LCD codes related to some combinatorial structures

Ivona Traunkar
(inovak@math.uniri.hr)
joint work with Vedrana Mikulić Crnković

Faculty of Mathematics, University of Rijeka

CODESCO'24
Seville, Spain

This work has been supported by Croatian Science Foundation under the project 4571 and by the University of Rijeka under the project uniri-iskusni-prirod-23-155.

UNIVERSITY OF RIJEKA
FACULTY OF MATHEMATICS

Codes

We will talk only about linear codes, i.e. subspaces of the ambient vector space over a field $\mathbb{F}_q$ of order $q = p^l$, where p is prime.
A code C of length n and dimension k is denoted by $[n, k]$.
By $[n, k, d]_q$, we denote code C with minimal distance d over the field $\mathbb{F}_q$.
Specially, if $q = 2$, parameters of code C are denoted by $[n, k, d]$.

A generator matrix of a $[n, k]$ code C is a $k \times n$ matrix whose rows generate all the words of C.

The dual code of a code C is code $C^\perp$, $C^\perp = \{v \in (\mathbb{F}_q)^n \mid (v, c) = 0,\ \forall c \in C\}$.

A code is self-orthogonal if $C \subseteq C^\perp$. A code is self-dual if $C = C^\perp$.
A code is LCD (Linear code with complementary dual) if $C \cap C^\perp = \{\mathbf{0}\}$.

A code C with generator matrix G is LCD code if and only if $\det(G \cdot G^T) \neq 0$.

UNIVERSITY OF RIJEKA
FACULTY OF MATHEMATICS

Weakly self-orthogonal designs

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is called a $t-(v,k,\lambda)$ design, if $\mathcal{P}$ contains v points, every block $B \in \mathcal{B}$ is incident with k points, and every t distinct points are incident with λ blocks.

The incidence matrix of a design is a $b \times v$ matrix $[m_{ij}]$ where b and v are the numbers of blocks and points respectively, such that $m_{ij} = 1$ if the point P_j and the block B_i are incident, and $m_{ij} = 0$ otherwise.

A design is weakly p-self-orthogonal (p-WSO) if all block intersection numbers gives the same residue modulo p.
A weakly p-self-orthogonal design is p-self-orthogonal if block intersection numbers and the block sizes are multiples of p.

Specially, weakly 2-self-orthogonal design is called weakly self-orthogonal (WSO) design, and 2-self-orthogonal design is called self-orthogonal.

Examples of some known families of p-WSO

- Symmetric designs

1. Point-hyperplane designs.
   $v = \dfrac{(q^{m+1} - 1)}{q - 1}, \ k = \dfrac{q^m - 1}{q - 1}, \ \lambda = \dfrac{q^{m-1} - 1}{q - 1}, \ q = p^l$ a prime power and $m \geqslant 2$.
   These designs are weakly p-self-orthogonal ($k, \lambda \equiv 1 \mod p$)

2. Menon designs.

$$v = 4t^2, k = 2t^2 - t, \lambda = t^2 - t.$$

   These designs are p-self-orthogonal for every prime p dividing t.

   A design with this parameters exists if and only if there exists a regular Hadamard matrix of order $4t^2$. It is conjectured that these designs exist for all values of t. The incidence matrix of a Menon design is given by $M = \frac{1}{2}(J_{4t^2} - H)$, and H is a regular Hadamard matrix in which the sum of every row is equal to $2t$.

Examples of some known families of p-WSO

- Quasi-symmetric designs

1. Blokhuis and Haemers constructed an infinite family of quasi-symmetric 2-$(q^3, q^2(q-1)/2, q(q^3-q^2-2)/4)$ designs with block intersection numbers $q^2(q-2)/4$ and $q^2(q-1)/4$, where q is a power of 2.
   For $q > 2$ these designs are self-orthogonal (k and block intersection numbers are even).

2. V. Krčadinac, R. Vlahović. New quasi-symmetric designs by the Kramer-Mesner method, Discrete Math. 339 (2016), no. 12, 2884-2890.

   The authors found many new quasi-symmetric 2-$(28, 12, 11)$ and 2-$(36, 16, 12)$ designs. For example, the 2-$(28, 12, 11)$ quasi-symmetric design with intersection numbers 4 and 6 is self-orthogonal.

Examples of some known families of p-WSO

- Strongly regular graphs

1. The adjacency matrix of a $SRG(n, k, \lambda, \mu)$ such that $k \equiv a \mod p$, $\lambda = \mu \mod p$ is the incidence matrix of a p-WSO.

2. The triangular graph $T(n)$, whose vertices are 2-element subsets of an n-elements set, two pairs being adjacent if and only if they have an element in common.
   $T(n)$ is a strongly regular graph with parameters $\left(\binom{n}{2}, 2(n-2), n-2, 4\right)$.

   If $n > 2$ is even, the adjacency matrix of $T(n)$ is the incidence matrix of a self-orthogonal 1-design (k and the block intersection numbers are even)

Orbit matrices of a design

Let $\mathcal{D}$ be a 1-(v,k,λ) design and G be an automorphism group of the design. Let $v_1 = |\mathcal{V}_1|, \ldots, v_n = |\mathcal{V}_n|$ be the sizes of point orbits and $b_1 = |\mathcal{B}_1|, \ldots, b_m = |\mathcal{B}_m|$ be the sizes of block orbits under the action of the group G. We define an orbit matrix as $m \times n$ matrix

$$O = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix},$$

where a_{ij} is the number of points of the orbit $\mathcal{V}_j$ incident with a block of the orbit $\mathcal{B}_i$. It is easy to see that the matrix is well-defined and that $k = \sum_{j=1}^{n} a_{ij}$.

For $x \in \mathcal{B}_s$, by counting the incidence pairs (P, x') such that $x' \in \mathcal{B}_t$ and P is incident with the block x, we obtain

$$\sum_{x' \in \mathcal{B}_t} |x \cap x'| = \sum_{j=1}^{m} \frac{b_t}{v_j} a_{sj} a_{tj} = \frac{b_t}{v_j} O[s] \cdot O[t],$$

where $O[s]$ is the s-th row of the matrix O.

Let $\mathcal{D}$ be a weakly p-self-orthogonal design such that

$$k \equiv a \pmod{p}$$

and

$$|B_i \cap B_j| \equiv d \pmod{p},$$

for all $i, j \in \{1, \dots, b\},\ i \neq j$, where B_i and B_j are two blocks of a design $\mathcal{D}$.

Let G be an automorphism group of the design which acts on $\mathcal{D}$ with n point orbits of length w and block orbits of length $b_1, b_2, \dots, b_m$, and let O be an orbit matrix of a design $\mathcal{D}$ under the action of a group G.

Let $q = p^n$ be prime power and let $\mathbb{F}_q$ be a finite field of order q.
In $\mathbb{F}_q$, for $x \in \mathcal{B}_s,\ s \neq t$ it follows that

$$\frac{b_t}{w} O[s] \cdot O[t] = b_t d, \tag{1}$$

$$\frac{b_s}{w} O[s] \cdot O[s] = a + (b_s - 1)d. \tag{2}$$

Self-orthogonal codes from p-WSO designs

Mikulić Crnković, V., Traunkar, I. Self-orthogonal codes constructed from weakly self-orthogonal designs invariant under an action of M_{11}. AAECC 34, 139–156 (2023). https://doi.org/10.1007/s00200-020-00484-2

Let $\mathbb{F}_q$ be a finite field of order $q = p^l$, where p is a prime.

- Construction of SO codes obtained from incidence matrix of p-WSO designs, using suitable extensions of incidence matrix of a design.
- Construction od SO codes obtained from orbit matrix of p-WSO designs under the action of group G which acts on design with n point orbits of length w and m block orbits of length w, using suitable extensions of orbit matrix.
- Construction of SO codes obtained from submatrices of orbit matrix of of p-WSO designs under the action of group G which acts on design with f_1 fixed points and n point orbits of length p^α, and with f_2 fixed blocks and m block orbits of length p^α, $1 \leqslant \alpha \leqslant n$.

UNIVERSITY OF RIJEKA
FACULTY OF MATHEMATICS

LCD codes from p-WSO designs

Let a and d be elements of finite filed $\mathbb{F}_q$, where $q = p^l$ is prime power. Then

$$\det \begin{bmatrix} a & d & \cdots & d \\ d & a & \cdots & d \\ \vdots & \vdots & \ddots & \vdots \\ d & d & \cdots & a \end{bmatrix}_{n \times n} = (a-d)^{n-1}[a+(n-1)d].$$

Let M be $b \times v$ incidence matrix of 1-(v, k, λ) design $\mathcal{D}$ which has b blocks $B_1, ..., B_b$. Let $B_{i,j} = |B_i \cap B_j|$, for all $i, j \in \{1, ..., b\}$. It follows that

$$[M, x \cdot I_b, y \cdot \mathbf{1}] \cdot [M, x \cdot I_b, y \cdot \mathbf{1}]^T = \begin{bmatrix} B_{1,1} + x^2 + y^2 & B_{1,2} + y^2 & \cdots & B_{1,b} + y^2 \\ B_{2,1} + y^2 & B_{2,2} + x^2 + y^2 & \cdots & B_{2,b} + y^2 \\ \vdots & \vdots & \ddots & \vdots \\ B_{b,1} + y^2 & B_{b,2} + y^2 & \cdots & B_{b,b} + x^2 + y^2 \end{bmatrix}$$

LCD codes from p-WSO designs

Let $\mathcal{D}$ be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design $\mathcal{D}$.

1. If $a = d = 0$ then
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x \neq 0$, and
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $y \neq 0$ and $x^2 + b \cdot y^2 \neq 0$
   generate an LCD code over the field $\mathbb{F}_q$.
2. If $a = 0$ and $d \neq 0$ then
   the matrix $\mathbf{M}$ for $(b-1) \cdot d \neq 0$ and if $\mathbf{M}$ is of full rank,
   the matrix $[\mathbf{M}, y\mathbf{1}]$ for $by^2 + (b-1) \cdot d \neq 0$ and if $\mathbf{M}$ is of full rank,
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x - d \neq 0$ and $x^2 + (b-1) \cdot d \neq 0$, and
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x^2 - d \neq 0$ and $b \cdot y^2 + x^2 + (b-1) \cdot d \neq 0$
   generate an LCD code over the field $\mathbb{F}_q$.
3. If $a \neq 0$ and $d = 0$ then
   the matrix $\mathbf{M}$ if $\mathbf{M}$ is of full rank,
   the matrix $[\mathbf{M}, y\mathbf{1}]$ for $b \cdot y^2 + a \neq 0$ and if $\mathbf{M}$ is of full rank,
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x^2 + a \neq 0$, and
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x^2 + a \neq 0$ and $b \cdot y^2 + x^2 + a \neq 0$
   generate an LCD code over the field $\mathbb{F}_q$.

LCD codes from p-WSO designs

Let $\mathcal{D}$ be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design $\mathcal{D}$.

4. If $a = d \neq 0$ then
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x \neq 0$ and $x^2 + ba \neq 0$, and
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x \neq 0$ and $b \cdot y^2 + x^2 + b \cdot d \neq 0$
   generate an LCD code over the field $\mathbb{F}_q$.
5. If $a \neq 0$, $d \neq 0$, $a \neq d$ then
   the matrix $\mathbf{M}$ for $a + (b-1) \cdot d \neq 0$ and if $\mathbf{M}$ is of full rank,
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x^2 - d + a \neq 0$ and $x^2 + a + (b-1) \cdot d \neq 0$,
   the matrix $[\mathbf{M}, y\mathbf{1}]$ for $by^2 + a + (b-1) \cdot d \neq 0$ and if $\mathbf{M}$ is of full rank, and
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x^2 - d + a \neq 0$ and $b \cdot y^2 + x^2 + a + (b-1) \cdot d \neq 0$
   generate an LCD code over $\mathbb{F}_q$.

D. Crnković, VMC: Unitals, projective planes and other combinatorial structures constructed from the unitary groups $\mathcal{U}(3,q)$, $q = 3,4,5,7$, Ars. Combin. 110 (2013), pp. 3-13

Theorem

Let $G < S_v$ be a transitive permutation group, let P be a subgroup of the group G and $\Delta = \cup_{i=1}^{s} \delta_i P$ for distinct representatives $\delta_1, ..., \delta_s$. Then the set $\mathcal{B} = \{\Delta g \mid g \in G\}$ is a set of blocks of a 1-design with parameters $1-(v, |\Delta|, \frac{|P|}{|G_\Delta|} \sum_{i=1}^{s} \frac{|G_{\delta_i}|}{|P \cap G_{\delta_i}|})$ and $b = \frac{|G|}{|G_\Delta|}$ blocks on which the group G acts transitively and faithfully on the set of points and the set of blocks.

LCD codes from A_5

We constructed all weakly self-orthogonal 1-designs and corresponding binary LCD codes constructed from transitive permutation group $G \cong A_5$ and $P < G,\ P \neq I$.

We obtained 4 optimal LCD codes and 2 near-optimal LCD codes.

Design	C
1-$(10, 5, 3)$	$[10, 6, 3]*$
1-$(10, 6, 3)$	$[11, 5, 4]*$
1-$(20, 15, 9)$	$[20, 12, 4]*$
1-$(20, 12, 3)$	$[25, 5, 11]*$
1-$(12, 10, 5)$	$[18, 6, 6] **$
1-$(20, 14, 7)$	$[30, 10, 9] **$

LCD from orbit matrices of p-WSO designs

Let $q = p^l$ and let $\mathbb{F}_q$ be the finite field of order q.
Let $\mathcal{D}$ be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \ldots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design $\mathcal{D}$.

Let G be an automorphism group of $\mathcal{D}$ which is acting od $\mathcal{D}$ with n point orbits of size w and m block orbits of size w. Let $\mathbf{O}$ be an $m \times n$ orbit matrix under the action of G. Let x and y be nonzero elements of the field $\mathbb{F}_q$.

1. If $a = d = 0$ then
   the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ and
   the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $x^2 + m \cdot y^2 \neq 0$
   generate an LCD codes over the field $\mathbb{F}_q$.
2. If $a = 0, d \neq 0$ then we differ several cases.
   - If $q \nmid w, q \nmid w - 1$, then
     the matrix $\mathbf{O}$ if O is of full rank and $mw - q \neq 0$,
     the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ if $x^2 - d \neq 0$ and $x^2 - d + mw \cdot d \neq 0$,
     the matrix $[\mathbf{O}, y \cdot \mathbf{1}]$ for $m \cdot y^2 - d + mw \cdot d \neq 0$ and if O is of full rank, and
     the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $x^2 - d \neq 0$ and $x^2 + m \cdot y^2 - mw \cdot d + d \neq 0$
     generate an LCD code over the field $\mathbb{F}_q$.
   - If $q \mid w$, then
     the matrix $\mathbf{O}$ if O is of full rank,
     the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ if $x^2 - d \neq 0$,
     the matrix $[\mathbf{O}, y \cdot \mathbf{1}]$ if $d - m \cdot y^2 \neq 0$ and O is of full rank and
     the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ if $x^2 - d \neq 0$ and $x^2 - d + m \cdot y^2 \neq 0$
     generate an LCD code over the field $\mathbb{F}_q$.

Let $\mathcal{D}$ be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \ldots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design $\mathcal{D}$.

Let G be an automorphism group of $\mathcal{D}$ which is acting od $\mathcal{D}$ with n point orbits of size w and m block orbits of size w. Let $\mathbf{O}$ be an $m \times n$ orbit matrix under the action of G.

2. $a = 0,\ d \neq 0$
   - If $q \mid w - 1$, then
     the matrix $\mathbf{O}$ for $m - 1 \neq 0$ and if O is of full rank,
     the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$,
     the matrix $[\mathbf{O}, y \cdot \mathbf{1}]$ for $m \cdot y^2 - w \cdot d + mw \cdot d \neq 0$ and if O is of full rank, and
     the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $x^2 - w \cdot d \neq 0$ and $x^2 + m \cdot y^2 - w \cdot d + mw \cdot d \neq 0$
     generate an LCD code over the field $\mathbb{F}_q$.
3. If $a \neq 0, d = 0$, then
   the matrix $\mathbf{O}$ if O is of full rank,
   the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ for $x^2 + a \neq 0$,
   the matrix $[\mathbf{O}, y \cdot \mathbf{1}]$ for $m \cdot y^2 + a \neq 0$ and if O is of full rank, and
   the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $x^2 + a \neq 0$ and $x^2 + m \cdot y^2 + a \neq 0$
   generate an LCD code over the field $\mathbb{F}_q$.

Let $\mathcal{D}$ be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design $\mathcal{D}$.

Let G be an automorphism group of $\mathcal{D}$ which is acting od $\mathcal{D}$ with n point orbits of size w and m block orbits of size w. Let O be an $m \times n$ orbit matrix under the action of G.

4. If $a = d \neq 0$, then
   the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ for $x^2 + mw \cdot d \neq 0$, and
   the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $x^2 + m \cdot y^2 + mw \cdot d \neq 0$
   generate an LCD code over the field $\mathbb{F}_q$.
5. If $a \neq 0, d \neq 0, a \neq d$, then
   the matrix $\mathbf{O}$ if O is of full rank and for $a - d \neq 0$,
   the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ for $d - q \neq 0$ and $d - a - mw \cdot d \neq 0$,
   the matrix $[\mathbf{O}, y \cdot \mathbf{1}]$ for $d - a - mw \cdot d - m \cdot y^2 \neq 0$ and if O is of full rank, and
   the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $d - a - x^2 \neq 0$ and
   $d - a - x^2 - mw \cdot d - m \cdot y^2 \neq 0$
   generate an LCD code over the field $\mathbb{F}_q$.

UNIVERSITY OF RIJEKA
FACULTY OF MATHEMATICS

Some examples...

We constructed examples of weakly 3-self-orthogonal designs and weakly 5-self-orthogonal designs from permutation representation of the group $S_4(9)$ on 1640 points.
The orbit matrices of the constructed designs were obtained under the action of the cyclic group of order 5 which acts on the points of the design, i.e. in orbits of length 5.

Design	C
1-$(1640, 729, 729)$	$[657, 328, 2]_3$
1-$(1640, 1458, 729)$	$[492, 164, 2]_3$ $[493, 164, 2]_3$
1-$(1640, 1638, 819)$	$[329, 164, 1]_3$
1-$(1640, 2, 1)$	$[328, 164, 2]_3$ $[329, 164, 3]_3$
1-$(1640, 182, 91)$	$[493, 164, 4]_3$
1-$(1640, 911, 911)$	$[656, 328]_3$
1-$(1640, 1458, 729)$	$[328, 164, 2]_5$ $[492, 164, 12]_5$ $[493, 164, 12]_5$ $[329, 164, 3]_5$
1-$(1640, 1638, 819)$	$[493, 164, 3]_5$ $[493, 164, 4]_5$

LCD codes obtained using submatrices of orbit matrix of p-WSO design

Let $q = p^l$ and let $\mathbb{F}_q$ be the finite field of order q.
Let $\mathcal{D}$ 1-(v, k, r) be a design sunch that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design $\mathcal{D}$.

Let G be an automorphism group of the design $\mathcal{D}$ which acts on the point set of $\mathcal{D}$ with f_1 fixed points and n orbits pf length p^α, $1 \leqslant \alpha \leqslant l$, and which acts of block set of the design $\mathcal{D}$ with f_2 fixed blocks and m orbits of length p^α.

$$O = \left[\begin{array}{cccc|cccc}
 & & & & a_{1,f_1+1} & a_{1,f_1+2} & \cdots & a_{1,f_1+n} \\
 & \text{OM1} & & & a_{2,f_1+1} & a_{2,f_1+2} & \cdots & a_{2,f_1+n} \\
 & & & & \vdots & \vdots & \ddots & \vdots \\
 & & & & a_{f_2,f_1+1} & a_{f_2,f_1+2} & \cdots & a_{f_2,f_1+n} \\
\hline
a_{f_2+1,1} & a_{f_2+1,2} & \cdots & a_{f_2+1,f_1} & & & & \\
a_{f_2+2,1} & a_{f_2+1,2} & \cdots & a_{f_2+2,f_1} & & & & \\
\vdots & \vdots & \ddots & \vdots & & \text{OM2} & & \\
a_{f_2+m,1} & a_{f_2+m,2} & \cdots & a_{f_2+m,f_1} & & & &
\end{array}\right]$$

Using suitable extensions of $OM1$ and $OM2$, one can construct an LCD code over the field $\mathbb{F}_q$.

UNIVERSITY OF RIJEKA
FACULTY OF MATHEMATICS

LCD codes obtained using submatrices of orbit matrix of p-WSO design ($OM1$)

Let $q = p^l$ and let $\mathbb{F}_q$ be the finite field of order q. Let $\mathcal{D}$ 1-(v, k, r) be a design sunch that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \ldots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design $\mathcal{D}$. Let G be an automorphism group of the design $\mathcal{D}$ which acts on the point set of $\mathcal{D}$ with f_1 fixed points and n orbits pf length p^{α}, $1 \leqslant \alpha \leqslant l$, and which acts of block set of the design $\mathcal{D}$ with f_2 fixed blocks and m orbits of length p^{α}. Let x and y be non-zero elements of $\mathbb{F}_q$.

1. For $a = d$, we conclude the following.
   - For $x^2 + f_1 \cdot a \neq 0$, linear code over the filed $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}]$ is LCD code.
   - For $x^2 + f_1 \cdot y^2 + f_1 \cdot a \neq 0$, linear code over $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}, y \cdot \mathbf{1}]$ is LCD code.
2. For $a \neq d$, we conclude the following.
   - If $OM1$ is of full rank, linear code over the filed $\mathbb{F}_q$ generated by matrix $\mathbf{OM1}$ is LCD code.
   - For $x^2 + a \neq 0$, linear code over the filed $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}]$ is LCD code.
   - If $OM1$ is of full rank and for $a + f_1 \cdot y^2 \neq 0$, linear code over the filed $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, y \cdot \mathbf{1}]$ is LCD code.
   - For $x^2 + a \neq 0$ and $x^2 + f_1 \cdot y^2 + f_1 \cdot a \neq 0$, linear code over the filed $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}, y \cdot \mathbf{1}]$ is LCD code.

UNIVERSITY OF RIJEKA
FACULTY OF MATHEMATICS

LCD codes obtained using submatrices of orbit matrix of p-WSO design ($OM2$)

Let $q = p^l$ and let $\mathbb{F}_q$ be the finite field of order q. Let $\mathcal{D}$ 1-(v, k, r) be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design $\mathcal{D}$. Let G be an automorphism group of the design $\mathcal{D}$ which acts on the point set of $\mathcal{D}$ with f_1 fixed points and n orbits pf length p^α, $1 \leqslant \alpha \leqslant l$, and which acts of block set of the design $\mathcal{D}$ with f_2 fixed blocks and m orbits of length p^α. Let x and y be non-zero elements of $\mathbb{F}_q$.

1. For $a = d$, we conclude the following.
    - Linear code over the fileld $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m]$ is LCD code.
    - For $x^2 + m \cdot y^2 \neq 0$, linear code over the field $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ is LCD code.
2. For $a \neq d$, we conclude the following.
    - If $OM2$ is of full rank, linear code over the field $\mathbb{F}_q$ generated by matrix $\mathbf{OM2}$ is LCD code.
    - For $x^2 + a - d \neq 0$, linear code over the field $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m]$ is LCD code.
    - If $OM2$ is of full rank and for $a - d + m \cdot y^2 \neq 0$, linear code over the field $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, y \cdot \mathbf{1}]$ is LCD code.
    - For $x^2 + a - d \neq 0$ and $x^2 + m \cdot y^2 + a - d \neq 0$, linear code over the field $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ is LCD code.

UNIVERSITY OF RIJEKA
FACULTY OF MATHEMATICS

Some examples...

Using weakly 3-self-orthogonal 1-designs constructed from the group $S_4(9)$ on 1640 points, we have constructed LCD codes using Theorem 3. The orbit matrices of the designs are obtained under the action of a cyclic group of order 3 acting on the points of the designs of lengths 1 and 3.

Design	C
1-$(1640, 182, 91)$	$[57, 19, 1]_3$ $[58, 19, 2]_3$ $[1068, 534]_3$ $[1069, 534]_3$
1-$(1640, 729, 729)$	$[76, 38, 1]_3$ $[801, 267]_3$ $[802, 267]_3$
1-$(1640, 1638, 819)$	$[39, 19, 1]_3$ $[534, 267, 2]_3$ $[535, 267, 3]_3$
1-$(1640, 2, 1)$	$[38, 19, 2]_3$ $[534, 267, 2]_3$ $[535, 267, 3]_3$
1-$(1640, 182, 91)$	$[58, 19, 2]_3$ $[801, 267]_3$ $[802, 267]_3$
1-$(1640, 911, 911)$	$[76, 38, 2]_3$ $[77, 38, 2]_3$ $[1068, 534]_3$ $[1069, 534]_3$

¡Muchas gracias!