2,x^2y),(2,x)\}$;
  $C = G \setminus (A \cup B)$.
- $G = D_6 \times D_6$

  $A = \{(y,xy),(xy,xy),(x^2y,xy)\}$;
  $B = \{(1,x^2y),(x,y),(x^2,1),(x^2,x),(x^2,x^2),(x^2,xy),(y,1),(xy,x^2),(x^2y,x)\}$;
  $C = G \setminus (A \cup B)$.
- $G = \mathbb{Z}_6 \times D_6$

  $A = \{(1,xy),(3,xy),(5,xy)\}$;
  $B = \{\{0,x),(1,x^2),(2,1),(3,1),(4,x^2),(4,y),(4,xy),(4,x^2y),(5,x)\}$;

$C = G \setminus (A \cup B)$.

- $(40, [1, 3, 9, 27], 20)$

| | Group $G$ |
|---|---|
| 1. | $C_5 \rtimes_2 C_8$ |
| 2. | $C_{40}$ |
| 3. | $C_5 \rtimes C_8$ |
| 4. | $\mathrm{Dic}_{10}$ |
| 5. | $C_4 \times D_{10}$ |
| 6. | $D_{20}$ |
| 7. | $C_2 \times \mathrm{Dic}_5$ |

| | Group $G$ |
|---|---|
| 8. | $C_5 \rtimes D_4$ |
| 9. | $C_{20} \times C_2$ |
| 10. | $C_5 \times D_4$ |
| 11. | $C_5 \times Q_8$ |
| 12. | $C_2 \times F_5$ |
| 13. | $C_2^2 \times D_5$ |
| 14. | $C_{10} \times C_2^2$ |

## Example

- $D$ is a $(\mathbb{Z}_{40}, 13, 4)$-DS

$$D = \{1, 2, 3, 5, 6, 9, 14, 15, 18, 20, 25, 27, 35\}$$

$$D_1 = \{1\}, \quad D_2 = \{2, 5, 14\}, \quad D_3 = \{3, 6, 9, 15, 18, 20, 25, 27, 35\}$$

- $\mathbb{Z}_{40} \setminus D$ is a $(40, 27, 8)$-DS

  $\{0, 4, 7, 8, 10, 11, 12, 13, 16, 17, 19, 21, 22, 23, 24, 26, 28, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39\}$

- $\{D_1, D_2, D_3, \mathbb{Z}_{40} \setminus D\}$ is a $(40, [1, 3, 9, 27], 20)$-HPDF

# Open problem

The parameter set of $(40, [1, 3, 9, 27], 20)$-HPDF can be written as

$$\left(\frac{3^4-1}{2}, [3^0, 3^1, 3^2, 3^3], \frac{3^4-1}{4}\right).$$

Inspired by this, we have noticed that

$$\left(\frac{q^{2n}-1}{q-1}, [q^0, q^1, q^2, q^3, \ldots, q^{2n-1}], \frac{q^{2n}-1}{q+1}\right)$$

is an admissible parameter set of a PDF for every positive integer $q$ (not necessarily a prime power!).

## Question

*Given positive integers $q$ and $n$, does there exist a PDF whose $K$ is*

$$[q^0, q^1, q^2, q^3, \ldots, q^{2n-1}]?$$

### Theorem (Buratti, 2018)

*If there exists a* $(G, [k_1, \ldots, k_t], \lambda)$*-HPDF and all the components of* $2n+1$ *are greater than* $2 \cdot \max\{k_1, \ldots, k_t\}$*, then there exists a* $(2\lambda(2n+1), [(2k_1)^n, \ldots, (2k_t)^n, 2\lambda], 2\lambda)$*-PDF in* $G \times \mathbb{F}_{2n+1}$*.*

### Corollary

- $(24, [17, {}^{2}2, {}^{3}1], 12)$

*If all the components of* $2n+1$ *are greater than* $34$*, then there exists a* $(48n+24, [34^n, 4^{2n}, 2^{3n}, 24], 24)$*-PDF in* $G \times \mathbb{F}_{2n+1}$ *for each of the three groups* $G$ *considered earlier.*

*The first possible value of* $n = 18$ *gives* $(984, [34^{18}, 4^{36}, 2^{54}, 24], 24)$*-PDF in* $G \times \mathbb{F}_{37}$*.*

- $(36, [24, 9, 3], 18)$

*If all the components of* $2n+1$ *are greater than* $48$*, then there exists a* $(72n+36, [6^n, 18^n, 48^n, 36], 36)$*-PDF in* $G \times \mathbb{F}_{2n+1}$ *for each of the nine groups* $G$ *considered earlier.*

*The first possible value of* $n = 24$ *gives* $(1764, [6^{24}, 18^{24}, 48^{24}, 36], 36], 36)$*-PDF in* $G \times \mathbb{F}_{49}$*.*

- $(40, [27, 9, 3, 1], 20)$

*If all the components of* $2n+1$ *are greater than* $54$*, then there exists a* $(80n+40, [2^n, 6^n, 18^n, 54^n, 40], 40)$*-PDF in* $\mathbb{Z}_{40} \times \mathbb{F}_{2n+1}$*.*

*The first value of* $n = 29$ *gives* $(2360, [2^{29}, 6^{29}, 18^{29}, 54^{29}, 40], 40)$*-PDF in* $\mathbb{Z}_{40} \times \mathbb{F}_{59}$*.*

## Theorem (Ding, Yin, 2005)

$$(v, [\lambda_0, \lambda_1, \ldots \lambda_{q-1}]; \lambda)\text{-}PDF$$
$$\Downarrow$$
$$\textit{optimal } (v, v, v-\lambda, [\lambda_0, \lambda_1, \ldots, \lambda_{q-1}])_q\text{-}CCC$$

## Corollary

- $(48n + 24, [34^n, 4^{2n}, 2^{3n}, 24], 24)$-*PDF*

*If the maximal prime power divisors of* $2n+1$ *are all greater than 34, then there exists an optimal* $(48n+24, 48n+24, 48n, [34^n, 4^{2n}, 2^{3n}, 24])_{6n+1}$-*CCC.*

- $(72n + 36, [6^n, 18^n, 48^n, 36], 36)$-*PDF*

*If the maximal prime power divisors of* $2n+1$ *are all greater than 48, then there exists an optimal* $(72n+36, 72n+36, 72n, [6^n, 18^n, 48^n, 36])_{3n+1}$-*CCC.*

- $(80n + 40, [2^n, 6^n, 18^n, 54^n, 40], 40)$-*PDF*

*If the maximal prime power divisors of* $2n+1$ *are all greater than 54, then there exists an optimal* $(80n+40, 80n+40, 80n, [2^n, 6^n, 18^n, 54^n, 40])_{4n+1}$-*CCC.*

Thank you for your attention!