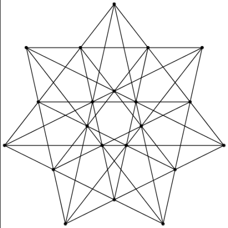

Contextual configurations

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Joint work with Adán Cabello and Petr Lisoněk

CODESCO’24

Background

Hypergraphs and Configurations

Hypergraph: $H = (V, E)$

- set of vertices (points) V
- set of hyperedges (lines) E
- $e \in E$ (multi)set of vertices from V, if **proper hypergraph** each is a set

Eulerian hypergraph: each vertex has even degree

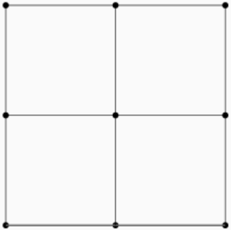
Configuration: (v_k, b_ℓ) is a hypergraph with:

- v vertices (points) each in k hyperedges (lines)
- b hyperedges (lines) each containing ℓ vertices (points)

Our general setting is **Eulerian hypergraphs**, but all "nice" examples are **configurations** with k **even**

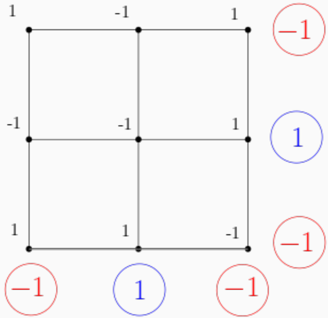
Puzzle

Label the points of the following configuration with either 1 or -1 so that an odd number of lines have product -1.



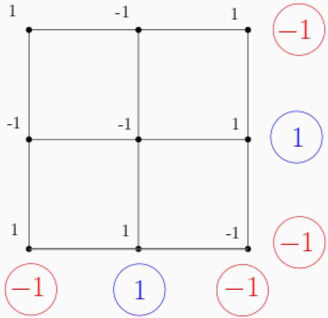
Puzzle

Label the points of the following configuration with either 1 or -1 so that an odd number of lines have product -1.



Puzzle

Label the points of the following configuration with either 1 or -1 so that an odd number of lines have product -1.



This is impossible since each point is in an even number of lines!

Contextuality

- Einstein-Podolski-Rosen 1935: Hidden variable model
- Kochen-Specker 1967: First proof of contextuality (measurements depend on context!)
- Mermin 1993: provides simple proof of contextuality via a contextual configuration
- M. Howard, J. Wallman, V. Veitch, J. Emerson, Contextuality supplies the ‘magic’ for quantum computation. Nature **510** (2014), 351-355.

Magic (contextual) assignments

$H = (V, E)$ - Hypergraph with each vertex even degree (*Eulerian* hypergraph)
$\alpha : V \to GL(\mathcal{H})$ - Assignment

1. $\alpha(v)^2 = I$ and $\alpha(v)$ is Hermitian for all $v \in V$.
2. $\alpha(v)\alpha(w) = \alpha(w)\alpha(v)$ whenever v, w are in a common hyperedge $e \in E$.
3. $\prod_{v \in e} \alpha(v) = \pm I$ for each hyperedge $e \in E$.
4. $\prod_{v \in e} \alpha(v) = -I$ for an odd number of hyperedges $e \in E$.

Magic: α satisfies 1-4

Valid: α satisfies 1-3, but maybe not 4

Pauli-based: $\alpha(v) \in \mathcal{P}_k$ for each $v \in V$. We say α is a k-qubit assignment.

Magic (contextual) assignments

Goal: Given proper Eulerian hypergraph (or configuration) H, check if H has magic assignment α. If so, we say that H is *magic*.

Why: Classical assignments (those given by hidden variable models, i.e classical physics) CANNOT satisfy 1-4 because of a parity argument.

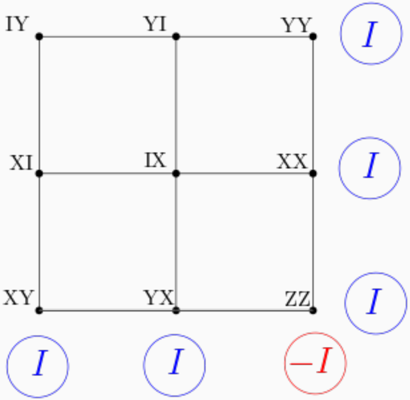
Therefore: The pair (H, α) is a proof of contextuality.

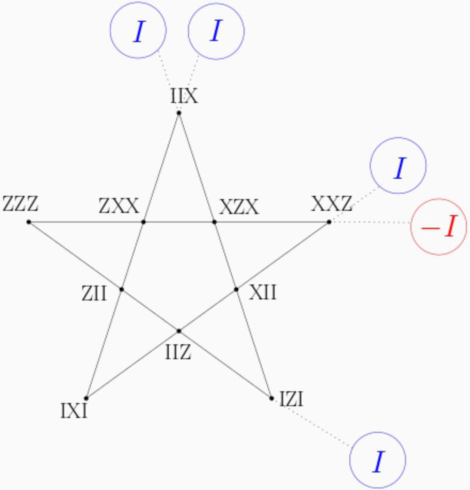
Pauli matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},\ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- I, X, Y, Z Hermitian
- $I^2 = X^2 = Y^2 = Z^2 = I$
- X, Y, Z pairwise anti-commute: (i.e $XY = -XY$, etc)
- $XYZ = iI$
- $\mathcal{P}_k$ is set of k-fold tensor products of I, X, Y, Z
    - elements of $\mathcal{P}_k$ are also Hermitian and square to the identity
    - easy to multiply and check commutativity “qubit-wise”
    - notation: $X \otimes Y \otimes I \to XYI$

Mermin square & Peres-Mermin Pentagram





2-regular hypergraphs

Theorem (Arkhipov 2012)

Let $H = (V, E)$ be a 2-regular hypergraph. Then H is magic if and only if the dual graph of H is non-planar.

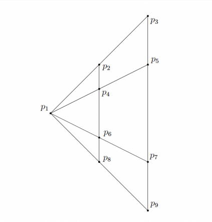
- Every magic 2-regular hypergraph can be “reduced” to square or pentagram. Dual of square is $K_{3,3}$, dual of pentagram is K_5.
- Every magic 2-regular hypergraph has a Pauli-based assignment with 2 or 3 qubits
- Only the square and pentagram cannot be labeled using either **repeated** operators or **identity** operators (i.e only square and pentagram are *irreducible*)

Extending Arkhipov's result

Motivation

When we began this work:

- the **only known** irreducible Eulerian hypergraphs in the literature were the **square** and **pentagram**.
- it was not known whether there was an Eulerian hypergraph necessitating more than **3** qubits
- given an Eulerian hypergraph H there was **no known algorithm** to check if H has a Pauli-based assignment
- Assignments were found from **ad-hoc methods** or **external constructions**



Our contribution

Developed an algorithm to:

- check if an Eulerian hypergraph H admits a Pauli-based assignment
- compute minimum number of qubits (can be computationally expensive)
- iterate through different Pauli-based assignments
- check if H is irreducible or not
- if H is reducible, can “farm” it to create irreducible instances

We have now found:

- over 8000 irreducible hypergraphs (necessitating from 3 to 6 qubits)
- only 4 vertex-transitive cases
- each of the 4 is a configuration!

We have also done an exhaustive search of (v_4, b_ℓ) configurations with $v \leq 20$ points, $\ell \in \{4, 5\}$

Gram matrices

Given assignment $\alpha : V \to \mathcal{P}_n$, the $|V| \times |V|$ Gram matrix M records commutativity:

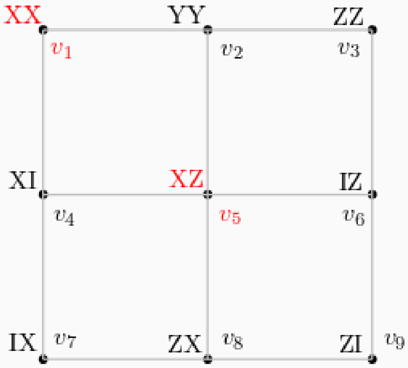
$$M_{i,j} = \begin{cases} 0 & \text{if } \alpha(v_i), \alpha(v_j) \text{ commute} \\ 1 & \text{if } \alpha(v_i), \alpha(v_j) \text{ do not commute} \end{cases}$$

We say that α *respects M*.

Many assignments respect the same Gram matrix

Question: What does Gram matrix M tell us about the assignments α respecting it?

Gram matrices (II)



	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	0	1	1
2	0	0	0	1	0	1	1	0	1
3	0	0	0	1	1	0	1	1	0
4	0	1	1	0	0	0	0	1	1
5	1	0	1	0	0	0	1	0	1
6	1	1	0	0	0	0	1	1	0
7	0	1	1	0	1	1	0	0	0
8	1	0	1	1	0	1	0	0	0
9	1	1	0	1	1	0	0	0	0

$\alpha(v_1) = XX$ and $\alpha(v_5) = XZ$ do not commute, so $M_{1,5} = 1$

Gram matrices (III)

Main idea. Gram matrices capture everything we need!

- computing valid assignments
  ***valid Gram matrix* binary rank $2k \iff \alpha$ k-qubit valid assignment**
- valid Gram matrices are easy to compute
  form subspace of $\mathbb{R}^{v \times v}$ that we call *valid Gram space*
- can check if Pauli-based magic assignment exists respecting Gram matrix
  affine subspace of valid Gram space.
  These Gram matrices are *magic Gram matrices*.
- compute number of qubits
  binary rank
- explicitly compute magic assignments
  embed corresponding graph in symplectic graph

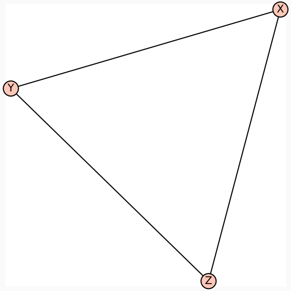
Constructing magic assignments?

Main question: how do we actually generate the k-qubit magic assignments once we've found a magic Gram matrix of binary rank $2k$ in the valid Gram space of H?

Symplectic graphs

Symplectic graph $SP(2k)$:

- vertices are non-identity k-qubit Pauli operators
- edge between pair of vertices if corresponding Pauli operators do not commute



Reduced graph :

- no isolated (degree 0) vertices
- no pair of vertices has same neighbourhood

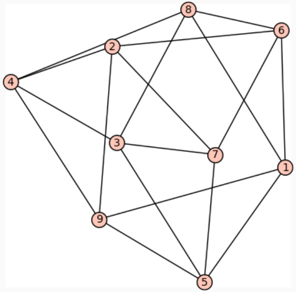
Theorem (Godsil/Royle 2001)

If a graph G is reduced and its adjacency matrix has binary rank at most $2k$ for some $k \in \mathbb{Z}_{\geq 0}$, then G is an induced subgraph of $SP(2k)$.

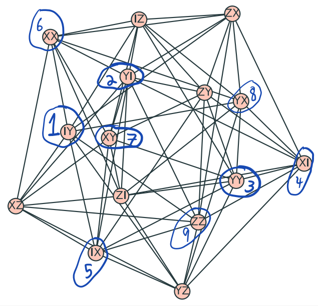
Magic Gram matrices to magic assignments

	1	2	3	4	5	6	7	8	9
1	0	0	0	0	1	1	0	1	1
2	0	0	0	1	0	1	1	0	1
3	0	0	0	1	1	0	1	1	0
4	0	1	1	0	0	0	0	1	1
5	1	0	1	0	0	0	1	0	1
6	1	1	0	0	0	0	1	1	0
7	0	1	1	0	1	1	0	0	0
8	1	0	1	1	0	1	0	0	0
9	1	1	0	1	1	0	0	0	0

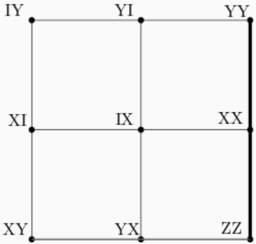
compute G



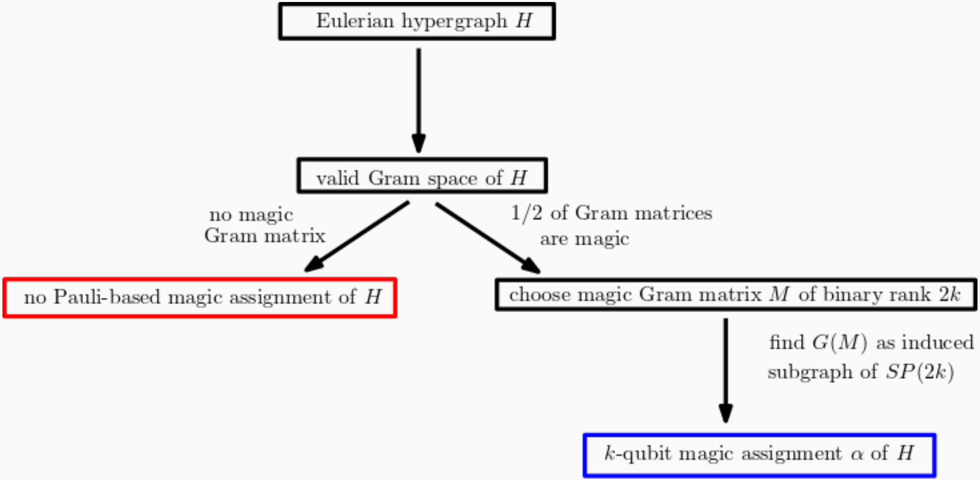
find G as induced subgraph of $SP(2k)$



translate to H



Algorithm overview



Planarity

Note that this (along with Arkhipov's result) implicitly defines a linear algebraic algorithm for checking planarity and producing K_5 or $K_{3,3}$ minor.

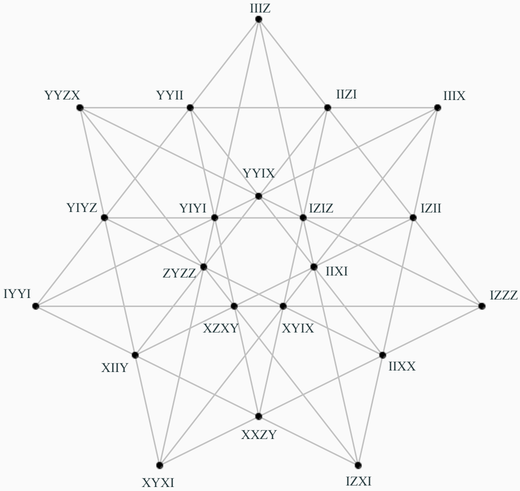
Graph $G \rightarrow$ dual (2-regular) hypergraph $H \rightarrow$ set of magic Gram matrices.

One of the magic Gram matrices encodes the $K_{3,3}$ or K_5 minor.

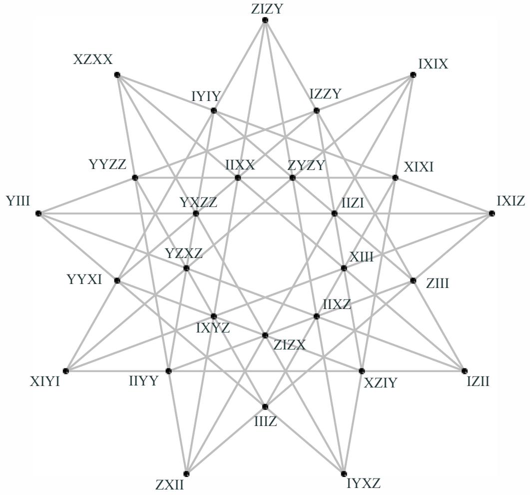
This forms a potentially interesting link between Algebraic Graph Theory and Topological Graph Theory.

New irreducible hypergraphs (configurations)

MS4-21

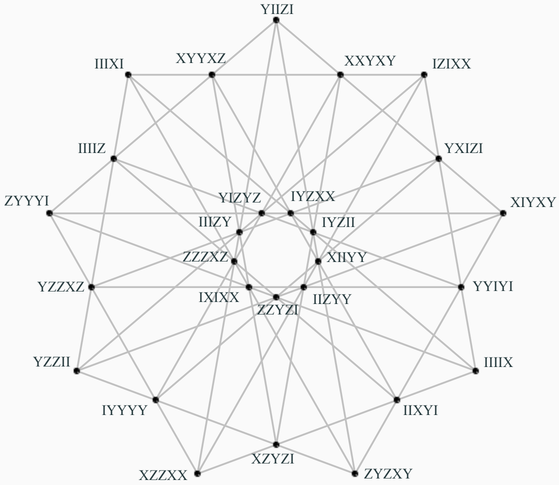


- Grünbaum-Rigby (or Klein) configuration
- (21_4)-configuration
- is self-dual
- needs 4 qubits

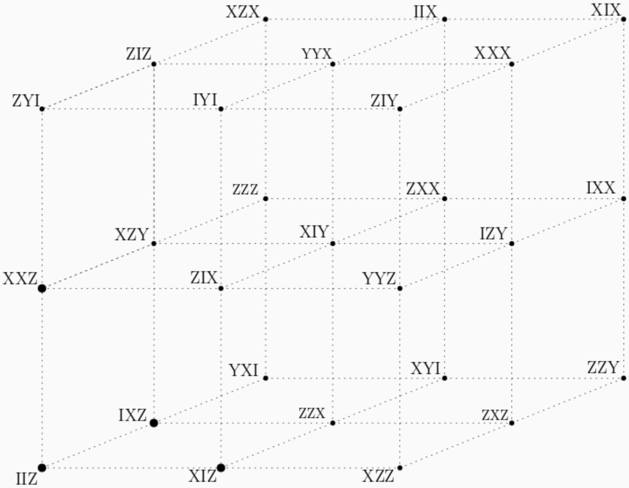


- 3-astral, 4-configuration (Fig. 3.7.2(b) of Grünbaum *Configurations of points and lines*)
  We thank T. Pisanski for pointing this out to us
- (27_4)-configuration
- is self-dual
- needs 4 qubits

MS5-27



- smallest known weakly flag-transitive configuration (described in Marušič, Pisanski *Weakly flag-transitive configurations and half-arc-transitive graphs*)
  We thank T. Pisanski for pointing this out to us
- (27_4)-configuation
- is self-dual
- needs 5 qubits



- lines given by "shifting" starter line via translation

  $T_{a,b,c} : \mathbb{Z}_3^3 \to \mathbb{Z}_3^3$

  $(x, y, z) \to (x + a, y + b, z + c)$

- (27_4)-configuration
- is self-dual
- needs 3 qubits
- **have not found in literature**

Notable non vertex transitive irreducible hypergraphs

n	Magic set	Observables	Contexts	b/Q	ε
3	MS3-15	$10_4 + 5_6$	$10_3 + 10_4$	14/20	0.3
3	MS3-18	$3_2 + 15_4$	$6_3 + 12_4$	12/18	0.33
3	MS3-27b	27_4	27_4	17/27	0.37
3	MS3-29	$27_4 + 2_{12}$	33_4	19/33	0.424
4	MS4-20	$5_2 + 15_4$	$6_3 + 13_4$	17/19	0.105
4	MS4-21b	$11_2 + 10_4$	$2_3 + 14_4$	14/16	0.125
4	MS4-21c	$1_2 + 19_4 + 1_6$	21_4	19/21	0.095
4	MS4-24	$3_2 + 11_4 + 9_6 + 1_{10}$	$2_3 + 12_4 + 12_5$	20/26	0.23
5	MS5-26	$25_4 + 1_{10}$	$10_3 + 20_4$	24/30	0.2
5	MS5-29	$23_4 + 5_6 + 1_8$	$6_3 + 28_4$	28/34	0.176
5	MS5-31	$3_2 + 23_4 + 2_6 + 3_8$	$2_3 + 12_4 + 16_5$	24/30	0.2
6	MS6-35	$30_4 + 5_8$	$3_3 + 14_4 + 19_5$	30/36	0.167

Open problems

Open problems

Find new irreducible contextual configurations

(i.e irreducible magic Eulerian hypergraphs that are configurations)

Infinite family of irreducible contextual configurations?

Would be even more interesting if minimum number of qubits $\to \infty$

Thank you!!

If you have any configurations that you think may be good candidates, please send them to me at **strandafir@us.es**

Stefan Trandafir, Petr Lisoněk, and Adán Cabello.
Irreducible magic sets for n-qubit systems.
Physical Review Letters, 129(20):200401, 2022.