On designs of degree 3 *

Lucija Relić

University of Zagreb, Croatia

(joint work with Vedran Krčadinac)

Combinatorial Designs and Codes (CODESCO'24)

July 8-12, 2024, Sevilla, Spain

* This work was supported by the Croatian Science Foundation under the project 9752.

Definition

A set P of v points with a set of k-subsets of P (blocks) is called t-(v, k, λ) design if every t-subset of P is contained in exactly λ blocks. Strength of the design is the largest t for which it is a design.

Definition

A set P of v points with a set of k-subsets of P (blocks) is called t-(v, k, λ) design if every t-subset of P is contained in exactly λ blocks. Strength of the design is the largest t for which it is a design.

The degree of a design is the number of distinct block intersection sizes.

$$d = \left| \left\{ |B_1 \cap B_2| : B_1 \neq B_2 \text{ are blocks} \right\} \right|$$

$\rightsquigarrow$ intersection numbers

Introduction

Definition

A set P of v points with a set of k-subsets of P (blocks) is called t-(v, k, λ) design if every t-subset of P is contained in exactly λ blocks. Strength of the design is the largest t for which it is a design.

The degree of a design is the number of distinct block intersection sizes.

$$d = \left| \{ |B_1 \cap B_2| : B_1 \neq B_2 \text{ are blocks} \} \right|$$

$\rightsquigarrow$ intersection numbers

- We consider simple designs.
- $k \leq \frac{v}{2}$

Introduction

$\boxed{\text{d=1}}$ $\longrightarrow$ symmetric designs

$v = b$, $t = 2$
intersection number: $x = \lambda$

Introduction

$\boxed{\text{d=1}} \longrightarrow$ symmetric designs

$v = b$, $t = 2$
intersection number: $x = \lambda$

$\boxed{\text{d=2}} \longrightarrow$ quasi-symmetric designs

$t \leq 4$ (Ray-Chaudhuri, Wilson)

Designs of degree 3

d=3

$t \leq 6$ (Ray-Chaudhuri, Wilson)
intersection numbers: $x < y < z$

Designs of degree 3

d=3

$t \leq 6$ (Ray-Chaudhuri, Wilson)
intersection numbers: $x < y < z$

- $t = 6$: do not exist

  C. Peterson, *On tight 6-designs*, Osaka J. Math. **14** (1977), 417–435.

d=3

$t \leq 6$ (Ray-Chaudhuri, Wilson)
intersection numbers: $x < y < z$

- $t = 6$: do not exist

  C. Peterson, *On tight 6-designs*, Osaka J. Math. **14** (1977), 417–435.

- $t = 5$: the Witt 5-$(24, 8, 1)$ is the only one?

  Y. J. Ionin, M. S. Shrikhande, 5-*designs with three intersection numbers*, J. Combin. Theory Ser. A **69** (1995), no. 1, 36–50.

Association schemes

Definition

A (symmetric) association scheme with d classes is a set of undirected graphs $G_0, G_1, \dots, G_d$ with common set of n vertices X such that:

- graph G_0 contains all loops and no other edges,
- $G_1, \dots, G_d$ partition the complete graph K_n,
- for every edge $\{x, y\}$ in G_k, the number of vertices z such that $\{x, z\}$ is an edge in G_i and $\{z, y\}$ an edge in G_j depends only on indices i, j, k.

Theorem (Cameron, Delsarte, 1973)

The blocks of a design of degree d and strength $t \geq 2d - 2$ form a symmetric association scheme with d classes.

Designs of degree 3

- $t = 4$: table of admissible parameters for $v \leq 1000$ and an infinite family of admissible parameters

  V. Krčadinac, R. Vlahović Kruc, *Schematic 4-designs*, Discrete Math. **346** (2023), no. 7, Paper No. 113385, 7 pp.

Designs of degree 3

- $t = 4$: table of admissible parameters for $v \leq 1000$ and an infinite family of admissible parameters

  V. Krčadinac, R. Vlahović Kruc, *Schematic 4-designs*, Discrete Math. **346** (2023), no. 7, Paper No. 113385, 7 pp.

- $t = 3$: the theorem does not hold

$$3 < 2 \cdot 3 - 2 = 4$$

Computing the admissible parameters

- Ray-Chaudhuri, Wilson: $b \leq \binom{v}{d}$
- $v \leq 130$
- $0 \leq x < y < z < k$

Computing the admissible parameters

- Ray-Chaudhuri, Wilson: $b \leq \binom{v}{d}$
- $v \leq 130$
- $0 \leq x < y < z < k$
- more conditions?

Computing the admissible parameters

- Ray-Chaudhuri, Wilson: $b \leq \binom{v}{d}$
- $v \leq 130$
- $0 \leq x < y < z < k$
- more conditions?

Let n_1, n_2 and n_3 be the number of blocks intersecting some fixed block B_0 in x, y and z points, respectively.

$$\begin{cases} n_1 + n_2 + n_3 = b - 1 \\ x\, n_1 + y\, n_2 + z\, n_3 = k(r-1) \\ \binom{x}{2} n_1 + \binom{y}{2} n_2 + \binom{z}{2} n_3 = \binom{k}{2}(\lambda_2 - 1) \end{cases}$$

System matrix has $\neq 0$ determinant so it has unique solution.

Computing the admissible parameters

Numbers n_1, n_2, n_3 don't depend on the fixed block choice.

$$n_1 = \frac{(\lambda_2 - 1)k^2 + (y + z - r(y + z - 1) - \lambda_2)k + (b - 1)yz}{(y - x)(z - x)}$$

$$n_2 = \frac{(\lambda_2 - 1)k^2 + (x + z - r(x + z - 1) - \lambda_2)k + (b - 1)xz}{(x - y)(z - y)}$$

$$n_3 = \frac{(\lambda_2 - 1)k^2 + (x + y - r(x + y - 1) - \lambda_2)k + (b - 1)xy}{(x - z)(y - z)}$$

Computing the admissible parameters

Numbers n_1, n_2, n_3 don't depend on the fixed block choice.

$$n_1 = \frac{(\lambda_2 - 1)k^2 + (y + z - r(y + z - 1) - \lambda_2)k + (b - 1)yz}{(y - x)(z - x)}$$
$$n_2 = \frac{(\lambda_2 - 1)k^2 + (x + z - r(x + z - 1) - \lambda_2)k + (b - 1)xz}{(x - y)(z - y)}$$
$$n_3 = \frac{(\lambda_2 - 1)k^2 + (x + y - r(x + y - 1) - \lambda_2)k + (b - 1)xy}{(x - z)(y - z)}$$

Another condition because we have $t = 3$:

$$\binom{x}{3} n_1 + \binom{y}{3} n_2 + \binom{z}{3} n_3 = \binom{k}{3}(\lambda_3 - 1)$$

Table of admisible parameters

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	10	4	1	0	1	2	3	8	18
2	10	5	3	1	2	3	5	10	20
3	11	5	2	0	2	3	2	20	10
4	11	5	4	1	2	3	15	20	30
5	14	4	1	0	1	2	20	40	30
6	16	4	1	0	1	2	39	64	36
7	16	6	2	0	2	3	5	30	20
8	16	6	4	1	2	3	36	15	60
9	16	8	45	0	3	5	1	224	224
10	17	5	1	0	1	2	12	15	40
11	17	8	28	2	4	6	69	243	27
12	18	6	10	0	2	4	47	315	45
13	18	8	21	2	4	6	85	205	15
14	18	9	14	3	5	6	42	81	12
15	18	9	35	1	4	6	9	234	96

(...)

Table of admisible parameters

No.	v	k	λ	x	y	z	n_1	n_2	n_3
					(...)				
409	127	7	1	0	1	2	6360	2660	504
410	127	36	612	6	9	13	2574	15470	10530
411	127	36	6800	6	11	20	59280	253344	4875
412	127	63	1891	27	31	35	3472	7938	4464
413	128	4	1	0	1	2	75051	9920	372
414	128	8	1	0	1	2	3615	1920	560
415	128	16	5	0	2	4	567	1920	560
416	128	32	155	0	8	16	375	9920	372
417	128	56	660	16	24	30	360	6615	1152
418	128	56	660	20	24	28	1512	4095	2520
419	130	4	1	0	1	2	78813	10248	378
420	130	65	23940	27	33	40	30000	154375	11704

Table of admisible parameters

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	10	4	1	0	1	2	3	8	18
2	10	5	3	1	2	3	5	10	20
3	11	5	2	0	2	3	2	20	10
4	11	5	4	1	2	3	15	20	30
5	14	4	1	0	1	2	20	40	30
6	16	4	1	0	1	2	39	64	36
7	16	6	2	0	2	3	5	30	20
8	16	6	4	1	2	3	36	15	60
9	16	8	45	0	3	5	1	224	224
10	17	5	1	0	1	2	12	15	40
11	17	8	28	2	4	6	69	243	27
12	18	6	10	0	2	4	47	315	45
13	18	8	21	2	4	6	85	205	15
14	18	9	14	3	5	6	42	81	12
15	18	9	35	1	4	6	9	234	96

(...)

Steiner 3-designs

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	10	4	1	0	1	2	3	8	18
2	10	5	3	1	2	3	5	10	20
3	11	5	2	0	2	3	2	20	10
4	11	5	4	1	2	3	15	20	30
5	14	4	1	0	1	2	20	40	30
6	16	4	1	0	1	2	39	64	36
7	16	6	2	0	2	3	5	30	20
8	16	6	4	1	2	3	36	15	60
9	16	8	45	0	3	5	1	224	224
10	17	5	1	0	1	2	12	15	40
11	17	8	28	2	4	6	69	243	27
12	18	6	10	0	2	4	47	315	45
13	18	8	21	2	4	6	85	205	15
14	18	9	14	3	5	6	42	81	12
15	18	9	35	1	4	6	9	234	96

(...)

Steiner 3-designs

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	10	4	1	0	1	2	3	8	18
5	14	4	1	0	1	2	20	40	30
6	16	4	1	0	1	2	39	64	36
10	17	5	1	0	1	2	12	15	40
18	20	4	1	0	1	2	108	128	48
23	22	4	1	0	1	2	162	168	54
41	26	4	1	0	1	2	319	264	66
42	26	5	1	0	1	2	84	105	70
43	26	6	1	0	1	2	30	24	75
49	28	4	1	0	1	2	426	320	72
62	32	4	1	0	1	2	707	448	84

(. . .)

Steiner 3-designs

Proposition

Every Steiner 3-designs has intersection numbers in $\{0, 1, 2\}$.

E. Witt, *Über Steinersche Systeme*, Abh. Math. Semin. Univ. Hambg. **12** (1937), 265-275.

Theorem

For every prime power q *and every* $n \geq 2$ *exists* $3\text{-}(q^n + 1, q + 1, 1)$ *design with automorphism group* $PGL(2, q^n)$.

For special case of q even and $n = 2$ we get series of **schematic** 3-designs.

Table of admisible parameters

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	10	4	1	0	1	2	3	8	18
2	10	5	3	1	2	3	5	10	20
3	11	5	2	0	2	3	2	20	10
4	11	5	4	1	2	3	15	20	30
5	14	4	1	0	1	2	20	40	30
6	16	4	1	0	1	2	39	64	36
7	16	6	2	0	2	3	5	30	20
8	16	6	4	1	2	3	36	15	60
9	16	8	45	0	3	5	1	224	224
10	17	5	1	0	1	2	12	15	40
11	17	8	28	2	4	6	69	243	27
12	18	6	10	0	2	4	47	315	45
13	18	8	21	2	4	6	85	205	15
14	18	9	14	3	5	6	42	81	12
15	18	9	35	1	4	6	9	234	96

(...)

$x = 0,\ \lambda > 1$

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	10	4	1	0	1	2	3	8	18
2	10	5	3	1	2	3	5	10	20
3	11	5	2	0	2	3	2	20	10
4	11	5	4	1	2	3	15	20	30
5	14	4	1	0	1	2	20	40	30
6	16	4	1	0	1	2	39	64	36
7	16	6	2	0	2	3	5	30	20
8	16	6	4	1	2	3	36	15	60
9	16	8	45	0	3	5	1	224	224
10	17	5	1	0	1	2	12	15	40
11	17	8	28	2	4	6	69	243	27
12	18	6	10	0	2	4	47	315	45
13	18	8	21	2	4	6	85	205	15
14	18	9	14	3	5	6	42	81	12
15	18	9	35	1	4	6	9	234	96

(...)

$x = 0$, $\lambda > 1$

No.	v	k	λ	x	y	z	n_1	n_2	n_3
3	11	5	2	0	2	3	2	20	10
7	16	6	2	0	2	3	5	30	20
9	16	8	45	0	3	5	1	224	224
12	18	6	10	0	2	4	47	315	45
20	20	10	16	0	4	6	1	75	75
22	21	6	4	0	2	5	49	210	6
24	22	7	2	0	2	3	10	42	35
25	22	8	12	0	2	4	7	168	154
29	22	10	12	0	4	6	3	105	45
33	23	7	25	0	2	4	130	924	210
34	23	8	16	0	2	4	15	280	210
37	23	11	45	0	5	8	9	418	55

$(\dots)$

Quasi-symmetric designs

Proposition

Every 3-design with degree 3 and intersection number $x = 0$ is extension of quasi-symmetric 2-design.

$$3\text{-}(v, k, \lambda),\ x = 0,\ y,\ z \quad \xrightarrow{\text{der}} \quad 2\text{-}(v-1, k-1, \lambda),\ y-1,\ z-1$$

This eliminates parameters for which the corresponding parameters of quasi-symmetric designs are not admissible.

$AG_{m+1}(m+3, q)$

No.	v	k	λ	x	y	z	n_1	n_2	n_3
6	16	4	1	0	1	2	39	64	36
63	32	8	7	0	2	4	87	448	84
160	64	16	35	0	4	8	183	2240	180
416	128	32	155	0	8	16	375	9920	372

- the affine spaces over $\mathbb{F}_q$ with subspaces of codimension 2 as blocks
- generally, every pair of subspaces (of dimension $m+1$) is either parallel or intersects in a subspace of dimension $m-1$ or m $\rightsquigarrow$ 2-design

$$x = 0,\ y = q^{m-1},\ z = q^m$$

$AG_{m+1}(m+3, q)$

No.	v	k	λ	x	y	z	n_1	n_2	n_3
6	16	4	1	0	1	2	39	64	36
63	32	8	7	0	2	4	87	448	84
160	64	16	35	0	4	8	183	2240	180
416	128	32	155	0	8	16	375	9920	372

- the affine spaces over $\mathbb{F}_q$ with subspaces of codimension 2 as blocks
- generally, every pair of subspaces (of dimension $m+1$) is either parallel or intersects in a subspace of dimension $m-1$ or m $\rightsquigarrow$ 2-design

$$x = 0,\ y = q^{m-1},\ z = q^m$$

- $q = 2$: any 3 points are non-collinear $\rightsquigarrow$ 3-design

$$v = 2^{m+3}, \quad k = 2^{m+1}, \quad \lambda = \frac{1}{3}(2^m - 1)(2^{m+1} - 1)$$
$$x = 0, \quad y = 2^{m-1}, \quad z = 2^m$$

Table of admisible parameters

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	10	4	1	0	1	2	3	8	18
2	10	5	3	1	2	3	5	10	20
3	11	5	2	0	2	3	2	20	10
4	11	5	4	1	2	3	15	20	30
5	14	4	1	0	1	2	20	40	30
6	16	4	1	0	1	2	39	64	36
7	16	6	2	0	2	3	5	30	20
8	16	6	4	1	2	3	36	15	60
9	16	8	45	0	3	5	1	224	224
10	17	5	1	0	1	2	12	15	40
11	17	8	28	2	4	6	69	243	27
12	18	6	10	0	2	4	47	315	45

(...)

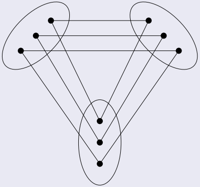
$v = 2^m$, $x > 0$?

LSSDs

Definition

Linked system of symmetric designs (LSSD) is a multipartite graph on $n = vr$ points with vertex set being partitioned into r sets of v vertices called *fibers*

$$X = X_1 \cup X_2 \cup \ldots \cup X_r,$$



satisfying these conditions:

- every edge has its ends in distinct fibers
- for all $1 \leq i, j \leq r$, $i \neq j$, the induced subgraph on $X_i \cup X_j$ is the incidence graph of some (v, k, λ) design
- there exist constants μ and ν such that for distinct $i, j, \ell \in \{1, \ldots, r\}$

$$x \in X_i, y \in X_j \implies |N(x) \cap N(y) \cap X_\ell| = \begin{cases} \mu & x \sim y \\ \nu & x \not\sim y \end{cases}.$$

LSSDs

Theorem (Noda's inequality)

If the LSSD$(v, k, \lambda; r)$ exists, then

$$(r-1)\left[(k-2)\lambda\binom{k}{3} - (v-2)\left[(v-k)\binom{\nu}{3} + k\binom{\mu}{3}\right]\right] \le$$
$$\le (v-2)\left[(v-1)\binom{\lambda}{3} + \binom{k}{3} - \left[(v-k)\binom{\nu}{3} + k\binom{\mu}{3}\right]\right].$$

The equality holds if and only if $(X_1, X_2 \cup \ldots \cup X_r)$ *is 3-design.*

$v = 2^m$, $m \geq 4$ even

Known series of examples:

P. J. Cameron, J. J. Seidel, *Quadratic forms over $GF(2)$*, Nederl. Akad. Wetensch. Proc. Ser. A **76** Indag. Math. **35** (1973), 1-8.

J.-M. Goethals, *Nonlinear codes defined by quadratic forms over $GF(2)$*, Information and Control **31** (1976), no. 1, 43-74.

Constructions use Kerdock sets and m has to be even.

$\rightsquigarrow$ 3-designs arising from LSSD-s for which equality holds

$\rightsquigarrow$ **schematic** designs

$$v = 2^m$$
$$k = 2^{m-1} - 2^{\frac{m-2}{2}}$$
$$\lambda = 2^{\frac{m-8}{2}} \left(2^{\frac{m}{2}} - 2\right) \left(2^m - 2^{\frac{m}{2}} - 4\right)$$

$$x = 2^{\frac{m-4}{2}} \left(2^{\frac{m}{2}} - 3\right)$$
$$y = 2^{\frac{m-4}{2}} \left(2^{\frac{m}{2}} - 2\right)$$
$$z = 2^{\frac{m-4}{2}} \left(2^{\frac{m}{2}} - 1\right)$$

$v = 2^m$, $m \geq 5$ odd

Kerdock sets in odd dimensions:

P. Delsarte, J. M. Goethals, *Alternating bilinear forms over $GF(q)$*, J. Combin. Theory Ser. A **19** (1975), 26-50.

J.-M. Goethals, *Nonlinear codes defined by quadratic forms over $GF(2)$*, Information and Control **31** (1976), no. 1, 43-74.

Not schematic

$$v = 2^m$$
$$k = 2^{m-1} - 2^{\frac{m-1}{2}}$$
$$\lambda = 2^{\frac{m-7}{2}} \left(2^{\frac{m-1}{2}} - 2\right) \left(2^m - 2^{\frac{m+1}{2}} - 2\right)$$

$$x = 2^{\frac{m-3}{2}} \left(2^{\frac{m-1}{2}} - 3\right)$$
$$y = 2^{\frac{m-3}{2}} \left(2^{\frac{m-1}{2}} - 2\right)$$
$$z = 2^{\frac{m-3}{2}} \left(2^{\frac{m-1}{2}} - 1\right)$$

Equidistant intersection numbers, $x > 0$

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	10	4	1	0	1	2	3	8	18
2	10	5	3	1	2	3	5	10	20
3	11	5	2	0	2	3	2	20	10
4	11	5	4	1	2	3	15	20	30
5	14	4	1	0	1	2	20	40	30
6	16	4	1	0	1	2	39	64	36
7	16	6	2	0	2	3	5	30	20
8	16	6	4	1	2	3	36	15	60
9	16	8	45	0	3	5	1	224	224
10	17	5	1	0	1	2	12	15	40
11	17	8	28	2	4	6	69	243	27
12	18	6	10	0	2	4	47	315	45
13	18	8	21	2	4	6	85	205	15
14	18	9	14	3	5	6	42	81	12
15	18	9	35	1	4	6	9	234	96

(...)

Equidistant intersection numbers, $x > 0$

No.	v	k	λ	x	y	z	n_1	n_2	n_3
2	10	5	3	1	2	3	5	10	20
4	11	5	4	1	2	3	15	20	30
8	16	6	4	1	2	3	36	15	60
11	17	8	28	2	4	6	69	243	27
13	18	8	21	2	4	6	85	205	15
16	18	9	56	3	5	7	165	351	27
27	22	8	30	1	3	5	145	574	105
31	22	10	48	2	4	6	45	360	210
32	22	11	72	3	5	7	55	396	220
38	23	11	120	3	5	7	165	792	330
40	25	9	42	1	3	5	135	744	270
51	29	8	20	1	3	5	555	714	35

$(\dots)$

Equidistant intersection numbers, $x > 0$

No.	v	k	λ	x	y	z	n_1	n_2	n_3
2	10	5	3	1	2	3	5	10	20
4	11	5	4	1	2	3	15	20	30
8	16	6	4	1	2	3	36	15	60
11	17	8	28	2	4	6	69	243	27
13	18	8	21	2	4	6	85	205	15
16	18	9	56	3	5	7	165	351	27
27	22	8	30	1	3	5	145	574	105
31	22	10	48	2	4	6	45	360	210
32	22	11	72	3	5	7	55	396	220
38	23	11	120	3	5	7	165	792	330
40	25	9	42	1	3	5	135	744	270
51	29	8	20	1	3	5	555	714	35

$(\ldots)$

More (sporadic) examples?

Non-equidistant intersection numbers

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	10	4	1	0	1	2	3	8	18
2	10	5	3	1	2	3	5	10	20
3	11	5	2	0	2	3	2	20	10
4	11	5	4	1	2	3	15	20	30
5	14	4	1	0	1	2	20	40	30
6	16	4	1	0	1	2	39	64	36
7	16	6	2	0	2	3	5	30	20
8	16	6	4	1	2	3	36	15	60
9	16	8	45	0	3	5	1	224	224
10	17	5	1	0	1	2	12	15	40
11	17	8	28	2	4	6	69	243	27
12	18	6	10	0	2	4	47	315	45
13	18	8	21	2	4	6	85	205	15
14	18	9	14	3	5	6	42	81	12
15	18	9	35	1	4	6	9	234	96

(...)

Non-equidistant intersection numbers

No.	v	k	λ	x	y	z	n_1	n_2	n_3
14	18	9	14	3	5	6	42	81	12
15	18	9	35	1	4	6	9	234	96
17	19	9	56	3	5	6	270	315	60
19	20	8	42	2	4	5	376	414	64
21	20	10	28	3	4	6	20	105	140
26	22	8	18	1	2	4	32	224	238
28	22	10	6	2	4	5	10	10	56
30	22	10	18	1	4	6	8	150	72
35	23	8	56	1	2	4	160	840	770
36	23	9	12	2	4	5	84	126	42

(...)

Non-equidistant intersection numbers

No.	v	k	λ	x	y	z	n_1	n_2	n_3
14	18	9	14	3	5	6	42	81	12
15	18	9	35	1	4	6	9	234	96
17	19	9	56	3	5	6	270	315	60
19	20	8	42	2	4	5	376	414	64
21	20	10	28	3	4	6	20	105	140
26	22	8	18	1	2	4	32	224	238
28	22	10	6	2	4	5	10	10	56
30	22	10	18	1	4	6	8	150	72
35	23	8	56	1	2	4	160	840	770
36	23	9	12	2	4	5	84	126	42

$(\dots)$

Trying to eliminate these?

To sum up...

- computing admissible parameters for $v \leq 130$
- identifying few series of existing examples
- more existing series or sporadic examples?
- eliminating sets of parameters with non-equidistant intersection numbers?

The End

Thank you for your attention!