

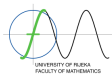
LCD codes related to some combinatorial structures

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Preliminaries

Self-orthogonal codes from p -WSO designs

LCD codes from p -WSO designs

Codes

We will talk only about **linear codes**, i.e. subspaces of the ambient vector space over a field \mathbb{F}_q of order $q = p^l$, where p is prime.

A code C of length n and dimension k is denoted by $[n, k]$.

By $[n, k, d]_q$, we denote code C with minimal distance d over the field \mathbb{F}_q .

Specially, if $q = 2$, parameters of code C are denoted by $[n, k, d]$.

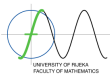
A **generator matrix** of a $[n, k]$ code C is a $k \times n$ matrix whose rows generate all the words of C .

The **dual code** of a code C is code C^\perp , $C^\perp = \{v \in (\mathbb{F}_q)^n \mid (v, c) = 0, \forall c \in C\}$.

A code is **self-orthogonal** if $C \subseteq C^\perp$. A code is **self-dual** if $C = C^\perp$.

A code is **LCD (Linear code with complementary dual)** if $C \cap C^\perp = \{\mathbf{0}\}$.

A code C with generator matrix G is LCD code if and only if $\det(G \cdot G^T) \neq 0$.



Weakly self-orthogonal designs

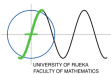
An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set \mathcal{P} , block set \mathcal{B} and incidence \mathcal{I} is called a $t - (v, k, \lambda)$ design, if \mathcal{P} contains v points, every block $B \in \mathcal{B}$ is incident with k points, and every t distinct points are incident with λ blocks.

The incidence matrix of a design is a $b \times v$ matrix $[m_{ij}]$ where b and v are the numbers of blocks and points respectively, such that $m_{ij} = 1$ if the point P_j and the block B_i are incident, and $m_{ij} = 0$ otherwise.

A design is weakly p -self-orthogonal (p -WSO) if all block intersection numbers gives the same residue modulo p .

A weakly p -self-orthogonal design is p -self-orthogonal if block intersection numbers and the block sizes are multiples of p .

Specially, weakly 2-self-orthogonal design is called weakly self-orthogonal (WSO) design, and 2-self-orthogonal design is called self-orthogonal.



Examples of some known families of p -WSO

► Symmetric designs

1. Point-hyperplane designs.

$$v = \frac{(q^{m+1} - 1)}{q - 1}, \quad k = \frac{q^m - 1}{q - 1}, \quad \lambda = \frac{q^{m-1} - 1}{q - 1}, \quad q = p^l \text{ a prime power and } m \geq 2.$$

These designs are weakly p -self-orthogonal ($k, \lambda \equiv 1 \pmod{p}$)

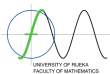
2. Menon designs.

$$v = 4t^2, \quad k = 2t^2 - t, \quad \lambda = t^2 - t.$$

These designs are p -self-orthogonal for every prime p dividing t .


A design with this parameters exists if and only if there exists a regular Hadamard matrix of order $4t^2$. It is conjectured that these designs exist for all values of t .

The incidence matrix of a Menon design is given by $M = \frac{1}{2}(J_{4t^2} - H)$, and H is a regular Hadamard matrix in which the sum of every row is equal to $2t$.



Examples of some known families of p -WSO

► Quasi-symmetric designs

1. Blokhuis and Haemers constructed an infinite family of quasi-symmetric $2-(q^3, q^2(q-1)/2, q(q^3 - q^2 - 2)/4)$ designs with block intersection numbers $q^2(q-2)/4$ and $q^2(q-1)/4$, where q is a power of 2.
For $q > 2$ these designs are self-orthogonal (k and block intersection numbers are even).
2.  V. Krčadinac, R. Vlahović. New quasi-symmetric designs by the Kramer-Mesner method, *Discrete Math.* 339 (2016), no. 12, 2884-2890.

The authors found many new quasi-symmetric $2-(28, 12, 11)$ and $2-(36, 16, 12)$ designs. For example, the $2-(28, 12, 11)$ quasi-symmetric design with intersection numbers 4 and 6 is self-orthogonal.

Examples of some known families of p -WSO

► Strongly regular graphs

1. The adjacency matrix of a $SRG(n, k, \lambda, \mu)$ such that $k \equiv a \pmod{p}$, $\lambda = \mu \pmod{p}$ is the incidence matrix of a p -WSO.
2. The triangular graph $T(n)$, whose vertices are 2-element subsets of an n -elements set, two pairs being adjacent if and only if they have an element in common.

$T(n)$ is a strongly regular graph with parameters $\left(\binom{n}{2}, 2(n-2), n-2, 4 \right)$.

If $n > 2$ is even, the adjacency matrix of $T(n)$ is the incidence matrix of a self-orthogonal 1-design (k and the block intersection numbers are even)

Orbit matrices of a design

Let \mathcal{D} be a $1-(v, k, \lambda)$ design and G be an automorphism group of the design. Let $v_1 = |\mathcal{V}_1|, \dots, v_n = |\mathcal{V}_n|$ be the sizes of point orbits and $b_1 = |\mathcal{B}_1|, \dots, b_m = |\mathcal{B}_m|$ be the sizes of block orbits under the action of the group G . We define an **orbit matrix** as $m \times n$ matrix

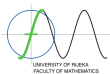
$$O = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

where a_{ij} is the number of points of the orbit \mathcal{V}_j incident with a block of the orbit \mathcal{B}_i . It is easy to see that the matrix is well-defined and that $k = \sum_{j=1}^n a_{ij}$.

For $x \in \mathcal{B}_s$, by counting the incidence pairs (P, x') such that $x' \in \mathcal{B}_t$ and P is incident with the block x , we obtain

$$\sum_{x' \in \mathcal{B}_t} |x \cap x'| = \sum_{j=1}^m \frac{b_t}{v_j} a_{sj} a_{tj} = \frac{b_t}{v_j} O[s] \cdot O[t],$$

where $O[s]$ is the s -th row of the matrix O .



Let \mathcal{D} be a weakly p -self-orthogonal design such that

$$k \equiv a \pmod{p}$$

and

$$|B_i \cap B_j| \equiv d \pmod{p},$$

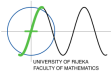
for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of a design \mathcal{D} .

Let G be an automorphism group of the design which acts on \mathcal{D} with n point orbits of length w and block orbits of length b_1, b_2, \dots, b_m , and let O be an orbit matrix of a design \mathcal{D} under the action of a group G .

Let $q = p^n$ be prime power and let \mathbb{F}_q be a finite field of order q . In \mathbb{F}_q , for $x \in \mathcal{B}_s$, $s \neq t$ it follows that

$$\frac{b_t}{w} O[s] \cdot O[t] = b_t d, \tag{1}$$

$$\frac{b_s}{w} O[s] \cdot O[s] = a + (b_s - 1)d. \tag{2}$$



Self-orthogonal codes from p -WSO designs



Mikulić Crnković, V., Traunkar, I. Self-orthogonal codes constructed from weakly self-orthogonal designs invariant under an action of M_{11} . AAECC 34, 139–156 (2023). <https://doi.org/10.1007/s00200-020-00484-2>

Let \mathbb{F}_q be a finite field of order $q = p^l$, where p is a prime.

- ▶ Construction of SO codes obtained from incidence matrix of p -WSO designs, using suitable extensions of incidence matrix of a design.
- ▶ Construction of SO codes obtained from orbit matrix of p -WSO designs under the action of group G which acts on design with n point orbits of length w and m block orbits of length w , using suitable extensions of orbit matrix.
- ▶ Construction of SO codes obtained from submatrices of orbit matrix of p -WSO designs under the action of group G which acts on design with f_1 fixed points and n point orbits of length p^α , and with f_2 fixed blocks and m block orbits of length p^α , $1 \leq \alpha \leq n$.

LCD codes from p -WSO designs

Let a and d be elements of finite field \mathbb{F}_q , where $q = p^l$ is prime power. Then

$$\det \begin{bmatrix} a & d & \cdots & d \\ d & a & \cdots & d \\ \vdots & \vdots & \ddots & \vdots \\ d & d & \cdots & a \end{bmatrix}_{n \times n} = (a - d)^{n-1} [a + (n - 1)d].$$

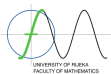
Let M be $b \times v$ incidence matrix of 1 -(v, k, λ) design \mathcal{D} which has b blocks B_1, \dots, B_b .
Let $B_{i,j} = |B_i \cap B_j|$, for all $i, j \in \{1, \dots, b\}$. It follows that

$$[M, x \cdot \mathbf{1}_b, y \cdot \mathbf{1}] \cdot [M, x \cdot \mathbf{1}_b, y \cdot \mathbf{1}]^T = \begin{bmatrix} B_{1,1} + x^2 + y^2 & B_{1,2} + y^2 & \cdots & B_{1,b} + y^2 \\ B_{2,1} + y^2 & B_{2,2} + x^2 + y^2 & \cdots & B_{2,b} + y^2 \\ \vdots & \vdots & \ddots & \vdots \\ B_{b,1} + y^2 & B_{b,2} + y^2 & \cdots & B_{b,b} + x^2 + y^2 \end{bmatrix}$$

LCD codes from p -WSO designs

Let \mathcal{D} be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design \mathcal{D} .

1. If $a = d = 0$ then
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x \neq 0$, and
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $y \neq 0$ and $x^2 + b \cdot y^2 \neq 0$
 generate an LCD code over the field \mathbb{F}_q .
2. If $a = 0$ and $d \neq 0$ then
 - the matrix \mathbf{M} for $(b-1) \cdot d \neq 0$ and if \mathbf{M} is of full rank,
 - the matrix $[\mathbf{M}, y\mathbf{1}]$ for $by^2 + (b-1) \cdot d \neq 0$ and if \mathbf{M} is of full rank,
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x - d \neq 0$ and $x^2 + (b-1) \cdot d \neq 0$, and
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x^2 - d \neq 0$ and $b \cdot y^2 + x^2 + (b-1) \cdot d \neq 0$
 generate an LCD code over the field \mathbb{F}_q .
3. If $a \neq 0$ and $d = 0$ then
 - the matrix \mathbf{M} if \mathbf{M} is of full rank,
 - the matrix $[\mathbf{M}, y\mathbf{1}]$ for $b \cdot y^2 + a \neq 0$ and if \mathbf{M} is of full rank,
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x^2 + a \neq 0$, and
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x^2 + a \neq 0$ and $b \cdot y^2 + x^2 + a \neq 0$
 generate an LCD code over the field \mathbb{F}_q .



LCD codes from p -WSO designs

Let \mathcal{D} be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design \mathcal{D} .

4. If $a = d \neq 0$ then
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x \neq 0$ and $x^2 + ba \neq 0$, and
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x \neq 0$ and $b \cdot y^2 + x^2 + b \cdot d \neq 0$
 generate an LCD code over the field \mathbb{F}_q .
5. If $a \neq 0$, $d \neq 0$, $a \neq d$ then
 - the matrix \mathbf{M} for $a + (b-1) \cdot d \neq 0$ and if \mathbf{M} is of full rank,
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x^2 - d + a \neq 0$ and $x^2 + a + (b-1) \cdot d \neq 0$,
 - the matrix $[\mathbf{M}, y\mathbf{1}]$ for $by^2 + a + (b-1) \cdot d \neq 0$ and if \mathbf{M} is of full rank, and
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x^2 - d + a \neq 0$ and $b \cdot y^2 + x^2 + a + (b-1) \cdot d \neq 0$
 generate an LCD code over \mathbb{F}_q .



D. Crnković, VMC: Unitals, projective planes and other combinatorial structures constructed from the unitary groups $\mathcal{U}(3, q)$, $q = 3, 4, 5, 7$, Ars. Combin. 110 (2013), pp. 3-13

Theorem

Let $G < S_v$ be a transitive permutation group, let P be a subgroup of the group G and $\Delta = \cup_{i=1}^s \delta_i P$ for distinct representatives $\delta_1, \dots, \delta_s$. Then the set $\mathcal{B} = \{\Delta g \mid g \in G\}$ is a set of blocks of a 1-design with parameters

$1 - (v, |\Delta|, \frac{|P|}{|G_\Delta|} \sum_{i=1}^s \frac{|G_{\delta_i}|}{|P \cap G_{\delta_i}|})$ and $b = \frac{|G|}{|G_\Delta|}$ blocks on which the group G acts transitively and faithfully on the set of points and the set of blocks.

LCD codes from A_5

We constructed all weakly self-orthogonal 1-designs and corresponding binary LCD codes constructed from transitive permutation group $G \cong A_5$ and $P < G$, $P \neq I$.

We obtained 4 optimal LCD codes and 2 near-optimal LCD codes.

Design	C
1-(10, 5, 3)	[10, 6, 3] *
1-(10, 6, 3)	[11, 5, 4] *
1-(20, 15, 9)	[20, 12, 4] *
1-(20, 12, 3)	[25, 5, 11] *
1-(12, 10, 5)	[18, 6, 6] * *
1-(20, 14, 7)	[30, 10, 9] * *

LCD from orbit matrices of p -WSO designs

Let $q = p^l$ and let \mathbb{F}_q be the finite field of order q .

Let \mathcal{D} be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design \mathcal{D} .

Let G be an automorphism group of \mathcal{D} which is acting on \mathcal{D} with n point orbits of size w and m block orbits of size w . Let \mathbf{O} be an $m \times n$ orbit matrix under the action of G . Let x and y be nonzero elements of the field \mathbb{F}_q .

1. If $a = d = 0$ then

the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ and

the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $x^2 + m \cdot y^2 \neq 0$

generate an LCD code over the field \mathbb{F}_q .

2. If $a = 0, d \neq 0$ then we differ several cases.

- ▶ If $q \nmid w, q \nmid w - 1$, then

the matrix \mathbf{O} if O is of full rank and $mw - q \neq 0$,

the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ if $x^2 - d \neq 0$ and $x^2 - d + mw \cdot d \neq 0$,

the matrix $[\mathbf{O}, y \cdot \mathbf{1}]$ for $m \cdot y^2 - d + mw \cdot d \neq 0$ and if O is of full rank, and

the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $x^2 - d \neq 0$ and $x^2 + m \cdot y^2 - mw \cdot d + d \neq 0$

generate an LCD code over the field \mathbb{F}_q .

- ▶ If $q \mid w$, then

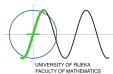
the matrix \mathbf{O} if O is of full rank,

the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ if $x^2 - d \neq 0$,

the matrix $[\mathbf{O}, y \cdot \mathbf{1}]$ if $d - m \cdot y^2 \neq 0$ and O is of full rank and

the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ if $x^2 - d \neq 0$ and $x^2 - d + m \cdot y^2 \neq 0$

generate an LCD code over the field \mathbb{F}_q .



Let \mathcal{D} be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design \mathcal{D} .

Let G be an automorphism group of \mathcal{D} which is acting on \mathcal{D} with n point orbits of size w and m block orbits of size w . Let O be an $m \times n$ orbit matrix under the action of G .

2. $a = 0$, $d \neq 0$

▶ If $q \mid w - 1$, then

the matrix O for $m - 1 \neq 0$ and if O is of full rank,

the matrix $[O, x \cdot I_m]$,

the matrix $[O, y \cdot \mathbf{1}]$ for $m \cdot y^2 - w \cdot d + mw \cdot d \neq 0$ and if O is of full rank, and

the matrix $[O, x \cdot I_m, y \cdot \mathbf{1}]$ for $x^2 - w \cdot d \neq 0$ and $x^2 + m \cdot y^2 - w \cdot d + mw \cdot d \neq 0$ generate an LCD code over the field \mathbb{F}_q .

3. If $a \neq 0$, $d = 0$, then

the matrix O if O is of full rank,

the matrix $[O, x \cdot I_m]$ for $x^2 + a \neq 0$,

the matrix $[O, y \cdot \mathbf{1}]$ for $m \cdot y^2 + a \neq 0$ and if O is of full rank, and

the matrix $[O, x \cdot I_m, y \cdot \mathbf{1}]$ for $x^2 + a \neq 0$ and $x^2 + m \cdot y^2 + a \neq 0$ generate an LCD code over the field \mathbb{F}_q .



Let \mathcal{D} be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design \mathcal{D} .

Let G be an automorphism group of \mathcal{D} which is acting on \mathcal{D} with n point orbits of size w and m block orbits of size w . Let O be an $m \times n$ orbit matrix under the action of G .

4. If $a = d \neq 0$, then
 - the matrix $[O, x \cdot I_m]$ for $x^2 + mw \cdot d \neq 0$, and
 - the matrix $[O, x \cdot I_m, y \cdot \mathbf{1}]$ for $x^2 + m \cdot y^2 + mw \cdot d \neq 0$
 generate an LCD code over the field \mathbb{F}_q .
5. If $a \neq 0, d \neq 0, a \neq d$, then
 - the matrix O if O is of full rank and for $a - d \neq 0$,
 - the matrix $[O, x \cdot I_m]$ for $d - a \neq 0$ and $d - a - mw \cdot d \neq 0$,
 - the matrix $[O, y \cdot \mathbf{1}]$ for $d - a - mw \cdot d - m \cdot y^2 \neq 0$ and if O is of full rank, and
 - the matrix $[O, x \cdot I_m, y \cdot \mathbf{1}]$ for $d - a - x^2 \neq 0$ and $d - a - x^2 - mw \cdot d - m \cdot y^2 \neq 0$
 generate an LCD code over the field \mathbb{F}_q .

Some examples...

We constructed examples of weakly 3-self-orthogonal designs and weakly 5-self-orthogonal designs from permutation representation of the group $S_4(9)$ on 1640 points.

The orbit matrices of the constructed designs were obtained under the action of the cyclic group of order 5 which acts on the points of the design, i.e. in orbits of length 5.

Design	C
1-(1640, 729, 729)	$[657, 328, 2]_3$
1-(1640, 1458, 729)	$[492, 164, 2]_3$ $[493, 164, 2]_3$
1-(1640, 1638, 819)	$[329, 164, 1]_3$
1-(1640, 2, 1)	$[328, 164, 2]_3$ $[329, 164, 3]_3$
1-(1640, 182, 91)	$[493, 164, 4]_3$
1-(1640, 911, 911)	$[656, 328]_3$
1-(1640, 1458, 729)	$[328, 164, 2]_5$ $[492, 164, 12]_5$ $[493, 164, 12]_5$
1-(1640, 1638, 819)	$[329, 164, 3]_5$ $[493, 164, 3]_5$ $[493, 164, 4]_5$

LCD codes obtained using submatrices of orbit matrix of p -WSO design

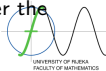
Let $q = p^l$ and let \mathbb{F}_q be the finite field of order q .

Let $\mathcal{D} 1-(v, k, r)$ be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design \mathcal{D} .

Let G be an automorphism group of the design \mathcal{D} which acts on the point set of \mathcal{D} with f_1 fixed points and n orbits of length p^α , $1 \leq \alpha \leq l$, and which acts on the block set of the design \mathcal{D} with f_2 fixed blocks and m orbits of length p^α .

$$O = \left[\begin{array}{cccc|cccc} & & & & a_{1,f_1+1} & a_{1,f_1+2} & \cdots & a_{1,f_1+n} \\ & & & & a_{2,f_1+1} & a_{2,f_1+2} & \cdots & a_{2,f_1+n} \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & a_{f_2,f_1+1} & a_{f_2,f_1+2} & \cdots & a_{f_2,f_1+n} \\ \hline & OM1 & & & & & & \\ \hline a_{f_2+1,1} & a_{f_2+1,2} & \cdots & a_{f_2+1,f_1} & & & & \\ a_{f_2+2,1} & a_{f_2+2,2} & \cdots & a_{f_2+2,f_1} & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ a_{f_2+m,1} & a_{f_2+m,2} & \cdots & a_{f_2+m,f_1} & & & OM2 & \end{array} \right]$$

Using suitable extensions of $OM1$ and $OM2$, one can construct an LCD code over the field \mathbb{F}_q .



LCD codes obtained using submatrices of orbit matrix of p -WSO design (OM1)

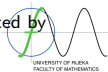
Let $q = p^l$ and let \mathbb{F}_q be the finite field of order q . Let $\mathcal{D} = (v, k, r)$ be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design \mathcal{D} . Let G be an automorphism group of the design \mathcal{D} which acts on the point set of \mathcal{D} with f_1 fixed points and n orbits of length p^α , $1 \leq \alpha \leq l$, and which acts on the block set of the design \mathcal{D} with f_2 fixed blocks and m orbits of length p^α . Let x and y be non-zero elements of \mathbb{F}_q .

1. For $a = d$, we conclude the following.

- ▶ For $x^2 + f_1 \cdot a \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}]$ is LCD code.
- ▶ For $x^2 + f_1 \cdot y^2 + f_1 \cdot a \neq 0$, linear code over \mathbb{F}_q generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}, y \cdot \mathbf{1}]$ is LCD code.

2. For $a \neq d$, we conclude the following.

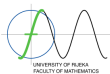
- ▶ If OM1 is of full rank, linear code over the field \mathbb{F}_q generated by matrix $\mathbf{OM1}$ is LCD code.
- ▶ For $x^2 + a \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}]$ is LCD code.
- ▶ If OM1 is of full rank and for $a + f_1 \cdot y^2 \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM1}, y \cdot \mathbf{1}]$ is LCD code.
- ▶ For $x^2 + a \neq 0$ and $x^2 + f_1 \cdot y^2 + f_1 \cdot a \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}, y \cdot \mathbf{1}]$ is LCD code.



LCD codes obtained using submatrices of orbit matrix of p -WSO design (OM2)

Let $q = p^l$ and let \mathbb{F}_q be the finite field of order q . Let $\mathcal{D} 1-(v, k, r)$ be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design \mathcal{D} . Let G be an automorphism group of the design \mathcal{D} which acts on the point set of \mathcal{D} with f_1 fixed points and n orbits of length p^α , $1 \leq \alpha \leq l$, and which acts on the block set of the design \mathcal{D} with f_2 fixed blocks and m orbits of length p^α . Let x and y be non-zero elements of \mathbb{F}_q .

- For $a = d$, we conclude the following.
 - Linear code over the field \mathbb{F}_q generated by matrix **[OM2, $x \cdot \mathbf{I}_m$]** is LCD code.
 - For $x^2 + m \cdot y^2 \neq 0$, linear code over the field \mathbb{F}_q generated by matrix **[OM2, $x \cdot \mathbf{I}_m, y \cdot \mathbf{1}$]** is LCD code.
- For $a \neq d$, we conclude the following.
 - If **OM2** is of full rank, linear code over the field \mathbb{F}_q generated by matrix **OM2** is LCD code.
 - For $x^2 + a - d \neq 0$, linear code over the field \mathbb{F}_q generated by matrix **[OM2, $x \cdot \mathbf{I}_m$]** is LCD code.
 - If **OM2** is of full rank and for $a - d + m \cdot y^2 \neq 0$, linear code over the field \mathbb{F}_q generated by matrix **[OM2, $y \cdot \mathbf{1}$]** is LCD code.
 - For $x^2 + a - d \neq 0$ and $x^2 + m \cdot y^2 + a - d \neq 0$, linear code over the field \mathbb{F}_q generated by matrix **[OM2, $x \cdot \mathbf{I}_m, y \cdot \mathbf{1}$]** is LCD code.



Some examples...

Using weakly 3-self-orthogonal 1-designs constructed from the group $S_4(9)$ on 1640 points, we have constructed LCD codes using Theorem 3. The orbit matrices of the designs are obtained under the action of a cyclic group of order 3 acting on the points of the designs of lengths 1 and 3.

Design	C
1-(1640, 182, 91)	$[57, 19, 1]_3$
	$[58, 19, 2]_3$
	$[1068, 534]_3$
	$[1069, 534]_3$
-----	-----
1-(1640, 729, 729)	$[76, 38, 1]_3$
	$[801, 267]_3$
	$[802, 267]_3$
1-(1640, 1638, 819)	$[39, 19, 1]_3$
	$[534, 267, 2]_3$
	$[535, 267, 3]_3$
1-(1640, 2, 1)	$[38, 19, 2]_3$
	$[534, 267, 2]_3$
	$[535, 267, 3]_3$
1-(1640, 182, 91)	$[58, 19, 2]_3$
	$[801, 267]_3$
	$[802, 267]_3$
-----	-----
1-(1640, 911, 911)	$[76, 38, 2]_3$
	$[77, 38, 2]_3$
	$[1068, 534]_3$
	$[1069, 534]_3$

¡Muchas gracias!