Graphs with prescribed edge-lengths: open problems and new results

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Conjectures

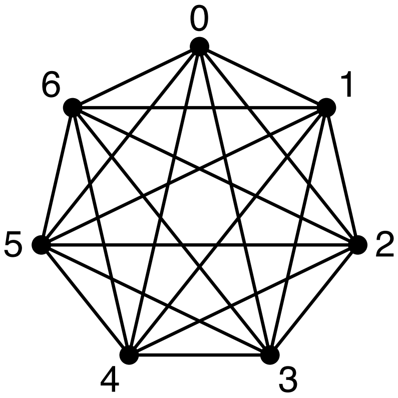
(1) Buratti (2007) $\rightarrow$ Horak, Rosa (2009)

(2) Bacher (2008) and Meszka (2012) $\rightarrow$ AP, Pellegrini (2015)

(3) Adamaszek (20??) $\rightarrow$ Meszka, AP, Pellegrini (202?)

Complete graphs

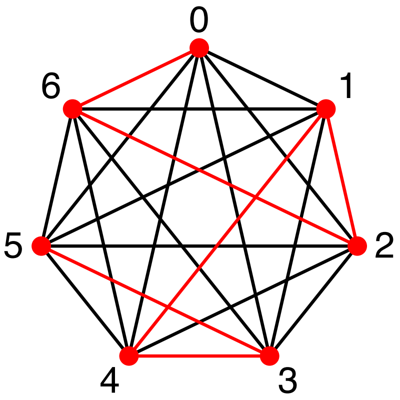
K_v = complete graph of order v, with $V(K_v) = \{0, 1, \ldots, v-1\}$



K_7

Hamiltonian paths

Hamiltonian path of K_v = path H such that $V(H) = V(K_v)$



$$H = [0, 6, 2, 1, 4, 3, 5]$$

Edge-lengths

Definition

The length of an edge $\{x, y\}$ of K_v is

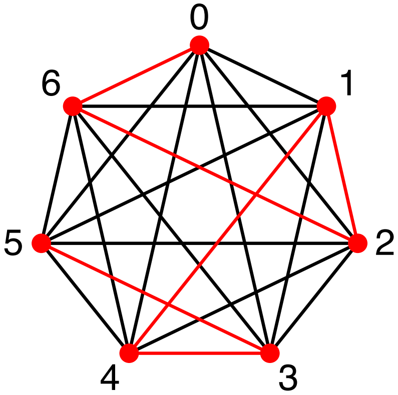
$$\ell(\{x, y\}) = min(|x - y|, v - |x - y|).$$

Given $G \leq K_v$, the list of edge-lengths of G is

$$\ell(G) = \{\ell(e) : e \in E(G)\}.$$

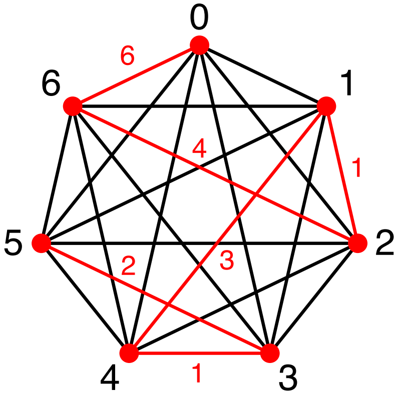
$e \in E(K_v) \Rightarrow \ell(e) \leq \lfloor \frac{v}{2} \rfloor$

Example



$H = [0\,,6\,,2\,,1\,,4\,,3\,,5]$

Example



$H = [0 \overset{6}{,} 6 \overset{4}{,} 2 \overset{1}{,} 1 \overset{3}{,} 4 \overset{1}{,} 3 \overset{2}{,} 5]$

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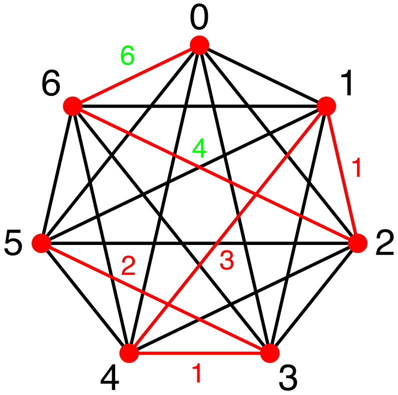

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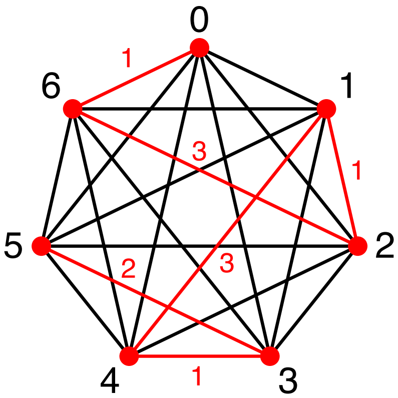

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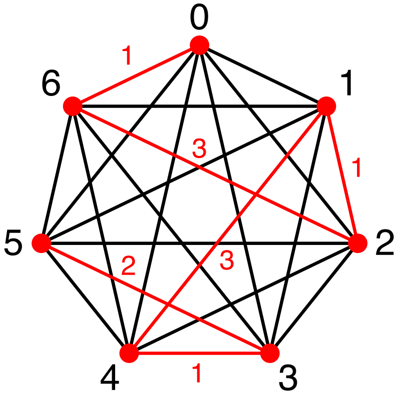
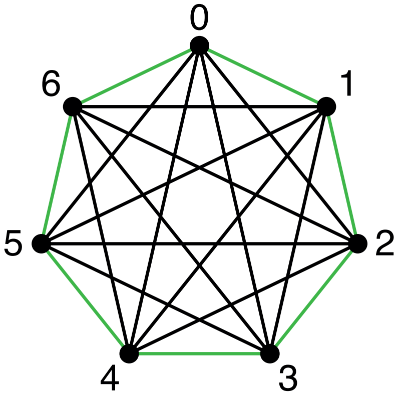


$$H = [0 \overset{1}{;} 6 \overset{3}{;} 2 \overset{1}{;} 1 \overset{3}{;} 4 \overset{1}{;} 3 \overset{2}{;} 5] \ \Rightarrow \ \ell(H) = \{1,3,1,3,1,2\} = \{1^3, 2, 3^2\}$$

Length of $\{x,y\}$ = distance of x and y in $(0,1,2,\ldots,v-1)$



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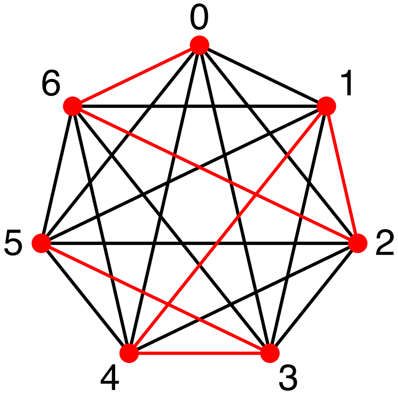

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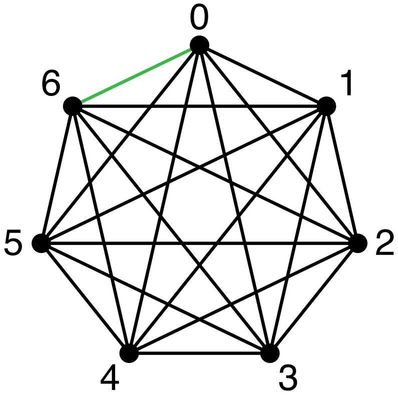

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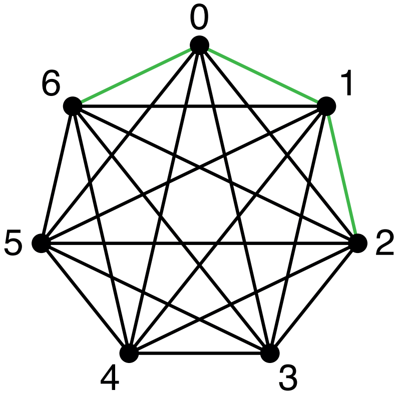

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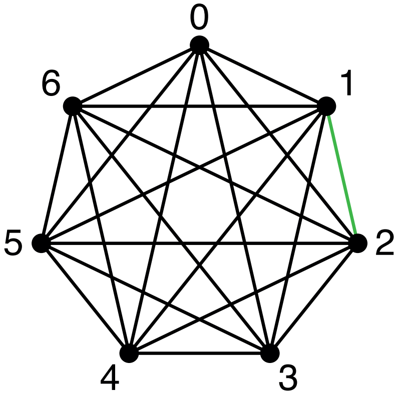

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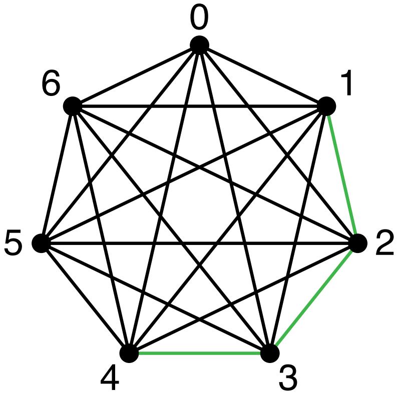

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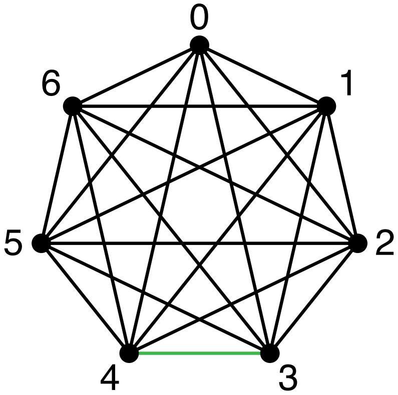

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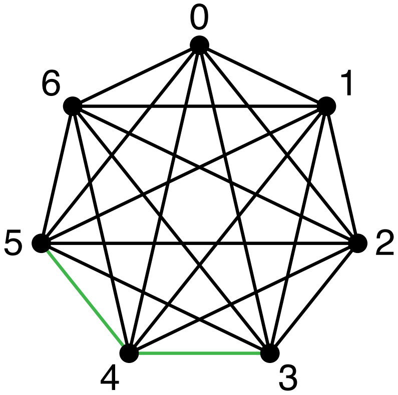


$H = [0\ ,6\ ,2\ ,1\ ,4\ ,3 \overset{2}{,} 5]$

Conjecture n.1

Buratti (2007) $\rightarrow$ Horak, Rosa (2009)

Buratti's conjecture

Conjecture [Buratti, 2007]

Given ANY prime $p = 2n + 1$ and ANY list L of $2n$ elements taken from $\{1, \ldots, n\}$, there exists a Hamiltonian path H of K_p such that $\ell(H) = L$.

Mail Marco-Alex

On Wed, 3 Jan 2007, Marco Buratti wrote:

> Dear Alex,
> the new year has just begun and already I need your help ...
> I wonder whether the following problem has been studied and, in the
> affirmative case, I would like to know what has been done. You maybe can
> suggest the name of some people who worked on this problem.
>
> The problem is:
>
> Given an odd prime p and a list L of p-1 elements in the set
> {1,2,...,(p-1)/2}, does there exist a hamiltonian path H of K(Z_p) (the
> complete graph on Z_p) such that the list of all differences between
> adjacent vertices of H is (\pm L)?
>
> I conjecture that the answer is always YES but, at the moment, I am not able
> to prove it.
>
> Thanks in advance and please forgive me for asking your help so frequently.
> Best regards,
> Marco

Mail Marco-Alex

Alexander Rosa <rosa@mcmaster.ca>

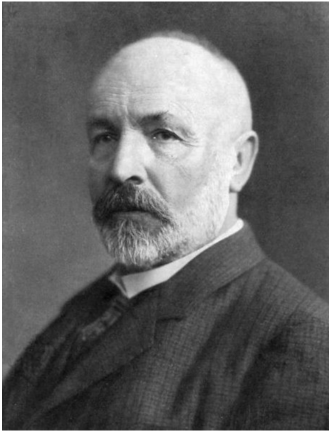
a Marco Buratti

Dear Marco,

this looks like a very nice problem. I am not aware of anyone having tried this, let alone solved it. I want to look at it myself!

Best regards, Alex

Georg Cantor (1845–1918)



To ask the right question is harder than to answer it.

Buratti's conjecture

Conjecture [Buratti, 2007]

Given ANY prime $p = 2n + 1$ and ANY list L of $2n$ elements taken from $\{1, \ldots, n\}$, there exists a Hamiltonian path H of K_p such that $\ell(H) = L$.

Alex Rosa:"This conjecture is a combinatorial disease!"
at "Combinatorics 2008", Costermano, Italy (2008).

A generalization of the Buratti's conjecture

Problem [Horak and Rosa, 2009]

Given a positive integer v, determine all lists L such that there exists a Hamiltonian path H of K_v with $\ell(H) = L$.

Conjecture [Horak and Rosa, 2009]

Let $L = \{\ell_1^{a_1}, \ell_2^{a_2}, \ldots, \ell_k^{a_k}\}$ with $|L| = v - 1$ and $1 \leq \ell_i \leq \lfloor \frac{v}{2} \rfloor$, then there exists a Hamiltonian path H of K_v such that $\ell(H) = L$ **if and only if** for all subsets $J \subseteq [1, k]$:

$$\sum_{j \in J} a_j \geq \gcd\{v, \ell_i : i \in [1, k] \setminus J\} - 1 \qquad (1)$$

A necessary condition

Proposition [AP and Pellegrini, 2014]

Condition (1) is equivalent to:

$$\text{for any divisor } d \text{ of } v\text{, the number of multiples of } d \text{ in } L \text{ does not exceed } v - d. \quad (2)$$

Proposition [AP and Pellegrini, 2014]

The list of the edge-lengths of any Hamiltonian path of K_v satisfies condition (2).

The BHR-conjecture

For ANY positive integer v and ANY list L with $v-1$ elements taken from $\{1,2,\ldots,\lfloor \frac{v}{2} \rfloor\}$ and satisfying condition (2) there exists a Hamiltonian path H of K_v such that $\ell(H) = L$.

Given $L \longrightarrow \text{BHR}(L)$.

First results on the BHR-conjecture

$U=$ underlying set of L

BHR(L) is true in each of the following cases:

- $|L| \leq 36$ [Meszka, 2008 + McKay-Peters, 2022]
- $L = M \cup \{1^a\}$ for any list M and $a > a_M$, where a_M is a **suitable** constant depending on M [Horak-Rosa, 2009]
- $|U| \leq 2$ [Horak-Rosa + Dinitz-Janiszewski, 2009]
- $U = \{1,2,3\}$ [Capparelli-Del Fra, 2010]
- $U \subseteq \{1,2,3,5\}$ [AP-Pellegrini, 2014]
- $L = \{1^a, 2^b, x^c\}$ when x is even and $a + b \geq x - 1$ [AP-Pellegrini, 2014]
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An explicit bound

Theorem [Ollis, AP, Pellegrini, Schmitt, 2022]

If M is a list with underlying set $U = \{x_1, \ldots, x_k\}$ with $1 < x_1 < \ldots < x_k$, then BHR(L) is true whenever $L = M \cup \{1^s\}$ with $s > 3x_k - 5 + \sum_{i=1}^{k} x_i$.

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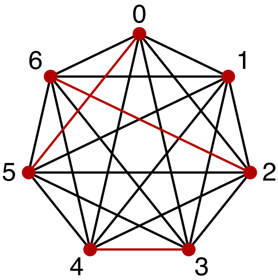
Conjecture n.2

Bacher (2008) and Meszka (2012) $\rightarrow$ AP, Pellegrini (2015)

G = a near 1-factor of K_{2n+1}



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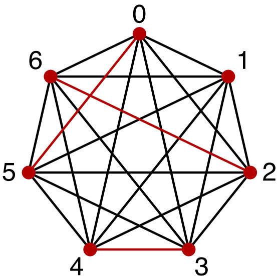

Conjecture [Meszka, 2012]

Given ANY prime $p = 2n + 1$ and ANY list L of n elements taken from $\{1, \ldots, n\}$, there exists a near 1-factor F of K_p such that $\ell(F) = L$.

Rosa, On a problem of Mariusz Meszka, Discrete Math. (2015)

From BHR to MPP

Buratti : Horak-Rosa = Meszka : Pasotti-Pellegrini

A generalization of Meszka's conjecture

MPP-Conjecture [AP, Pellegrini, 2015]

Let $v = 2n+1$ be ANY odd integer and let L be ANY list of n elements taken from $\{1,\dots,n\}$. Then there exists a near 1-factor F of K_v such that $\ell(F) = L$ **if and only if** the following condition holds:

$$\text{for any divisor } d \text{ of } v, \text{ the number of multiples of } d \text{ in } L \text{ does not exceed } \frac{v-d}{2}. \tag{3}$$

Proposition [AP, Pellegrini, 2015]

The list of edge-lengths of any near 1-factor of K_v satisfies condition (3).

Results about MPP-conjecture

Theorem [AP, Pellegrini, 2015]

MPP-conjecture is true for any list L with n elements such that:

- $2n+1 \leq 23$;
- $L = \{\ell_1^a\}$, $L = \{\ell_1^a, \ell_2^b\}$;
- $L = \{1, 2, \ldots, n\}$;
- $L = \{1^a, 2^b, t^c\}$ with 1) t not coprime with $2n+1$ OR
  2) $a+b \geq \lfloor \frac{t-1}{2} \rfloor$ OR
  3) $t \leq 11$.

The King's Table Problem

Seating Couples Around the King's Table and a New Characterization of Prime Numbers

Emmanuel Preissmann and Maurice Mischler

1. INTRODUCTION A king invites n couples for dinner at his round table containing $2n + 1$ seats, the king taking the last unoccupied chair. The king has to address the following problem [**1**]: Given an arbitrary set of n couples, no one married for more than n years, is it always possible to seat all n couples at his table according to the royal protocol stipulating that if the two spouses of a couple are in their ath year of marriage, they have to occupy two chairs at *circular distance* a? ("*Circular distance* a" means that the two chairs are separated by exactly $a - 1$ chairs.)

In other words, given an arbitrary set of n natural numbers $d_1, \ldots, d_n$ in $\{1, \ldots, n\}$, is it always possible to find an involution of $2n + 1$ circularly ordered points having a unique fixed point and consisting of n disjoint transpositions exchanging respectively two points at circular distance $d_1, d_2, \ldots, d_n$?

Theorem 1. *The king's problem for a table surrounded by* $2n + 1 \geq 3$ *seats has a solution for every set of distances between* 1 *and* n *if and only if* $2n + 1$ *is a prime number.*

The American Mathematical Monthly **116** (2009), 268–272.

With graph terminology

- $2n+1$ seats $\rightarrow$ $2n+1$ vertices
- round table $\rightarrow$ cycle of length $2n+1$
- king $\rightarrow$ isolated vertex
- a couple $\rightarrow$ an edge
- years of marriage $\rightarrow$ edge-length
- a solution $\rightarrow$ a near 1-factor of K_{2n+1}

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The King's Table Problem $\equiv$ Meszka's Problem

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Conjecture [Bacher, 2008]

There exists a solution to the king's table problem if all distances are invertible elements modulo the total number $2n+1$ of seats.

Conjecture [Bacher, 2008]

Let L be a list of n positive integers not exceeding n and coprime with $2n+1$. Then there exists a near 1-factor F of K_{2n+1} such that $\ell(F) = L$.

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In the prime case:

- The proof by Preissmann and Mischler is not constructive
- Alternative NON constructive proofs are given by:
  - Karasev and Petrov (2012)
  - Kohen and Sadofschi Costa (2016)

MPP-conjecture is still open!

MPP-Conjecture [AP, Pellegrini, 2015]

Let $v = 2n+1$ be ANY odd integer and let L be ANY list of n elements taken from $\{1, \ldots, n\}$. Then there exists a near 1-factor F of K_v such that $\ell(F) = L$ **if and only if** the following condition holds:

for any divisor d of v, the number of multiples of d in L does not exceed $\frac{v-d}{2}$.

And the Queen???

Question

Why doesn't the queen attend the dinner??

And the Queen???

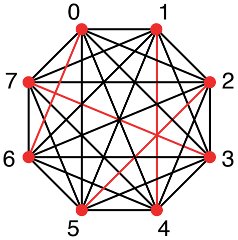
Question

Why doesn't the queen attend the dinner??

Add a place at the table.

$\Downarrow$

Take a 1-factor (perfect matching) in a complete graph of even order.



Conjecture n.3

Adamaszek (20??) $\rightarrow$ Meszka, AP, Pellegrini (202?)

The King's Table Problem in the even case

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This conjecture holds:

- for n prime, Mezei (2013)
- for any n, Kohen and Sadofschi Costa (2016)

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- for any n, Kohen and Sadofschi Costa (2016)

Theorem [Kohen, Sadofschi Costa, 2016]

Let L be a list of n positive integers not exceeding n and coprime with $2n$. Then there exists a 1-factor F of K_{2n} such that $\ell(F) = L$.

The King's Table Problem in the even case

Example

Take $n = 2$, $L = \{2^2\}$.
$F = \{\{0,2\},\{1,3\}\}$ is a 1-factor of K_4 with $\ell(F) = L$.

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Conjecture??

Let $v = 2n$ be ANY even integer and let L be ANY list of n elements taken from $\{1,\dots,n\}$. Then there exists a 1-factor F of K_v such that $\ell(F) = L$ **if and only if** the following condition holds:

for any divisor d of v, the number of multiples of d
in L does not exceed $\frac{v-d}{2}$.

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Problem

Find a "good" conjecture to state!

Some necessary conditions

Proposition [Meszka, AP, Pellegrini, 202?]

Let $v = 2n$ and L be a list of n positive integers not exceeding n. If there exists a 1-factor F of K_v such that $\ell(F) = L$, then:

(1) for any divisor d of v such that d does not divide n, the number of multiples of d in L does not exceed $\frac{v-d}{2}$

(2) L contains an even number of even integers.

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Necessary but not sufficient conditions!

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Take $n = 5$ and $L = \{2^2, 3, 5^2\}$

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Example

Take $n = 5$ and $L = \{2^2, 3, 5^2\} \Rightarrow v = 10$, $d = 2$.

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Necessary but not sufficient conditions!

Example

Take $n = 5$ and $L = \{2^2, 3, 5^2\} \Rightarrow v = 10$, $d = 2$.
There is no 1-factor F of K_{10} such that $\ell(F) = L$.

A complete solution for the extremal cases

Case 1: one edge-length with multiplicity n

Proposition [Meszka, AP, Pellegrini, 202?]

Let $1 \leq x \leq n$. There exists a 1-factor F of K_{2n} such that $\ell(F) = \{x^n\}$ **if and only if** $\gcd(x, 2n)$ is a divisor of n.

Case 2: n edge-lengths with multiplicity 1

Proposition [Meszka, AP, Pellegrini, 202?]

Let $L = \{1, 2, \ldots, n\}$. There exists a 1-factor F of K_{2n} such that $\ell(F) = L$ **if and only if** $n \equiv 0, 1 \pmod 4$.

A new open problem

Problem [Meszka, AP, Pellegrini, 202?]

Given an even integer $v = 2n$, determine all lists L satisfying conditions

(1) for any divisor d of v such that d does not divide n, the number of multiples of d in L does not exceed $\frac{v-d}{2}$

(2) L contains an even number of even integers

such that there exists a 1-factor F of K_v such that $\ell(F) = L$.

A complete solution for the two edge-lengths case

Theorem [Meszka, AP, Pellegrini, 202?]

Let $1 \leq x, y \leq n$, $x \neq y$ and $1 \leq a < n$. Let $d_x = \gcd(x, 2n)$, $d_y = \gcd(y, 2n)$ and $d = \gcd(x, y, 2n)$. There exists a 1-factor F of K_{2n} such that $\ell(F) = \{x^{n-a}, y^a\}$ **if and only if** d divides n and one of the following cases occurs:

(1) $\frac{x}{d}$ is even, $\frac{y}{d}$ is odd, $n - a$ is even and either
   - (a) d_x divides n; or
   - (b) d_x does not divide n and $2a \geq d_x$;

(2) $\frac{x}{d}$ is odd, $\frac{y}{d}$ is even, a is even and either
   - (a) d_y divides n; or
   - (b) d_y does not divide n and $2(n - a) \geq d_y$;

(3) $\frac{x}{d}$ and $\frac{y}{d}$ are both odd, and the following two conditions are both satisfied:
   - (a) a is even or $da \geq d_x$.
   - (b) $n - a$ is even or $d(n - a) \geq d_y$.

A new conjecture

At the moment we have no idea for a general conjecture, so we focus on the case n prime.

Conjecture [Meszka, AP, Pellegrini, 202?]

Let n be a prime and let L be a list of n positive integers less than n. There exists a 1-factor F of K_{2n} such that $\ell(F) = L$ **if and only if** the number of even integers in L is even.

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Let n be a prime and let L be a list of n positive integers less than n. There exists a 1-factor F of K_{2n} such that $\ell(F) = L$ **if and only if** the number of even integers in L is even.

Example

Take $n = 5$ and $L = \{2^2, 3, 5^2\} \Rightarrow v = 10$, $d = 2$.
There is no 1-factor F of K_{10} such that $\ell(F) = L$.

A tool for attacking the King's table problem

Definition [Skolem, 1957]

A Skolem sequence of order n is a sequence $S = (s_0, s_1, \dots, s_{2n-1})$ of $2n$ integers such that for every $k \in \{1, 2, \dots, n\}$ the following conditions hold:

(1) there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;

(2) if $s_i = s_j = k$ with $i < j$, then $j - i = k$.

$$n = 4 : \;\; S = (1, 1, 3, 4, 2, 3, 2, 4)$$

Skolem sequences

Skolem sequence of order n

$$\Downarrow$$

$$\{(a_i, b_i) \ : \ 1 \leq i \leq n, \ b_i - a_i = i\} \text{ with } \bigcup_{i=1}^{n} \{a_i, b_i\} = \{0, 1, \ldots, 2n-1\}$$

$$n = 4: \ S = (1,1,3,4,2,3,2,4) \ \rightarrow \ \{(0,1),(4,6),(2,5),(3,7)\}$$

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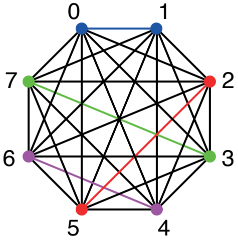
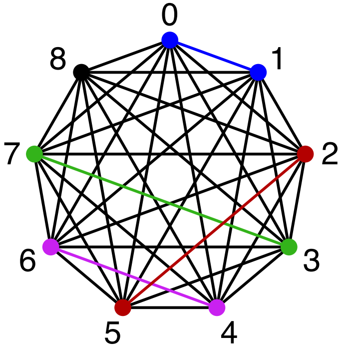


Example

$$S = (1,1,3,4,2,3,2,4) \;\rightarrow\; \{(0,1),(4,6),(2,5),(3,7)\}$$

$$\Downarrow$$

$$F = \{\{0,1\},\{4,6\},\{2,5\},\{3,7\}\} \cup \{8\}$$



Definition [Baker, 1995]

A k-extended Skolem sequence of order n is a sequence $S = (s_0, s_1, s_2, \ldots, s_{2n})$ of $2n+1$ integers satisfying conditions (1) and (2) and such that $s_k = 0$.

$k = 2n-1 \;\Rightarrow$ hooked Skolem sequence

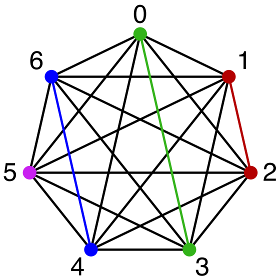
Example

$5-$extended Skolem sequence of order 3:

$$S = (3,1,1,3,2,0,2) \;\to\; \{(1,2),(4,6),(0,3)\} \cup \{5\}$$

$$\Downarrow$$

$$F = \{\{1,2\},\{4,6\},\{0,3\}\} \cup \{5\}$$



Skolem sequences of a list

Definition

Let $L = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}$, with $|L| = n$ and $a_i \geq 0$.
A Skolem sequence of L is a sequence $S = (s_0, s_1, \ldots, s_{2n-1})$ such that:

- $T_i := \{(x,y) \mid s_x = s_y = i = y - x\}$ has size a_i
- $\bigcup T_i = \{0, 1, \ldots, 2n-1\}$

Example

$L = \{2, 3^2, 4\}$

Example

$L = \{2, 3^2, 4\} \Rightarrow \ L = \{2, 3, 3, 4\}$

Skolem sequence of L: $S = (3, 3, 4, 3, 3, 2, 4, 2)$

Example

$L = \{2, 3^2, 4\} \Rightarrow \; L = \{2, 3, 3, 4\}$

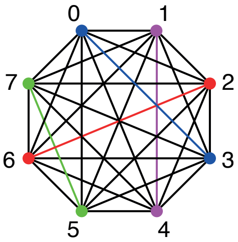
Skolem sequence of L: $S = (3, 3, 4, 3, 3, 2, 4, 2)$

$$T_1 = \emptyset, \quad T_2 = \{(5,7)\}, \quad T_3 = \{(0,3),(1,4)\}, \quad T_4 = \{(2,6)\}$$

$$\bigcup_{i=1}^{4} T_i = \{0, 1, \ldots, 7\}$$

$$\Downarrow$$

$F = \{\{5,7\},\{0,3\},\{1,4\},\{2,6\}\}$ 1-factor of K_8 s.t. $\ell(F) = L$



Example

$L = \{1, 2, 4^2\}$ and $k = 5$

5-extended Skolem sequence of L: $S = (4, 2, 4, 2, 4, 0, 4, 1, 1)$

Example

$L = \{1, 2, 4^2\}$ and $k = 5$

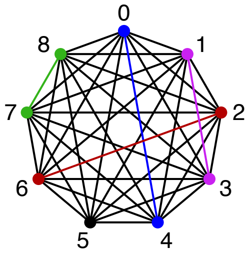
5-extended Skolem sequence of L: $S = (4, 2, 4, 2, 4, 0, 4, 1, 1)$

$$T_1 = \{(7, 8)\}, \quad T_2 = \{(1, 3)\}, \quad T_3 = \emptyset, \quad T_4 = \{(0, 4), (2, 6)\}$$

$$\bigcup_{i=1}^{4} T_i = \{0, 1, \ldots, 8\} \setminus \{5\}$$

$$\Downarrow$$

$F = \{\{7, 8\}, \{1, 3\}, \{0, 4\}, \{2, 6\}\} \cup \{5\}$ near 1-factor of K_9 s.t. $\ell(F) = L$



The mind revels in conjecture. Where information is lacking, it will gladly fill in the gaps.

James Geary