COMPLETE SOLUTIONS TO THE UNIFORM HAMILTON-WATERLOO PROBLEM

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Outline

Cycle Decompositions

- Let G be a graph and H be a subgraph of G. If all edges of G can be decomposed into edge disjoint copies of H, then this decomposition is called an *H-decomposition of G*.
- If all edges of G can be decomposed into edge disjoint copies of k-factors, then this decomposition is called a *k-factorization* and G is called *k-factorable*.
- A *parallel class* (or resolution class) of a decomposition of G is a subset of vertex disjoint graphs whose union partitions the vertex set of G.
- *Cycle decomposition* of a graph G is an H-decomposition in which all H's are cycles.
- A *resolvable cycle decomposition* is a cycle decomposition which forms a 2-factorization, in other words, it is a cycle decomposition which can be partitioned into parallel classes.

Cycle Decompositions

Example

A $\{C_7, C_6, C_5, C_3\}-$decomposition of K_7.

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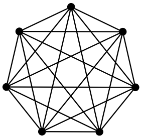

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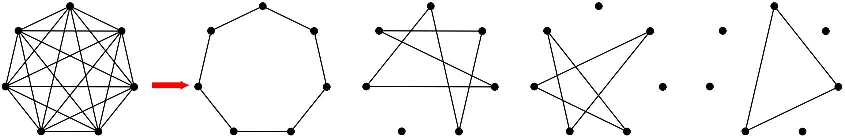

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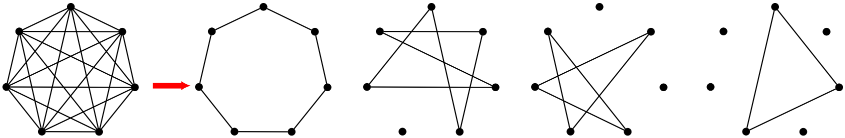

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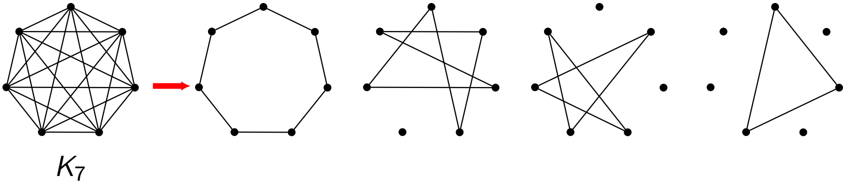

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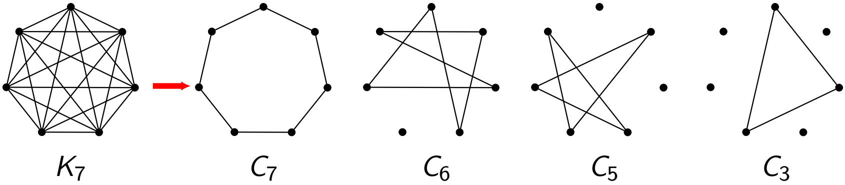

Cycle Decompositions

Obvious necessary conditions:

Lemma

Let G be a graph of order n, let $m_1, m_2, \ldots, m_k$ be a sequence of integers, and suppose that there is a decomposition $\{G_1, G_2, \ldots, G_k\}$ of G where G_i is an m_i-cycle for $i = 1, 2, \ldots, k$. Then

(i) $3 \leq m_i \leq n$ *for* $i = 1, 2, \ldots, k$.

(ii) *the number of edges in G is $m_1 + m_2 + \cdots + m_k$, and*

(iii) *Each vertex of G has even degree.*

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In 1981, Alspach conjectured that these are also sufficient for complete graphs and his conjecture is proven by Bryant and Horsley in 2010.

2-Factorizations

Definition

- A $\{F_1^{k_1}, F_2^{k_2}, \ldots, F_l^{k_l}\}-$*factorization* of a graph G is a decomposition which consists precisely of k_i factors isomorphic to F_i.

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- When each F_i factor consists of only n_i cycles for $i \in [1, t]$, then we will call the F_i factor as a C_{n_i}-*factor* and call this factorization as a $\left\{C_{n_1}^{r_1}, C_{n_2}^{r_2}, \ldots, C_{n_t}^{r_t}\right\}$-*factorization.*

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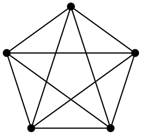

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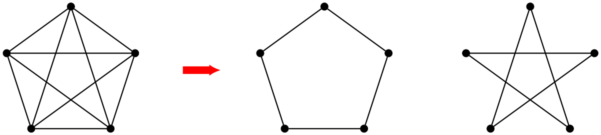

The Oberwolfach Problem

- It is motivated by seating arrangements at the meeting; is it possible to seat v participants of the conference in such a way that each person sits next to each other person exactly once over $\lfloor \frac{v-1}{2} \rfloor$ days, where there are a_i round tables with m_i seats for $i = 1, 2, \ldots, s$.

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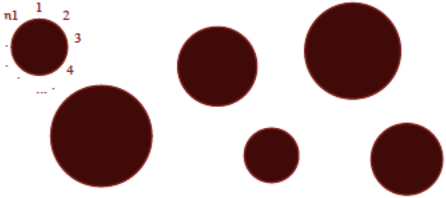

- It asks for a 2-factorization of the complete graph K_v (for even v, a 2-factorization of $K_v - F$ where F is a 1-factor) in which each 2-factor is isomorphic to $[m_1^{a_1}, m_2^{a_2}, \ldots, m_s^{a_s}]$.

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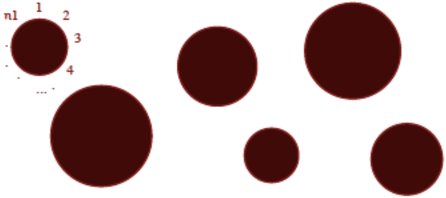

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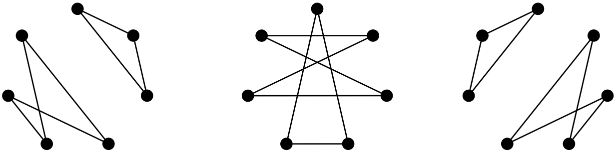
- In this case, the corresponding Oberwolfach problem is denoted by $\mathrm{OP}(m_1^{a_1}, m_2^{a_2}, \ldots, m_s^{a_s})$.

Example

A solution to OP$(3, 4)$.



Some Known Results

It is known that the solutions to the cases OP(3^2), OP(3^4) do not exist. The Oberwolfach Problem for a single cycle size OP(m^k) for all $m \geq 3$ has been solved.

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$\mathrm{OP}(m_1, m_2, \ldots, m_t)$ has a solution for all $m_1 + m_2 + \cdots + m_t \leq 40$ except for $\mathrm{OP}(3^2)$, $\mathrm{OP}(3^4)$, $\mathrm{OP}(3,4)$, $\mathrm{OP}(3^2,5)$.

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$\mathrm{OP}(m_1, m_2, \ldots, m_t)$ has a solution for all $m_1, m_2, \ldots, m_t$ all even.

(Bryant and Danziger–2011)

The Hamilton-Waterloo Problem

- One extension of the problem is the Hamilton-Waterloo problem, where the conference takes places in two venues (Hamilton and Waterloo) and one of them has r round tables, each seating m_i people for $i = 1, 2, \ldots, r$ and the second one has s round tables, each seating n_i people for $i = 1, 2, \ldots, s$ (necessarily $\sum_{i=1}^{r} m_i = \sum_{i=1}^{r} n_i = v$).

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- In other words, each 2-factor in the factorization is isomorphic to either $[m_1, m_2, \ldots, m_r]$ or to $[n_1, n_2, \ldots, n_s]$.
- If we let $m = m_1 = m_2 = \cdots = m_r$ and $n = n_1 = n_2 = \cdots = n_s$, then each 2-factor is composed of either m-cycles, C_m, or n-cycles, C_n . Then the Hamilton-Waterloo problem is same as uniformly resolvable $\{C_m, C_n\}$-decompositions of K_v (or $K_v - F$ for even v).

The Hamilton-Waterloo Problem

Notations $(m, n) - URD(v; r, s)$ or $(m, n) - HWP(v; r, s)$ or HWP$(v; C_n^r, C_m^s)$ are used to denote such a decomposition on v points with r factors of m-cycles and s factors of n-cycles.

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Lemma

(Adams, Billington, Bryant, El-Zanati - 2002) Let v, m, n, r, s be non-negative integers with $m, n \geq 3$. If there exists a $(m, n) - \mathrm{HWP}(v; r, s)$, then

(i) *if $r > 0$, m divides v, and if $s > 0$, n divides v;*

(ii) *if v is odd, then $r + s = (v-1)/2$; and*

(iii) *if v is even, then $r + s = (v-2)/2$.*

Example

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A solution to HWP$(12; C_3^2, C_4^3)$.

1, 6, 8	1, 2, 3	1, 9, 5, 12	1, 5, 7, 11	1, 4, 7, 10
2, 7, 11	4, 5, 6	2, 10, 8, 4	2, 6, 10, 9	2, 5, 8, 11
3, 5, 10	7, 8, 9	3, 11, 6, 7	3, 8, 12, 4	3, 6, 9, 12
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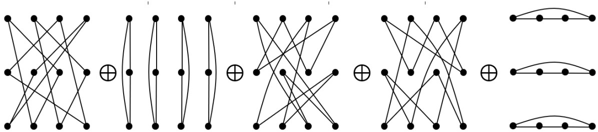

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- In 2002, Adams et al.solved the Hamilton-Waterloo problem for the cases $(m, n) \in \{(4,6),(4,8),(4,16),(8,16),(3,5),(3,15),(5,15)\}$ and settled the problem for all $v \leq 16$. Danziger et al. solved the problem for the case $(m, n) = (3,4)$ with a few exceptions.

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- Odabaşı and Özkan solved all the cases for $(4, m)$ in 2016 and $(4, 4m)$ in 2017.
- Burgess, Danziger and Traetta worked on the odd order cycles and different parity cycles leaving some possible exceptions (2017, 2018)

Main Results

Theorem (Burgess, Danziger, Traetta; 2018)

Let m and v be odd integers with such that $m \geq v \geq 3$, and $\alpha, \beta > 0$ be integers. Then $(\alpha, \beta) \in \mathrm{HWP}(K_{mv}; m, mv)$ if and only if $\alpha + \beta = \frac{mv-1}{2}$, except possibly when at least one of the following holds:

1. $\beta = 1$,
2. $\alpha < \frac{m-1}{2}$,
3. $\alpha - \frac{m-1}{2} \in \{1, 3\}$ and $m > v$;
4. $(m, v) = (5, 3)$ and $\alpha - \frac{m-1}{2} \equiv 1, 3 \pmod{m}$.

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In our main result we get rid off possible exceptions in the above theorem when $m \geq 7$, so provide a complete solution to the HWP when each 2-factor is either Hamiltonian cycle or consist of m-cycles only, where m is odd and $m \geq 7$.

Theorem

Let m, v be odd integers such that $m \geq 7$ and $v \geq 3$. Then, $(\alpha, \beta) \in \mathrm{HWP}(K_{mv}; m, mv)$ if and only if $\alpha + \beta = \frac{mv-1}{2}$ with $\alpha, \beta \geq 0$.

Another result, concerning 2-factorizations of complete graphs when one cycle length is a proper divisor of the other, proven by Burgess, Danziger and Traetta in 2018.

Theorem (Burgess, Danziger, Traetta; 2018)

Let m and v be odd integers such that $m, v \geq 3$, and $s, \alpha, \beta > 0$ be integers. Then $(\alpha, \beta) \in \mathrm{HWP}(K_{smv}; m, mv)$ if and only if $\alpha + \beta = \lfloor \frac{smv-1}{2} \rfloor$, except possibly when at least one of the following holds:

1. $\beta = 1$,
2. $(m, s) = (3, 6)$,
3. $s \in \{1, 2, 4\}$, *and either* $v > m$ *or one of the following subcases holds:*
   - 3*a*. $\alpha < \lfloor \frac{ms-1}{2} \rfloor$,
   - 3*b*. $\alpha - \lfloor \frac{ms-1}{2} \rfloor \in \{1, 3\}$ *and* $m > v$,
   - 3*c*. $(m, v) = (5, 3)$ *and* $\alpha - \lfloor \frac{ms-1}{2} \rfloor \equiv 1, 3 \pmod{m}$,
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   - 3c. $(m, v) = (5, 3)$ *and* $\alpha - \lfloor \frac{ms-1}{2} \rfloor \equiv 1, 3 \pmod{m}$,
   - 3d. $(v, s) = (3, 2)$.

Our result removes the most of possible exceptions in the above theorem when $m \geq 7$.

Theorem

Let $s \geq 1$ be an integer and m, v be odd integers such that $m \geq 7$ and $v \geq 3$. Then, $(\alpha, \beta) \in \mathrm{HWP}(K_{smv}; m,\ mv)$ if and only if $\alpha + \beta = \lfloor \frac{smv-1}{2} \rfloor$ with $\alpha, \beta \geq 0$.

Definitions, Notations

- For a positive integer v and a set $S \subseteq \{1, 2, \ldots, \lfloor \frac{v}{2} \rfloor\}$, a *circulant* $C(v; S)$ is a graph with vertex set $\mathbb{Z}_v$, and edge set $E = \{\{x, y\} : \delta(x, y) \in S\}$ where $\delta(x, y) = \pm|x - y| \mod v$. S is called a *connection set*.

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- Given a graph G, dG denotes a graph with d components, each of which is isomorphic to G.
- If G and H are two graphs such that $V(G) = V(H)$ but they are edge-disjoint, then $G \oplus H$ denotes the graph with the same vertex set and $E(G \oplus H) = E(G) \cup E(H)$.

Definitions and Notations

- The *cartesian* product of G and H is the graph $G \times H$ with the vertex set $V(G \times H) = V(G) \times V(H)$ such that two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ and $\{y_1, y_2\} \in E(H)$ or $y_1 = y_2$ and $\{x_1, x_2\} \in E(G)$.

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- We say that the edge $\{(x_1, y_1), (x_2, y_2)\}$ has *difference* $y_2 - y_1$.
- A given set of differences S, $S \subseteq \mathbb{Z}_v$, $G[S_v]$ denotes a spanning subgraph of $G[v]$ induced by the set of edges with differences in S.

First Result

Theorem (Burgess et al. 2018)

Let $m, v \geq 3$ be odd integers. Then, $(\alpha, \beta) \in \mathrm{HWP}(C_m[v]; m, mv)$ if and only if $\alpha + \beta = v$ with $\alpha, \beta \geq 0$, except possibly when $\beta = 1$.

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Let $m, v \geq 3$ *be odd integers. Then* $(v-1, 1) \in \mathrm{HWP}(C_m[v]; m, mv)$ *except when* $v = 3$ *or* $v = m + 2 = 5$*.*

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Thus, combining the above theorems together, we get a complete solution.

Corollary

Let $m, v \geq 3$ be odd integers. Then, $(\alpha, \beta) \in \mathrm{HWP}(C_m[v]; m, mv)$ if and only if $\alpha + \beta = v$ with $\alpha, \beta \geq 0$, except when $\beta = 1$ and either $v = 3$ or $v = m + 2 = 5$. □

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- The vertex set of K_n, where $n = mv$, is the set $\mathbb{Z}_v \times \{1, 2, \ldots, m\}$. All labels are taken modulo v. All indices are read modulo m where 0 replaced with m.

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- Let the $x_l \sim^i x_p$ denote the path $< x_l, (x+i)_{l+1}, x_{l+2}, (x+i)_{l+3}, x_{l+4}, \ldots, x_p >$ of length $p - l$ if $p > l$ and the single vertex x_l otherwise.

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- $l_i \overset{\delta}{\sim} l - 2j_{i+j}$ is the converse of the previous one, i.e., $< l_i, (l-2)_{i+1}, (l-4)_{i+2}, (l-6)_{i+3}, (l-8)_{i+4}, \ldots, (l-2j)_{i+j} >$.

Preliminary Results

Lemma

For each odd $v \geq 9$ *and each odd* $m \geq 3$*, there exists a 2-factorization of the graph* $C_m[\pm\{0,1,2\}_v]$ *which contains four* C_m*-factors and one Hamiltonian cycle.*

Proof

Let $v = 2k + 1$. Then $k \geq 4$. We construct a required 2-factorization $\{T_1, T_2, T_3, T_4, H\}$ of $C_m[\pm\{0, 1, 2\}_v]$. Let C_m-factors be:

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Moreover, let $H = (P_1 \cup P_2)$ if $k = 4$ and
$H = (P_1 \cup R^2_{k-4} \cup R^2_{k-5} \cup \cdots \cup R^2_1 \cup P_2 \cup R^1_1 \cup R^1_2 \cup \cdots \cup R^1_{k-4})$

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$T_4 : \{(1_1, 0_2, 0_3 \sim^{-2} 0_m), (2_1, 3_2, 4_3 \sim^{-2} 4_m), (3_1, 2_2, 2_3 \sim^{-2} 2_m), (4_1, 4_2, 3_3 \sim^{-2} 3_m),$
$(5_1, 6_2, 6_3 \sim^{-2} 6_m)\} \cup$
$\bigcup_{i=1}^{k-2}\{((4+2i)_1, (6+2i)_2, (6+2i)_3 \sim^{-2}$
$(6+2i)_m), ((5+2i)_1, (3+2i)_2, (3+2i)_3 \sim^{-2} (3+2i)_m)\}.$

Moreover, let $H = (P_1 \cup P_2)$ if $k = 4$ and
$H = (P_1 \cup R^2_{k-4} \cup R^2_{k-5} \cup \cdots \cup R^2_1 \cup P_2 \cup R^1_1 \cup R^1_2 \cup \cdots \cup R^1_{k-4})$

Preliminary Results

Theorem (Alspach et al. 1989, Burgess et al. 2017)

If $v \geq m \geq 3$ are odd integers then $C_m[\pm\{0, 1, 2\}_v]$ has a 2-factorization into five C_v-factors.

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If $v \geq m \geq 3$ are odd integers then $C_m[\pm\{0,1,2\}_v]$ has a 2-factorization into five C_v-factors.

Lemma

If $v \geq 5$ is an odd integer then $C_3[\pm\{0,1,2\}_v]$ has a 2-factorization into one Hamiltonian cycle and four C_v-factors.

Proof

Let $v = 2k + 1$. Then $k \geq 2$. We construct a 2-factorization $\{T_1, T_2, T_3, T_4, H\}$ of $C_3[\pm\{0, 1, 2\}_v]$. Let C_v-factors be:

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$T_1 : \{(0_1, 0_2, 2_3 \overset{\beta}{\sim} (2k)_3), (0_3, 1_1, 1_2 \overset{\beta}{\sim} (2k-1)_2), (1_3, 2_1, 2_2 \overset{\alpha}{\sim} (2k)_2)\}$,

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Moreover, $H = (0_1 \overset{\delta}{\sim} 3_k, 2_{k+2} \overset{\gamma}{\sim} 1_2, 0_3 \overset{\delta}{\sim} 3_{k+2}, 2_{k+1} \overset{\gamma}{\sim} 1_1, 0_2 \overset{\delta}{\sim} 3_{k+1}, 2_k \overset{\gamma}{\sim} 1_3)$ is an Hamiltonian cycle. □

Signed Langford Sequences

Definition

A *signed Langford sequence* of order t and defect d is a sequence $\pm\mathcal{L}_d^t = (l_{-2t}, l_{-2t+1}, \ldots, l_{-1}, *, l_1, l_2, \ldots, l_{2t})$ of length $4t+1$ that satisfies the following conditions:
(1) for every $k \in \pm\{d, d+1, \ldots, t+d-1\}$ there are exactly two elements $l_i, l_j \in \pm\mathcal{L}_d^t$ such that $l_i = l_j = k$, and
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The existence of signed Langford sequences has been completely settled by Jordon and Mitchell [31].

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Order 5 and defect 2,

$$(5, 6, 4, -2, -4, 3, -3, 2, -6, -5, *, 2, 3, 6, 4, 5, -5, -3, -6, -2, -4)$$

Preliminary Results

Signed Langford sequences are useful tools to construct 2-factorizations of graphs $C_m[\pm S_v]$, $C_m[\pm S_{2v}]$ and $C_m[\pm S_{4v}]$.

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Lemma

Let m, v, d and r be integers such that both m and v are odd, $m \geq 3$, $0 < d \leq \frac{v+3}{6}$, $0 \leq r \leq v - 2d + 1$ and moreover $r \neq 1$ when either $d \geq 2$ or $v = 3$ or $v = m + 2 = 5$. Let $S = \{d, d+1, \ldots, \frac{v-1}{2}\}$. Then there exists a 2-factorization of the graph $C_m[\pm S_v]$ into r Hamiltonian cycles and $(v - 2d - r + 1)$ C_m-factors.

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Let J be a set of positive integers. An (n, J)-*resolvable cycle design*, denoted (n, J)-RCD, is a 2-factorization of K_n (when n id odd) or $K_n \setminus I$ (when n is even) such that the length of any cycle in the 2-factorization belongs to J. The existence of $(n, \{3, 5\})$-resolvable cycle designs has been completely settled.

Theorem (Alspach et al.; 1989)

For each $n \geq 3$ there exists an $(n, \{3, 5\})$-RCD if and only if $n \notin \{4, 6, 7, 11, 12\}$.

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It is known that every connected circulant of valency four is decomposable into two Hamiltonian cycles.

Theorem (Bermond et al.; 1989)

Every 4-regular connected Cayley graph on a finite Abelian group can be decomposed into two Hamiltonian cycles.

Back to Main Results

Theorem

Let $m, v \geq 3$ *be odd integers. Then* $(v-1,1) \in \mathrm{HWP}(C_m[v]; m, mv)$ *except when* $v = 3$ *or* $v = m+2 = 5$.

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Factorize $C_m[v]$ into two factors: $C_m[\pm\{0, 1, 2\}_v]$ and $C_m[\pm\{3, 4, \ldots, \frac{v-1}{2}\}_v]$.

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By Lemma 15, there exists a 2-factorization of $C_m[\pm\{0,1,2\}_v]$ which contains four C_m-factors and one Hamiltonian cycle.

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By Lemma 20, $C_m[\pm\{3, 4, \ldots, \frac{v-1}{2}\}_v]$ is factorable into $v - 5$ C_m factors.

Back to Main Results

Theorem

Let m, v *be odd integers such that* $m \geq 7$ *and* $v \geq 3$*. Then,* $(\alpha, \beta) \in \mathrm{HWP}(K_{mv}; m, mv)$ *if and only if* $\alpha + \beta = \frac{mv-1}{2}$ *with* $\alpha, \beta \geq 0$*.*

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Proof

Let $n = mv$.

Case I: $\beta \geq \frac{v+5}{2}$. Factorize K_n into $\frac{m+1}{2}$ factors: $K_v \times C_m$, $C_m[\pm\{1, 2, \ldots, \frac{v-1}{2}\}_v]$, and $\frac{m-3}{2}$ copies of $C_m[v]$.

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Case II: $\beta \leq \frac{v+3}{2}$ and $\beta \neq 1$ when $v = 3$. The complete graph K_m is clearly decomposable into two circulants: $C(m; 1, 2)$ and $C(m; 3, 4, \ldots, \frac{m-1}{2})$.

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Case III: $\beta = 1$ and $v = 3$. We decompose K_n in exactly the same way as in Case II. $K_3[\pm\{0, 1, 2\}_m]$ is factorable, by Lemma 17, into one Hamiltonian cycle and four C_m-factors. All copies of $C_m[3]$ are factorized into C_m-factors. □

Thank You!

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