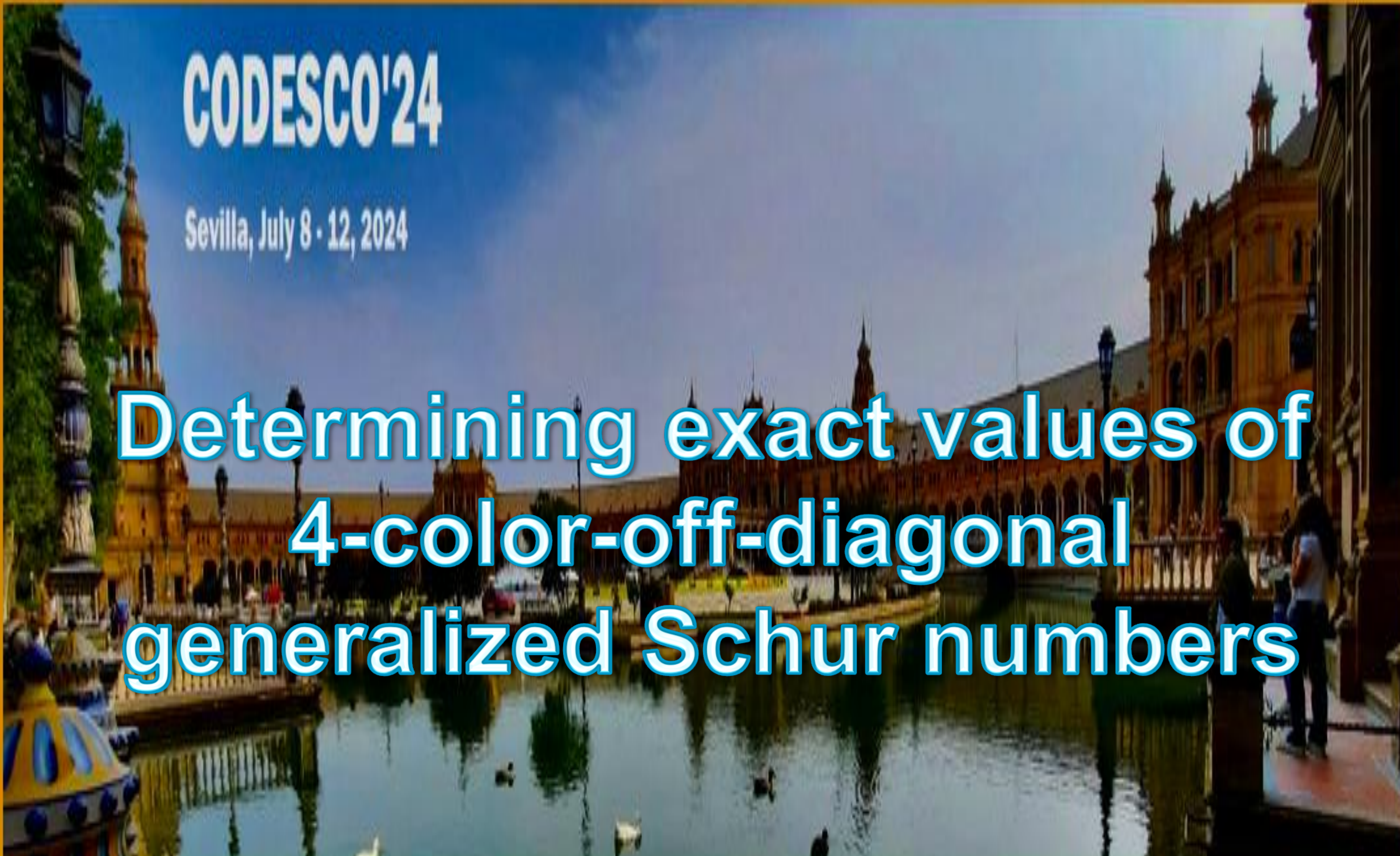

Determining exact values of 4-color-off-diagonal generalized Schur numbers

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- Schur numbers and Rado numbers

- Weak Schur numbers and weak Rado numbers

- Off-diagonal generalized Schur numbers and weak Schur numbers

- New results

Given a set A and a positive integer *n:*

A finite **n-coloring** *of A is a function*

$$\Delta : A \rightarrow \{1,2, \ldots, n\}$$

Equivalently, *it is a partition of A into n disjoint subsets*

$$A = A_1 \cup A_2 \cup \ldots \cup A_n$$

The equivalence is given by $A_i = \Delta^{-1}(i)$

That is, A_i *is the* ***monochromatic*** *subset of color* ***i***

Definition: *A set B of integers is* ***sumfree*** if the equation $x_1+x_2= x_3$ *has no solution in B*

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Counterexamples*: B'={1,2} or B"={2,3,5}*

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can we partition A={1,2,…,N} into n sumfree subsets?

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Question*: Given positive integers N, n,*
can we partition A={1,2,…,N} into n sumfree subsets?

Theorem *(Schur, 1916): The answer is* **no** if N is too large with respect to n.

***Examples**:*

- *$N=4$, $n=2$. $\{1,2,3,4\}=\{1,4\}\cup\{2,3\}$*

- *$N=5$, $n=2$. Impossible to partition $\{1,2,3,4,5\}$ in two sumfree subsets.*

***Examples**:*

- *N=4, n=2. {1,2,3,4}={1,4}∪{2,3}*

- *N=5, n=2. Impossible to partition {1,2,3,4,5} in two sumfree subsets.*

***Definition**: The **Schur number** S(n) is the least N such that the set {1,2,…..,N} **cannot be partitioned** into n sumfree subsets.*

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***Example**: S(2)=5*

Schur numbers and Rado numbers

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1		

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1	2	

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2		

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Example: *S(2)=5*

1		
2	**3**	

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1		
2	3	

1	3	4
2	4	

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2	3	5

Equivalently, S(n) is the least integer N, such that for all n-colouring Δ : {1,2,…,N} → {1,2, …,n}, there exists a monochromatic solution to $x_1+x_2= x_3$

Schur numbers and Rado numbers

For ***n=3***, we have ***S(3)=14***

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1	4	10	13	
2	3	11	12	
5	6	7	8	9

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Equivalently, for every 3-coloring of the set {1,2,…,14}, there exists a monochromatic solution to $x_1 + x_2 = x_3$

1	4	10	13	
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2	3	11	12		14
5	6	7	8	9	14

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Here, instead of equation $x_1+x_2=x_3$, we now consider the more general equation ***E(k,c):***

$$x_1+x_2+\ldots+x_k+c = x_{k+1}$$

where k,c are integers with k positive and c nonnegative.

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where *k,c* are integers with *k* positive and *c* nonnegative.

Definition: The ***Rado number*** $\mathbf{R_k}$ **(*n,c*)** *is the* ***least*** *integer* ***N,*** *such that, for every n-coloring of the set* **{1,2,…,*N*}**, *there exists a monochromatic solution to the equation* ***E(k,c).***

If there is no such *N*, set $\mathbf{R_k}$ **(*n,c*)** = ∞

Rado (1933) considered systems of linear Diophantine equations and the existence of monochromatic solutions thereof.

Here, instead of equation $x_1+x_2=x_3$, we now consider the more general equation ***E(k,0):***

$$x_1+x_2+\ldots+x_k+0 = x_{k+1}$$

where *k,c* are integers with *k* positive.

Definition: The ***Schur number*** ***R_k* (*n*,0)** *is the* ***least*** *integer* ***N****, such that, for every n-coloring of the set* **{1,2,…,*N*}**, *there exists a monochromatic solution to the equation* ***E(k,0).***

Rado (1933) considered systems of linear Diophantine equations and the existence of monochromatic solutions thereof.

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$$x_1+x_2+\ldots+x_k= x_{k+1}$$

where *k,c* are integers with *k* positive.

Definition: The ***Schur number*** S_k **(*n*)** *is the **least** integer **N**, such that, for every n-coloring of the set* **{*1,2,…,N*}**, *there exists a monochromatic solution to the equation **E(k).***

Results $R_k(n,0)=S_k(n)$

k	2	3	4	5	*k*
n=2	**5**	**11**	**19**	**29**	K^2+k-1
n=3	**14**	**43**	**94**	**173**	K^3+2k^2-2
n=4	**45**				
n=5	**161**				
n=6	**≥537**				
n=7	**≥1681**				

Baumert, 1961 ***Beutelspacher and Brestovansky, 1982***
Exo, 1994 ***Radziszowski, 1999*** ***Fredricksen and Sweet, 2000***
Boza, Marín, Revuelta and Sanz, 2010, 2014, 2016, 2019
Heule, 2018

Results on $R_k(n,c)$, $c>1$

k	2	3	4	k
$n=2$	$4c+5$			$(k+1)^2+(c-1)(k+2)$
$n=3$	$13c+14$	$21c+43$	$31c+94$	
$n=4$	$\geq 40c+41$ $\leq 44c+45$			
$n=5$	$\geq 121c+122$ $\leq 305c+306$			

Burr and Loo, 1992

Schaal, 1993,1995

Adhikari, Boza, Eliahou, Marín, Revuelta, Sanz, 2018

When is $R_k(n,c)$ *finite?*

When is $R_k(n,c)$ finite?

Conjecture (Adhikari, Boza, Eliahou, Marín, R, Sanz):
$\boldsymbol{R_k}$ **(*n*, *c*)** is finite **if and only if** every divisor $d \leq n$ of k-1 also divides c ($k \geq 2$, $n \geq 1$, $c \geq 0$).

- **True for $k \leq 7$**
- **Open for k ≥ 8**

Adhikari, Boza, Eliahou, Marín, Revuelta, Sanz, 2020

We now consider the equation ***E'(k,c):***

$$x_1+x_2+\ldots+x_k+c = x_{k+1}$$

$$x_i \neq x_j \;\; \forall\, i \neq j$$

Note: Every solution of ***E'(k,c)*** is a solution of ***E(k,c)***

Definition: The ***weak Rado number*** $WR_k(n,c)$ *is the* ***least*** *integer* ***N,*** *such that for every n-coloring of the set* **{1,2,…,*N*},** *there exists a monochromatic solution of the equation* ***E'(k,c)***.
If there is no such *N*, set $WR_k(n,c) = \infty$

Note: $R_k(n,c) \leq WR_k(n,c)$

We now consider the equation ***E'(k,0):***

$$x_1+x_2+\ldots+x_k+0 = x_{k+1}$$

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We now consider the equation ***E'(k):***

$$x_1+x_2+\ldots+x_k = x_{k+1}$$

$$x_i \neq x_j \quad \forall\, i \neq j$$

Note: Every solution of ***E'(k)*** is a solution of ***E(k)***

Definition: The ***weak Schur number*** $WS_k(n)$ *is the* ***least*** *integer* ***N****, such that for every n-coloring of the set* **{*1,2,…,N*}**, *there exists a monochromatic solution of the* equation ***E'(k)****.*

Results W$R_k(n,0)$ = W$S_k(n)$

Results $WR_k(n,0) = WS_k(n)$

k	2	3	4	5	*k*
n=2	9				
n=3	24				
n=4	67				
n=5	197?				
n=6					

Blanchard, Harary and Reis, 2006

Walker in 1952, claimed the value $WR_2(5,0) = 197$, without proof.

Results $WR_k(n,0) = WS_k(n)$

k	2	3	4	5	*k*
n=2	9	24	52	101	
n=3	24				
n=4	67				
n=5	≥197				
n=6	≥583				

Blanchard, Harary and Reis, 2006

Walker in 1952, claimed the value $WR_2(5,0) = 197$, without proof.

Eliahou, Marín, Revuelta, Sanz, 2013

Eliahou et al., 2013

Boza, Marín, Revuelta, Sanz, 2019

$WR_k(n,c)$, $c \geq 1$

k	2	3	4	5	*k*
$n=2$	$4c+8$	$5c+24$	$6c+52$	$7c+95$	Conjecture
$n=3$	$13c+22$				
$n=4$	$\geq 39c+62$				
$n=5$					

Boza, Marín, Revuelta, Sanz, 2020

Conjecture: $WR_k(2\ c) = (k+2)c+(k+2)(k+1)k/2-2k$

We now consider the system ***E(k1,k2,…,kr):***

$$x_1+x_2+\ldots+x_{k1} = x_{k1+1}$$

$$x_1+x_2+\ldots+x_{k2} = x_{k2+1}$$

$$\ldots\ldots$$

$$x_1+x_2+\ldots+x_{kr} = x_{kr+1}$$

Definition: The ***r-color off-diagonal generalized Schur number*** ***S*****(r; k1,k2,…,kr)** *is the* ***least*** *integer* ***M,*** *such that any r-coloring of the integer interval [1, M]***,** *must contain a r-colored solution to the system E(k1,k2, …, kr).*

Off-diagonal generalized Schur numbers

- These numbers are given their name because of their similarity to the classical off-diagonal Ramsey numbers.

- In dinamic survey of Radziszowski, the following is stated

  $$R(k1,\ldots, kr) > S(r; k1,\ldots, kr)-2$$

- In 2001, Robertson and Schaal determined all values of the 2-color off-diagonal Schur numbers, $S(2; k1, k2)$.

- Not much progress has been made since then due to the great difficulty of calculating these numbers.

- Advances in the computation of these numbers may be relevant considering their relation to the off-diagonal Ramsey numbers, mentioned above, given by Radziszowski.

Off-diagonal generalized Schur numbers

$$\begin{cases} E_2 : x_1 + x_2 = x_3, \\ E_2 : x_1 + x_2 = x_3, \\ E_k : x_1 + \ldots + x_k = x_{k+1}, \ k \geq 2 \end{cases}$$

Theorem (Ahmed, Boza, R, Sanz, 2024)

For all k >= 2, we have the exact values of 3-color off-diagonal generalized Schur number S(2,2,k)

$$S(2,2,k) = \begin{cases} 9k-4, & \text{si } k \notin 1+5N \\ 9k-5, & \text{si } k \in 1+5N \end{cases}$$

We now consider the system ***E'(k1,k2,…,kr):***

$$x_1+x_2+\ldots+x_{k1} = x_{k1+1}$$

$$x_1+x_2+\ldots+x_{k2} = x_{k2+1}$$

$$\ldots\ldots$$

$$x_1+x_2+\ldots+x_{kr} = x_{kr+1}$$

$$x_i \neq x_j \quad \forall\, i \neq j$$

Definition: The ***r-color off-diagonal generalized weak Schur number*** ***WS*****(2; k1,k2,…,kr)** *is the* ***least*** *integer* ***M,*** *such that any r-coloring of the integer interval [1, M]***,** *must contain a r-colored solution to the system E'(k1,k2, …, kr).*

Off-diagonal generalized weak Schur numbers

WS(a,b)	*b* = 2	3	4	5	6	7	8	9	10
a = 2	9	16	23	37	53	71	93	119	≥147
3		24	39	49	66	87	111	138	≥168
4			52	76	93	118	150	≥186	≥226
5				101	130	156	≥190	≥235	≥285
6					166	≥204	≥241	≥285	≥345
7						253	≥303	≥351	≥409

WS(2,a,b)	*b* = 2	3	4	5	6
a = 2	24	42	64	102	148
3		64	105	138	194
4			151	204	

WS(4,a,b)	*b* = 4
a = 4	259

WS(3,3,a,b)	*b* = 3
a = 3	≥369

WS(2,3,a,b)	*b* = 3
a =3	≥279

WS(3,a,b)	*b* = 3	4	5
a = 3	94	141	188
4		189	

WS(2,2,a,b)	*b* = 2	3	4
a = 2	67	117	≥177
3		≥183	

New Results: 4-color off-diagonal generalized Schur numbers

S(2,2,a,b)	a=2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
b=2	45	77	107	137	175	203	231	261	301	329	357	385	427	455	483	511
3		101	143	155	180	207	244	269	308	332	372	394	436			
4			174	221	244	274	311	347	393	423						
5				262	323	349	391	437								
6					372	437										

S(2,3,a,b)	a=3	4	5	6	7	8	9	10	11
b=3	135	191	197	235	265	317	349	399	431
4		239	285	331	379	426			
5			311	373	409				
6				432					

S(2,4,a,b)	a=4	5	6	7
b=4	296	358	409	467
5		454	501	

S(2,5,a,b)	a=5
b=5	539

Lower Bounds

We will partition the integer interval [1, M] into r subsets A_1, A2,…, Ar containing no a r-colored solution to the system E(k1,k2, …, kr).

Upper Bounds

Reformulation as a Boolean satisfiability problem, which can then be handled by a SAT solver.

- *Express the combinatorial constraints as Boolean satisfiability problems.*
- *We then use a SAT solver to determine whether the logical system is satisfiable or not.*

S(2,2,2,17) = 511

{1,4,10,13,15,18,24,27,29,32,38,41,43,46,52,55,57,60,66,69,71,74,83,85,88,99,102,113,116,127,130,141,144,158,189,203,217,231,238,245,252,259,266,273,280,294,308,322,353,367,370,381,384,395,398,409,412,423,426,428,437,440,442,445,451,454,456,459,465,468,470,473,479,482,484,487,493,496,498,501,507,510},

{2,3,11,12,16,17,25,26,30,31,39,40,44,45,53,54,58,59,67,68,72,73,81,82,86,87,95,96,100,101,109,110,114,115,123,124,128,137,
142,147,151,156,165,170,175,179,184,193,207,304,318,327,332,336,341,346,355,360,364,369,374,383,387,388,396,397,401,402,410,411,415,416,424,425,429,430,438,439,443,444,452,453,457,458,466,467,471,472,480,481,485,486,494,495,499,500,508,509},

[5,9]U[19,23]U[33,37]U[47,51]U[61,65]U[75,79]U[89,93]U[104,107]U[118,121]U[132,135]U{148,149,161,162,176,190,321,335,349,350,362,363}
U[376,379]U[390,393]U[404,407]U[418,422]U[432,436]U[446,450]U[460,464]U[474,478]U[488,492]U[502,506],

{14,28,42,56,70,80,84,94,97,98,103,108,111,112,117,122,125,126,129,131,136}U[138,140]U{143,145,146,150}
U[152,155]U{157,
159,160,163,164}U[166,169]U[171,174]U{177,178}U[180,183]U[185,188]U{191,192}
U[194,202]U[204,206]U[208,216]U[218,230]U[232,237]U[239,244]U[246,251]U[253,258]U[260,265]U[267,272]U
[274,279]U[281,293]U[295,303]U[305,307]U[309,317]U{319,
320}U[323,326]U[328,331]U{333,334}U[337,340]U[342,345]U{347,348,351,352,354}
U[356,359]U{361,365,366,368}U[371,373]U
{375,380,382,385,386,389,394,399,400,403,408,413,414,417,427,431,441,455,469,483,497}.

$S(2,2,3,14) = 436$

$J_1[1,39]U\{82,86,90,94,98,102,106\}$
$UJ_0[110,136]U\{140,144,148,152,156,160,190,367,371,375,379,383,387\}UJ_1[391,395]U$
$\{399,403,407,411,415,419,423,427,431,435\}$,

$\{2,6,10,14,18,22,26,41,45,49,53,56,57,60,61,64,65,68,72,76,80,84,88,92,95,96,99,100,103,104,107,108,111,115,119,123,$
$127,131,135,138,139,142,143,146,147,150,154,158,162,166,170,174,178,182,186,363,397,401,$
$405,406,409,410,413,414,$
$417,418,421,422,425,426,429,430,433,434\}$,

$\{121,125,129,133,137,141,145\}$
$UJ_1[149,161]U[163,165]U[167,169]U[171,173]U[175,177]U[179,181]U[183,185]U[187,189]U[191,362]U[364,366]U[368,370]U[372,374]U[376,378]U[380,382]U[384,386]U[388,390]UJ_0[392,408]U$
$\{412,416,420,424,428,432\}$,

$\{4,8,12,16,20,24\}UJ_0[28,40]U[42,44]U[46,48]U[50,52]U\{54,55,58,59,62,63,66,67\}$
$U[69,71]U[73,75]U[77,79]UJ_1[81,93]U\{97,101,105,109,113,117\}$.

$S(2,3,3,11) = 431$

$J_1[1,9] \cup J_0[22,42] \cup J_1[93,113] \cup J_0[318,338] \cup J_0[389,409] \cup J_0[422,430]$,

$J_1[31,41] \cup [43,92] \cup J_0[94,104] \cup J_1[327,337] \cup [339,388] \cup J_0[390,400]$,

$J_0[106,112] \cup [114,317] \cup J_1[319,325]$,

$J_0[2,8] \cup [10,21] \cup J_1[23,29] \cup J_0[402,408] \cup [410,421] \cup J_1[423,429]$.

$S(2,3,4,8) = 426$

$\{1,3,8\} \cup J_0[44,50] \cup J_0[376,382] \cup \{418,423,425\}$,

$[10,39] \cup [387,416]$,

$\{2\} \cup [4,7] \cup \{9\} \cup [40,43] \cup \{45,381\} \cup [383,386] \cup \{417\} \cup [419,422] \cup \{424\}$,

$\{47,49\} \cup [51,375] \cup \{377,379\}$.

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