uc3m | Universidad Carlos III de Madrid

Universitat d'Alacant
Universidad de Alicante

Combinatorial Designs and Codes – CODESCO24
Satellite event of the 9th European Congress of Mathematics

Linearly equivalent flag codes

Miguel Ángel Navarro-Pérez

Joint work with Xaro Soler-Escrivà

July 10, 2024

mignavar@math.uc3m.es

From subspace codes to flag codes

- Let q be a prime power and consider the finite field $\mathbb{F}_q$.

SUBSPACES

Given $1 \leqslant k < n$, the Grassmannian

$$\mathcal{G}_q(k, n)$$

is the set of k-dimensional vector subspaces of $\mathbb{F}_q^n$.

FLAGS

Given $1 \leqslant t_1 < \cdots < t_r < n$, a flag of type $T = (t_1, \ldots, t_r)$ is

$$\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$$

- $\mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_r$ and
- $\dim(\mathcal{F}_i) = t_i$, for $1 \leqslant i \leqslant r$.

The flag variety

$$\mathcal{F}_q(T, n).$$

From subspace codes to flag codes

SUBSPACES in $\mathcal{G}_q(k, n)$

- Given $\mathcal{U}, \mathcal{V}$

$$d_S(\mathcal{U}, \mathcal{V}) = 2\left(k - \dim(\mathcal{U} \cap \mathcal{V})\right).$$

- A constant dimension code is

$$\emptyset \neq \mathcal{C} \subseteq \mathcal{G}_q(k, n).$$

- Minimum distance:

$$d_S(\mathcal{C}) = \min_{\mathcal{U}, \mathcal{V} \in \mathcal{C},\ \mathcal{U} \neq \mathcal{V}} \{d_S(\mathcal{U}, \mathcal{V})\}.$$

FLAGS in $\mathcal{F}_q(T, n)$

- Given flags $\mathcal{F}, \mathcal{F}'$

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i).$$

- A flag code is

$$\emptyset \neq \mathcal{C} \subseteq \mathcal{F}_q(T, n).$$

- Minimum distance:

$$d_f(\mathcal{C}) = \min_{\mathcal{F}, \mathcal{F}' \in \mathcal{C},\ \mathcal{F} \neq \mathcal{F}'} \{d_f(\mathcal{F}, \mathcal{F}')\}.$$

- A flag code of length 1 is a constant dimension code in

$$\mathcal{F}_q((t_1), n) = \mathcal{G}_q(t_1, n).$$

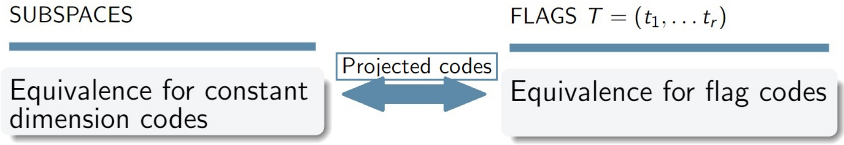
- For longer type vectors $T = (t_1, \dots, t_r)$:

Given $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \dots, t_r), n)$, its i-th projected code is

$$\mathcal{C}_i = \{\mathcal{F}_i \mid \mathcal{F} = (\dots, \mathcal{F}_i, \dots) \in \mathcal{C}\} \subseteq \mathcal{G}_q(t_i, n).$$

Information about flag codes in terms of constant dimension codes

Connection subspaces ⇔ flags



Linear equivalence: $A \in \mathrm{GL}(n, q)$

SUBSPACES

- $\mathcal{U} \in \mathcal{G}_q(k, n)$

$$\mathcal{U} \cdot A = \{uA \mid u \in \mathcal{U}\}$$

$C \subseteq \mathcal{G}_q(k, n)$

$$C \cdot A = \{\mathcal{U} \cdot A \mid \mathcal{U} \in C\}$$

linearly equivalent codes

FLAGS $T = (t_1, \dots t_r)$

- $\mathcal{F} \in \mathcal{F}_q(T, n)$

$$\begin{aligned} \mathcal{F} &= (\mathcal{F}_1, \dots, \mathcal{F}_r) \\ \mathcal{F} \cdot A &= (\mathcal{F}_1 \cdot A, \dots, \mathcal{F}_r \cdot A) \end{aligned}$$

$\mathcal{C} \subseteq \mathcal{F}_q(T, n)$

$$\mathcal{C} \cdot A = \{\mathcal{F} \cdot A \mid \mathcal{F} \in \mathcal{C}\}$$

linearly equivalent flag codes

Connection subspaces $\Leftrightarrow$ flags

SUBSPACES

Equivalence for constant dimension codes

Projected codes $\Longleftrightarrow$

FLAGS $T = (t_1, \dots t_r)$

Equivalence for flag codes

Flag equivalence $\Rightarrow$ Projected codes equivalence

- $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$, $A \in \mathrm{GL}(n, q)$
- $\mathcal{F} \cdot A = (\mathcal{F}_1, \ldots, \mathcal{F}_r) \cdot A = (\mathcal{F}_1 \cdot A, \ldots, \mathcal{F}_r \cdot A)$
- $\mathcal{C} \cdot A = \{\mathcal{F} \cdot A, \ \mathcal{F} \in \mathcal{C}\} \Rightarrow$ i-th subspaces $\mathcal{F}_i \cdot A$
- $(\mathcal{C} \cdot A)_i = \mathcal{C}_i \cdot A$, for all $i = 1, \ldots, r$.

Theorem (N.-P., Soler-Escrivà, 2024)

$\mathcal{C}$ a flag code, $A \in \mathrm{GL}(n, q)$.

$$\mathcal{C}' = \mathcal{C} \cdot A \Rightarrow \begin{cases} \mathcal{C}'_1 & = & \mathcal{C}_1 \cdot A \\ \mathcal{C}'_2 & = & \mathcal{C}_2 \cdot A \\ \vdots & \vdots & \vdots \\ \mathcal{C}'_r & = & \mathcal{C}_r \cdot A \end{cases}$$

SUBSPACES

Equivalence for constant dimension codes

FLAGS $T = (t_1, \dots t_r)$

Equivalence for flag codes

Linearly equivalent projected codes $\Longleftarrow$ linearly equivalent flag codes

With the same linear transformation
for **ALL** the projected codes.

The converse...

Linearly equivalent projected codes $\overset{?}{\Longrightarrow}$ linearly equivalent flag codes

$$\left.\begin{array}{ccc} \mathcal{C}'_1 & = & \mathcal{C}_1 \cdot A \\ \mathcal{C}'_2 & = & \mathcal{C}_2 \cdot A \\ \vdots & \vdots & \vdots \\ \mathcal{C}'_r & = & \mathcal{C}_r \cdot A \end{array}\right\} \overset{?}{\Rightarrow} \quad \mathcal{C}' = \mathcal{C} \cdot A$$

for an invertible $n \times n$ matrix A.

The converse...

Linearly equivalent projected codes $\overset{?}{\Longrightarrow}$ linearly equivalent flag codes

$$\left.\begin{array}{ccc} \mathcal{C}'_1 & = & \mathcal{C}_1 \cdot A = \mathbf{C}_1 \\ \mathcal{C}'_2 & = & \mathcal{C}_2 \cdot A = \mathbf{C}_2 \\ \vdots & \vdots & \vdots \\ \mathcal{C}'_r & = & \mathcal{C}_r \cdot A = \mathbf{C}_r \end{array}\right\} \overset{?}{\Rightarrow} \quad \mathcal{C}' = \mathcal{C} \cdot A = \mathbf{C}$$

for an invertible $n \times n$ matrix A.

The converse...

Same projected codes $\overset{?}{\Longrightarrow}$ Unique flag code

$$\left.\begin{array}{ccc} \mathcal{C}'_1 & = & \mathbf{C}_1 \\ \mathcal{C}'_2 & = & \mathbf{C}_2 \\ \vdots & \vdots & \vdots \\ \mathcal{C}'_r & = & \mathbf{C}_r \end{array}\right\} \overset{?}{\Rightarrow} \quad \mathcal{C}' = \mathbf{C}$$

The converse...

Same projected codes $\not\Rightarrow$ Unique flag code

Example

$\{e_1, e_2, e_3\}$ $\mathbb{F}_q$-basis of $\mathbb{F}_q^3$

$$\begin{aligned} \mathcal{F} &= (\langle e_1\rangle, \langle e_1, e_2\rangle) \\ \mathcal{F}' &= (\langle e_3\rangle, \langle e_1, e_3\rangle) \end{aligned} \qquad \mathcal{F}'' = (\langle e_1\rangle, \langle e_1, e_3\rangle)$$

$$\mathcal{C} = \{\mathcal{F}, \mathcal{F}'\} \qquad \mathcal{C}' = \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$$

$$\begin{aligned} \mathcal{C}_1 &= \mathcal{C}'_1 = \{\langle e_1\rangle, \langle e_3\rangle\}, \\ \mathcal{C}_2 &= \mathcal{C}'_2 = \{\langle e_1, e_2\rangle, \langle e_1, e_3\rangle\}. \end{aligned}$$

Same projected codes but different flag codes

The converse...

Same projected codes $\not\Rightarrow$ Unique flag code

Example

$\{e_1, e_2, e_3\}$ $\mathbb{F}_q$-basis of $\mathbb{F}_q^3$

$$\begin{aligned} \mathcal{F} &= (\langle e_1\rangle, \langle e_1, e_2\rangle) \\ \mathcal{F}' &= (\langle e_3\rangle, \langle e_1, e_3\rangle) \end{aligned} \qquad \mathcal{F}'' = (\langle e_1\rangle, \langle e_1, e_3\rangle)$$

$$\mathcal{C} = \{\mathcal{F}, \mathcal{F}'\} \qquad \mathcal{C}' = \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$$

$$\begin{aligned} \mathcal{C}_1 &= \mathcal{C}'_1 = \{\langle e_1\rangle, \langle e_3\rangle\}, \\ \mathcal{C}_2 &= \mathcal{C}'_2 = \{\langle e_1, e_2\rangle, \langle e_1, e_3\rangle\}. \end{aligned}$$

Same projected codes but different flag codes

The converse is not true

SUBSPACES

Equivalence for constant dimension codes

FLAGS $T = (t_1, \dots t_r)$

Equivalence for flag codes

Uniqueness generating flag codes

Definition

A flag code $\mathcal{C}$ is determined by its projected codes if $\mathcal{C}$ is the only flag code with projected codes $\mathcal{C}_1, \dots, \mathcal{C}_r$.

Example

$\{e_1, e_2, e_3\}$ $\mathbb{F}_q$-basis of $\mathbb{F}_q^3$

$$\begin{aligned} \mathcal{F} &= (\langle e_1 \rangle, \langle e_1, e_2 \rangle) \\ \mathcal{F}' &= (\langle e_3 \rangle, \langle e_1, e_3 \rangle) \end{aligned} \qquad \mathcal{F}'' = (\langle e_1 \rangle, \langle e_1, e_3 \rangle)$$

$$\mathcal{C} = \{\mathcal{F}, \mathcal{F}'\} \qquad \mathcal{C}' = \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$$

Definition

A flag code $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \dots, t_r), n)$ is subspace-inclusion-closed (SIC) if for every $1 < i \leqslant r$ and every pair of flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $\mathcal{F}_{i-1} \subset \mathcal{F}'_i$ the flag

$$(\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{F}'_i, \dots, \mathcal{F}'_r) \in \mathcal{C}.$$

Example

$\{e_1, e_2, e_3\}$ $\mathbb{F}_q$-basis of $\mathbb{F}_q^3$

$$\mathcal{F} = (\langle e_1 \rangle, \langle e_1, e_2 \rangle) \qquad \mathcal{F}'' = (\langle e_1 \rangle, \langle e_1, e_3 \rangle)$$

$$\mathcal{F}' = (\langle e_3 \rangle, \langle e_1, e_3 \rangle)$$

$\mathcal{C} = \{\mathcal{F}, \mathcal{F}'\}$ is not SIC $\qquad$ $\mathcal{C}' = \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$ is SIC

SIC flag codes

Theorem (N.-P., Soler-Escrivà, 2024)

Let $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ with projected codes $\mathcal{C}_1, \ldots, \mathcal{C}_r$. The following statements are equivalent:

1. $\mathcal{C}$ is a SIC flag code,
2. $\mathcal{C} = (\mathcal{C}_1 \times \cdots \times \mathcal{C}_r) \cap \mathcal{F}_q((t_1, \ldots, t_r), n)$.

Example

$\{e_1, e_2, e_3\}$ $\mathbb{F}_q$-basis of $\mathbb{F}_q^3$

$$\mathcal{F} = (\langle e_1 \rangle, \langle e_1, e_2 \rangle) \qquad \mathcal{F}'' = (\langle e_1 \rangle, \langle e_1, e_3 \rangle)$$

$$\mathcal{F}' = (\langle e_3 \rangle, \langle e_1, e_3 \rangle)$$

$\mathcal{C} = \{\mathcal{F}, \mathcal{F}'\}$ is not SIC $\qquad$ $\mathcal{C}' = \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$ is SIC

SIC flag code $\not\Rightarrow$ Unique flag code

Example

$\{e_1, e_2, e_3\}$ $\mathbb{F}_q$-basis of $\mathbb{F}_q^3$

$$\begin{aligned} \mathcal{F} &= (\langle e_1 \rangle, \langle e_1, e_2 \rangle) \\ \mathcal{F}' &= (\langle e_3 \rangle, \langle e_1, e_3 \rangle) \end{aligned} \qquad \mathcal{F}'' = (\langle e_1 \rangle, \langle e_1, e_3 \rangle)$$

$$\mathcal{C} = \{\mathcal{F}, \mathcal{F}'\} \qquad \mathcal{C}' = \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$$

Multiplicity

Definition

$\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ a flag code.

- $\mathcal{F} \in \mathcal{C}$
- for every $1 \leqslant i \leqslant r$: $m_{\mathcal{C}}(\mathcal{F}_i)$ multiplicity of $\mathcal{F}_i$ wrt. $\mathcal{C}$

(number of times that this subspace appears as the i-th subspace of different flags of $\mathcal{C}$).

Example

$\{e_1, e_2, e_3\}$ $\mathbb{F}_q$-basis of $\mathbb{F}_q^3$

$$\begin{aligned} \mathcal{F} &= (\langle e_1 \rangle, \langle e_1, e_2 \rangle) \\ \mathcal{F}' &= (\langle e_3 \rangle, \langle e_1, e_3 \rangle) \end{aligned} \qquad \mathcal{F}'' = (\langle e_1 \rangle, \langle e_1, e_3 \rangle)$$

$$\mathcal{C} = \{\mathcal{F}, \mathcal{F}'\} \qquad \mathcal{C}' = \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$$

Our goal

Theorem (N.-P., Soler-Escrivà, 2024)

A flag code $\mathcal{C}$ is determined by its projected codes if, and only if,

1. $\mathcal{C}$ is SIC and
2. every $\mathcal{F} \in \mathcal{C}$ has, at least, one subspace with multiplicity 1 (wrt. $\mathcal{C}$).

Example

$$\begin{array}{rcl} \mathfrak{F} & = & (\langle e_2 \rangle, \langle e_1, e_2 \rangle) \\ \mathfrak{F}' & = & (\langle e_3 \rangle, \langle e_1, e_3 \rangle) \end{array}$$

$\mathfrak{C} = \{\mathfrak{F}, \mathfrak{F}'\}$ is det. by its projected codes.

Corollary (N.-P., Soler-Escrivà, 2024)

Let $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ determined by its projected codes. If $\mathcal{C}' \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ satisfies

$$\mathcal{C}'_i = \mathcal{C}_i \cdot A, \quad 1 \leqslant i \leqslant r$$

for some $A \in \mathrm{GL}(n, q)$,

then $\mathcal{C}' = \mathcal{C} \cdot A$.

Optimum distance flag codes

Corollary (N.-P., Soler-Escrivà, 2024)

Let $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ and ODFC. If $\mathcal{C}'$ has projected codes $\mathcal{C}'_i = \mathcal{C}_i \cdot A$ and one of the following statements holds:

1. $t_1 \geqslant \frac{n}{2}$,
2. $t_r \leqslant \frac{n}{2}$,
3. $t_1 \leqslant \frac{n}{2} \leqslant t_r$ and $t_b < 2t_a$,
4. $t_1 \leqslant \frac{n}{2} \leqslant t_r$ and $2t_b < n + t_a$;

then $\mathcal{C}' = \mathcal{C} \cdot A$.

uc3m | Universidad Carlos III de Madrid

Universitat d'Alacant
Universidad de Alicante

Combinatorial Designs and Codes – CODESCO24
Satellite event of the 9th European Congress of Mathematics

Linearly equivalent flag codes

Miguel Ángel Navarro-Pérez

Joint work with Xaro Soler-Escrivà

July 10, 2024

mignavar@math.uc3m.es