Some constructions of strongly regular graphs and digraphs

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The outline of the talk:

1. Introduction and motivation
2. Main construction
3. Results

A **finite incidence structure** consists of a finite set $\mathcal{V}$, called points, a set $\mathcal{B}$ of subsets of $\mathcal{V}$, called blocks, and the incidence relation $\in$ (containment) between points and blocks.

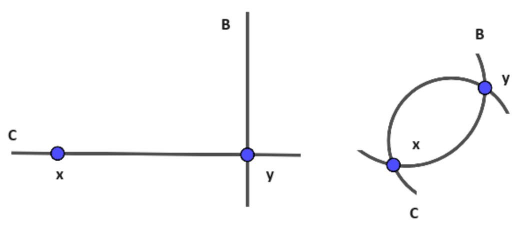
An incident point–block pair is called a **flag**, and a non–incident point–block pair is called an **antiflag**.

A tactical configuration with parameters (v, b, k, r) is a finite incidence structure $(\mathcal{V}, \mathcal{B})$ with $|\mathcal{V}| = v$, $|\mathcal{B}| = b$ such that every block contains k points and every point belongs to exactly r blocks.

Example: a 1-(v, k, λ) design with b blocks is a tactical configuration with parameters $(v, b, k, r = \lambda)$.

- R. C. Bose, S. S. Shrikhande, N. M. Singhi, Edge regular multigraphs and partial geometric designs, *Proc. Internat. Colloq. Combin. Theory*, 17 (1976), 49–81.
- A. Neumaier, $t\frac{1}{2}$–designs, *J. Combin. Theory, Ser. A*, 28 (1980), 226–248.

A partial geometric design or a $1\frac{1}{2}$**–design** with parameters $(v, b, k, r; \alpha, \beta)$ is a tactical configuration $(\mathcal{V}, \mathcal{B})$ with parameters (v, b, k, r) such that for every point $x \in \mathcal{V}$ and every block $B \in \mathcal{B}$, the number of flags (y, C) such that $y \in B \setminus \{x\}$, $x \in C \neq B$ equals α or β, according as $x \notin B$ or $x \in B$.



- Examples: 2-designs, partial geometries, complete bipartite graphs,...
- Remark: $t\frac{1}{2}$-design, for $t \geq 4$ is $(t+1)$–design.

- R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs. *Pacific J. Math.* **13** (1963), 389–419.

A **partial geometry** with parameters s, t, α, or shorty, $pg(s, t, \alpha)$, is a pair (P, L) of a set P of **points** and a set L of **lines**, with an incidence relation between points and lines, satisfying the following axioms:

1. A pair of distinct points is not incident with more than one line.
2. Every line is incident with exactly $s+1$ points ($s \geq 1$).
3. Every point is incident with exactly $t+1$ lines ($t \geq 1$).
4. For every point p not incident with a line l, there are exactly α lines ($\alpha \geq 1$) which are incident with p, and also incident with some point incident with l.

A partial geometry $pg(s,t,\alpha)$ is a $1\frac{1}{2}$-design with parameters

$$(v,b,k,r;\alpha',\beta) = ((s+1)c,(t+1)c,s+1,t+1;\alpha,r+k-1),$$

where $c = 1 + \frac{s\cdot t}{\alpha}$.

- D. Crnković, A. Švob, V. D. Tonchev, Strongly regular graphs with parameters (81,30,9,12) and a new partial geometry, J. Algebraic Combin. 53 (2021), 253–261.

A pg(5,5,2) gives rise to $1\frac{1}{2}$-design with parameters $(81,81,6,6;2,11)$.

- W. G. Bridges, M. S. Shrikhande, Special partially balanced incomplete block designs and associated graphs, *Discrete Math.*, 9 (1974), 1–18.

A special partially balanced incomplete block design (SPBIBD) with parameters $(v, b, k, r, \lambda_1, \lambda_2)$ of type (α_1, α_2), with $v, b, r, k \geq 2$, $\lambda_1, \lambda_2, \alpha_1, \alpha_2 \geq 0$, $\lambda_1 \neq \lambda_2$ and $r < b$, is a tactical configuration with parameters (v, b, k, r) such that

(i) Two distinct points are either in exactly λ_1 (when they are λ_1–associated) or in exactly λ_2 common blocks (when they are λ_2–associated).

(ii) A point x is λ_1–associated to exactly α_1 points of a block B if $x \in B$, and to α_2 points of B if $x \notin B$.

A SPBIBD is called **quasi–symmetric** if any two distinct blocks have either μ_1 or μ_2, $\mu_1 \neq \mu_2$, points in common.

A strongly regular graph (SRG) Γ with parameters (v, k, λ, μ) is a (connected, simple, undirected and loopless) k–regular graph with v vertices such that any two adjacent vertices have λ common neighbours and any two non–adjacent vertices have μ common neighbours.

Remark: If Γ is a strongly regular graph, then $V(\Gamma)$ will denote the set of its vertices.

A special class of SRGs

Let G be a group of permutations acting on a set Ω.

The **rank** of the action is the number of orbits of the subgroup G_x fixing $x \in \Omega$ on Ω.

The orbits of G on $\Omega \times \Omega$ are called **orbitals** and they are symmetric if for all $x, y \in \Omega$ the pairs (x, y) and (y, x) belong to the same orbital.

Let G be transitive of rank three. Then its orbitals, say $I = \{(x, x) \mid x \in \Omega\}, R, S$, are symmetric if and only if G has even order. In this case (Ω, R) and (Ω, S) form a pair of complementary strongly regular graphs, called **rank three strongly regular graph**.

In particular, they are connected if and only if G is primitive and the group G acts transitively on ordered pairs of adjacent vertices and on ordered pairs of non–adjacent vertices of each of these graphs.

- A. Duval, A directed graph version of strongly regular graphs, *J. Combin. Theory, Ser. A*, 47 (1988), 71–100.

A directed strongly regular graph with parameters (v, k, t, λ, μ) is a directed graph on v vertices without loops such that

(i) every vertex has in-degree and out-degree k,

(ii) every vertex x has t out-neighbours that are also in-neighbours of x,

(iii) the number of directed paths of length 2 from a vertex x to another vertex y is λ if there is an edge from x to y, and is μ if there is no edge from x to y.

- A. E. Brouwer, O. Olmez, S. Y. Song, Directed strongly regular graphs from $1\frac{1}{2}$–designs, *European J. Combin.*, 33 (2012), 1174–1177.

Directed strongly regular graphs can be **constructed** from **partial geometric designs**.

A partial geometric design with parameters $(v, b, k, r; \alpha, \beta)$ gives rise to two distinct DSRGs having parameters:

$$(b(v-k), r(v-k), kr-\alpha, kr-(k+r-1+\beta), kr-\alpha),$$
$$(vr, rk-1, \beta+r+k-2, \beta+r+k-3, \alpha).$$

We will consider **proper** partial geometric design, i.e., the design for which $\alpha > 0$, $3 \leq k \leq v-3$ and $3 \leq r \leq b-3$.

The pg(5,5,2) gives rise to $1\frac{1}{2}$–design with parameters $(81, 81, 6, 6; 2, 11)$.

The $1\frac{1}{2}$–design with parameters $(81, 81, 6, 6; 2, 11)$ gives rise to DSRGs:

$$(6075, 450, 34, 14, 34), (486, 35, 21, 20, 2).$$

Main idea

- We show that a SRG that has a "nice family" of subsets i.e. intriguing sets gives rise to SPBIBD.
- SPBIBDs form particular class of partial geometric designs.
- A partial geometric design with parameters gives rise to DSRGs.
- We apply the construction on rank three SRGs on at most 45 vertices.

SRGs

Let Γ be a strongly regular graph with parameters (v, k, λ, μ), and A be the adjacency matrix of Γ.

- The matrix A satisfies the equation $A^2 = kI + \lambda A + \mu(J - I - A)$, where I denotes the identity matrix of order v and J the all–ones matrix of order v.
- if A is a $v \times v$ matrix and there exist non–negative integers k, λ, μ such that $A^2 = kI + \lambda A + \mu(J - I - A) = (\lambda - \mu)A + (k - \mu)I + \mu J$, then A can be seen as the adjacency matrix of a strongly regular graph.
- **The matrix A has three distinct eigenvalues**: $\theta_0 > \theta_1 > \theta_2$, where $\theta_0 = k$, $\theta_1 = \left(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right)/2$ and $\theta_2 = \left(\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right)/2$.

- The matrices $A_0 := I, A_1 := A, A_2 := J - I - A$ are symmetric and they pairwise commute.
- $A_i A_j = \sum_{k=0}^{2} p_{ij}^k A_k$, where $p_{0j}^k = \delta_{j,k}, p_{11}^0 = k, p_{11}^1 = \lambda, p_{11}^2 = \mu, p_{12}^0 = 0, p_{12}^1 = k - \lambda - 1, p_{12}^2 = k - \mu, p_{22}^0 = v - k - 1, p_{22}^1 = v - 2k + \lambda, p_{22}^2 = v - 2k + \mu - 2$.
- The matrices A_0, A_1, A_2 are linearly independent, they generate a commutative 3–dimensional algebra $\mathcal{A}$ consisting of real symmetric matrices, called Bose–Mesner algebra of Γ.
- $\mathcal{A}$ admits a basis $\{E_0, E_1, E_2\}$, of so called *minimal idempotents*, where $E_i E_j = \delta_{i,j} E_i$ and $E_0 + E_1 + E_2 = I$.

$$E_0 = \frac{1}{v} J,$$
$$E_1 = \frac{1}{\theta_1 - \theta_2} \left(A - \theta_2 I - \frac{k - \theta_2}{v} J \right),$$
$$E_2 = \frac{1}{\theta_2 - \theta_1} \left(A - \theta_1 I - \frac{k - \theta_1}{v} J \right).$$

- P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Rep. Suppl.*, No. 10 (1973), 97 pp.

A subset $\mathcal{I}$ of vertices of strongly regular graph Γ, $0 < |\mathcal{I}| < v$, is said to be **intriguing** with parameters (h_1, h_2) if there exist constants h_1 and h_2 such that every vertex of $\mathcal{I}$ is adjacent to precisely h_1 vertices of $\mathcal{I}$ and every vertex of $V(\Gamma) \setminus \mathcal{I}$ is adjacent to precisely h_2 vertices of $\mathcal{I}$.

- J. Bamberg, F. De Clerck, N. Durante, Intriguing sets in partial quadrangles, *J. Combin. Des.*, 19 (2011), 217–245.
- B. De Bruyn, H. Suzuki, Intriguing sets of vertices of regular graphs, *Graphs Combin.*, 26 (2010), 629–646.
- B. De Bruyn, Intriguing sets of points of $\mathcal{Q}(2n, 2) \setminus \mathcal{Q}^+(2n-1, 2)$, *Graphs Combin.*, 28 (2012), no. 6, 791–805.

If $\mathcal{I}$ is intriguing with parameters (h_1, h_2), then $(h_1 - h_2 - k)\boldsymbol{j}_{\mathcal{I}} + h_2\boldsymbol{j}$ is an eigenvector of the adjacency matrix A with the eigenvalue $h_1 - h_2$.

Here and in the sequel $\boldsymbol{j}$ denotes the $v \times 1$ all–ones vector, $\mathbf{0}$ the $v \times 1$ all–zeros vector and $\boldsymbol{j}_{\mathcal{I}}$ the $v \times 1$ characteristic vector of $\mathcal{I}$.

Hence, either $h_1 - h_2$ is θ_1 and $\mathcal{I}$ is said to be **a positive intriguing set** or $h_1 - h_2$ is θ_2 and $\mathcal{I}$ is said to be **a negative intriguing set**.

For an intriguing set $\mathcal{I}$, we have that $|\mathcal{I}| = \frac{h_2 v}{k-\theta_i}$, where i equals 1 or 2 according as $\mathcal{I}$ is positive or negative, respectively.

Note that the complement of an intriguing set is an intriguing set of the same type; the union of two disjoint intriguing sets of the same type is an intriguing set of the same type; if A and B are intriguing sets of the same type and $A \subseteq B$, then $B \setminus A$ is an intriguing set of the same type.

Moreover, if Γ^c denotes the complement of Γ and $\mathcal{I}$ is a (positive or negative) intriguing set of Γ, then $\mathcal{I}$ is a (negative or positive) intriguing set of Γ^c.

Proposition

A self–complementary strongly regular graph has a positive intriguing set of size x if and only if it has a negative intriguing set of size x.

An equivalent definition of an intriguing set is the following:

Definition

$\mathcal{I}$ is a positive intriguing set of Γ if $E_2\boldsymbol{j}_{\mathcal{I}} = \mathbf{0}$, and $\mathcal{I}$ is a negative intriguing set of Γ if $E_1\boldsymbol{j}_{\mathcal{I}} = \mathbf{0}$.

Remark:
Since both h_1, h_2 are non–negative integers, the definition of an intriguing set does not make sense if Γ is a conference graph with non–integral eigenvalues.

SPBIBDs

Let $\mathcal{D}$ be a SPBIBD with parameters $(v, b, k, r, \lambda_1, \lambda_2)$ of type (α_1, α_2). Let $\Gamma_{\mathcal{D}}$ be the graph having as vertices the points of $\mathcal{D}$, where two distinct vertices are adjacent whenever the corresponding points of $\mathcal{D}$ are λ_1–associated.

- W. G. Bridges, M. S. Shrikhande, Special partially balanced incomplete block designs and associated graphs, *Discrete Math.*, 9 (1974), 1–18.

Lemma

The graph $\Gamma_{\mathcal{D}}$ is strongly regular.

Lemma

The block graph of a quasi-symmetric SPBIBD is strongly regular.

- SPBIBDs form a particular class of partial geometric designs.

Lemma

A SPBIBD with parameters $(v, b, k, r, \lambda_1, \lambda_2)$ of type (α_1, α_2) is a partial geometric design with parameters
$(v, b, k, r; \alpha_2(\lambda_1 - \lambda_2) + k\lambda_2, \alpha_1(\lambda_1 - \lambda_2) + (k-1)(\lambda_2 - 1))$.

The converse situation:

- R. C. Bose, S. S. Shrikhande, N. M. Singhi, Edge regular multigraphs and partial geometric designs, *Proc. Internat. Colloq. Combin. Theory*, 17 (1976), 49–81.
- E. R. van Dam, E. Spence, Combinatorial designs with two singular values. II. Partial geometric designs, *Linear Algebra Appl.*, 396 (2005), 303–316.

Proof:

Let x be a point and B a block. We count the number N of flags (y, C) such that $x \in C, y \in B$, with $y \neq x$ and $C \neq B$.

Assume first that $x \notin B$. Let $y \in B$ such that there are exactly λ_1 blocks containing both x, y; then y can be chosen in α_2 ways. The remaining $k - \alpha_2$ elements of B are λ_2–associated with x. Hence $N = \lambda_1\alpha_2 + (k - \alpha_2)\lambda_2 = k\lambda_2 + \alpha_2(\lambda_1 - \lambda_2)$.

Assume that $x \in B$. Let $y \in B$ such that there are exactly $\lambda_1 - 1$ blocks distinct from B and containing both x, y; then y can be chosen in α_1 ways. The remaining $k - \alpha_1 - 1$ elements of B are λ_2–associated with x. Then $N = (\lambda_1 - 1)\alpha_1 + (k - \alpha_1 - 1)(\lambda_2 - 1) = \alpha_1(\lambda_1 - \lambda_2) + (k - 1)(\lambda_2 - 1)$.

Main construction

- D. Crnković, F. Pavese, A. Švob, Intriguing sets of strongly regular graphs and their related structures, Contrib. Discrete Math. 18 (2023), 66–89.
- **SRG** $\rightarrow$ **Intriguing set** $\rightarrow$ **SPBIBD** $\rightarrow$ partial geometric design $\rightarrow$ DSRG

Theorem

Let Γ be a strongly regular graph and let $\mathcal{F}$ be a family of subsets of $V(\Gamma)$ such that

1) all elements of $\mathcal{F}$ have that same number z of elements, $0 < z < |V(\Gamma)|$;
2) there exist constants λ_i, $0 \leq i \leq 2$, such that $\forall x, y \in V(\Gamma)$, $d(x,y) = i$, then $\lambda_i = |\{\mathcal{I} \in \mathcal{F} \mid \{x,y\} \subset \mathcal{I}\}|$.

Then $(V(\Gamma), \mathcal{F})$ is a SPBIBD with parameter $(|V(\Gamma)|, |\mathcal{F}|, z, \lambda_0, \lambda_1, \lambda_2)$ of type $\left(\theta_i + \frac{k-\theta_i}{|V(\Gamma)|}z, \frac{k-\theta_i}{|V(\Gamma)|}z\right)$ if and only if $\mathcal{F}$ consists of intriguing sets of Γ with parameters $\left(\theta_i + \frac{k-\theta_i}{|V(\Gamma)|}z, \frac{k-\theta_i}{|V(\Gamma)|}z\right)$.

Theorem

Let Γ be a strongly regular graph admitting a rank three automorphism group G and let $\mathcal{I} \neq V(\Gamma)$ be a non–empty subset of vertices of Γ. Then $(V(\Gamma), \mathcal{I}^G)$ is a SPBIBD with parameters $(|V(\Gamma)|, b, k, r, r_1, r_2)$ of type $(\theta_i + h_2, h_2)$, with $b = |G|/|G_{\mathcal{I}}|$, $k = |\mathcal{I}|$, if and only if $\mathcal{I}$ is an intriguing set of Γ with parameters $(\theta_i + h_2, h_2)$.

Proof:

The group G has three orbits on $V(\Gamma) \times V(\Gamma)$, namely I, R, S, where $x, y \in V(\Gamma)$, $x \neq y$, are adjacent if and only if $(x, y) \in R$.

Let $\mathcal{I} \neq V(\Gamma)$ be a non–empty subset of vertices of Γ, hence $0 < |\mathcal{I}| = k < |V(\Gamma)|$, and let $b = |G|/|G_{\mathcal{I}}|$. Then each of the incidence structures $(I, \mathcal{I}^G)$, $(R, \mathcal{I}^G)$ and $(S, \mathcal{I}^G)$ is a tactical configuration.

Therefore, through a vertex of Γ there pass a constant number of elements of $\mathcal{I}^G$, say r, and through two distinct vertices x, y of Γ there pass either r_1 or r_2 elements of $\mathcal{I}^G$, according as x is adjacent to y or not. The result follows from Theorem.

Results

We consider a primitive rank three group G of even order and the strongly regular graph Γ obtained from one of its orbitals.

If Γ has at most 40 vertices, we completely classify its intriguing sets and compute the corresponding DSRGs.

Some partial results are obtained for Γ having 45 vertices. Most of them have a large number of vertices.

The Paley graph SRG$(25, 12, 5, 6)$

There are three rank three groups: $5^2 : Q(12)$, $5^2 : 12$, $3^2 : D(8) = Aut(\Gamma)$.

The eigenvalues of Γ are 2 and -3, and Γ has one positive and one negative intriguing set of size 5, both stabilized by a subgroup of $Aut(\Gamma)$ of order 40.

There are also two positive and two negative intriguing sets of size 10, invariant under a subgroup of $Aut(\Gamma)$ of order 6 and 20, respectively.

The corresponding DSRGs have parameters

$$(300, 60, 13, 8, 13), (1500, 600, 260, 210, 260), (1000, 399, 189, 188, 140),$$
$$(450, 180, 78, 63, 78), (300, 119, 56, 55, 42).$$

$\boxed{\mathrm{SRG}(36, 14, 4, 6)}$

In this case $G = \mathrm{P\Gamma U}(3, 9) = Aut(\Gamma)$ and $\theta_1 = 2, \theta_2 = -4$.

Concerning positive intriguing sets, there is one example of size 6 left invariant a group of order 96, four types of size 12 fixed by a group of order 6, 16, 24 and 192, respectively, and eight examples of size 18, four of which are fixed by a group of order 6, two by a group of order 12 and the remaining two by a group of order 24 and 216, respectively.

The corresponding DSRGs have parameters

$(3780, 630, 110, 80, 110), (756, 125, 45, 44, 16), (48384, 16128, 5632, 4864, 5632),$
$(24192, 8063, 3199, 3198, 2432), (18144, 6048, 2112, 1824, 2112),$
$(36288, 18144, 9504, 8640, 9504), (36288, 18143, 9503, 9502, 8640),$
$(18144, 9072, 4752, 4320, 4752), (18144, 9071, 4751, 4750, 4320),$
$(12096, 4032, 1408, 1216, 1408), (9072, 3023, 1199, 1198, 912), (6048, 2015, 799, 798, 608),$
$(1512, 504, 176, 152, 176), (9072, 4536, 2376, 2160, 2376), (756, 251, 99, 98, 76),$
$(9072, 4535, 2375, 2374, 2160), (1008, 504, 264, 240, 264), (1008, 503, 263, 262, 240).$

As for negative intriguing sets there is one example of size 12, admitting an automorphism group of order 192 and one example of size 18 fixed by a group of order 108.

The related DSRGs have parameters

$$(1512, 504, 180, 144, 180), (756, 251, 107, 106, 72),$$
$$(2016, 1008, 540, 468, 540), (2016, 1007, 539, 538, 468).$$

- A. E. Brouwer, D. Crnković, A. Švob, A construction of directed strongly regular graphs with parameters (63,11,8,1,2), Discrete Math. 347 (2024), 114146, 3 pages.

Theorem [D. Crnković, V. Mikulić Crnković, AŠ, 2014]

Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$ is the stabilizer of α and $\delta_1, \ldots, \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits on Ω_2. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, \ldots, \delta_s) = (\Omega_2, \mathcal{B})$ is a 1-$(n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|} \sum_{i=1}^{s} |\alpha G_{\delta_i}|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G / \bigcap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, transitively on points and blocks of the design.
If $\Delta_2 = \Omega_2$ then the set $\mathcal{B}$ consists of one block, and $\mathcal{D}(G, \alpha, \delta_1, \ldots, \delta_s)$ is a design with parameters 1-$(n, n, 1)$.

The construction described in Theorem gives us all simple designs on which the group G acts transitively on the points and blocks, *i.e.* if G acts transitively on the points and blocks of a simple 1-design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in Theorem.

Note that the construction from Theorem gives us 1-designs, and **the incidence matrices of some of these 1-designs may be the adjacency matrices of directed strongly regular graphs**.

Since the construction given in Theorem gives all designs having G as an automorphism group acting transitively on points and blocks, it gives us also **all directed strongly regular graphs admitting a transitive action** of the set of vertices.

Clearly, the adjacency matrix of a directed strongly regular graph with parameters (n, k, t, λ, μ) is the incidence matrix of a 1-(n, k, k) design. In that way, the neighbourhoods of a directed strongly regular graph correspond to the blocks of a design, where the neighbourhood of a vertex x is the set of all vertices y such that there is an arc $x \to y$.

The linear group $PSL(2, 8)$ is the simple group of order 504 and up to conjugation it has exactly one subgroup of order 8, which is isomorphic to the elementary abelian group E_8.

By taking $G = PSL(2, 8)$ and $G_\alpha = E_8$, we constructed two non-isomorphic directed strongly regular graphs with parameters $(63, 11, 8, 1, 2)$, both having $PSL(2, 8) : Z_3$ as the full automorphism group.

Theorem [A. E. Brouwer, D. Crnković, AŠ, 2024]

Up to isomorphism, there are exactly two directed strongly regular graphs with parameters $(63, 11, 8, 1, 2)$ on which the linear group $PSL(2, 8)$ acts transitively. These directed strongly regular graphs have $PSL(2, 8) : Z_3$ as the full automorphism group.

Muchas gracias por su atención!