 of Latin Squares

Ian Wanless

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What do these mathematical words have in common?

- group,
- graph,
- set,
- manifold,
- field,
- design,
- matrix,
- category,
- module,
- ring,
- sequence,
- space?

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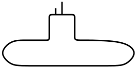

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Subsquares

A *latin square* of order n is an $n \times n$ matrix in which each of n symbols occurs exactly once in each row and once in each column.

e.g. $\begin{array}{|cccc} \hline 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{array}$ is a latin square of order 4.

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Intersectionality

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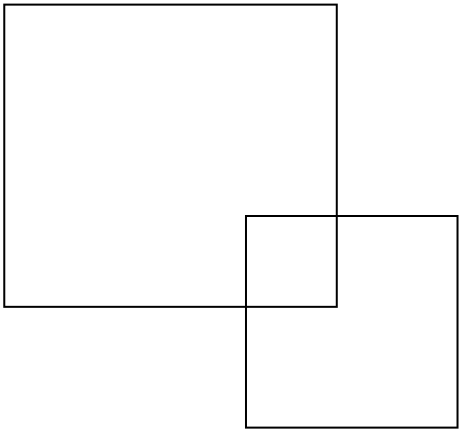

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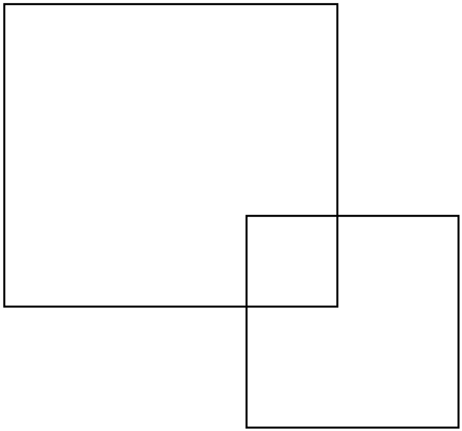

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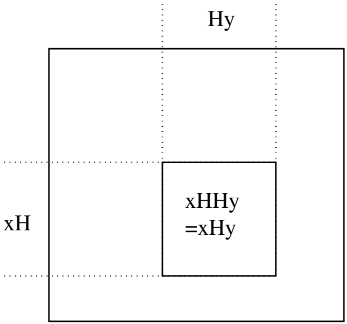
Group tables

Suppose H is a subgroup of order k in a group G of finite order n.



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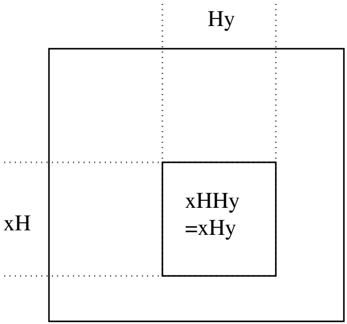

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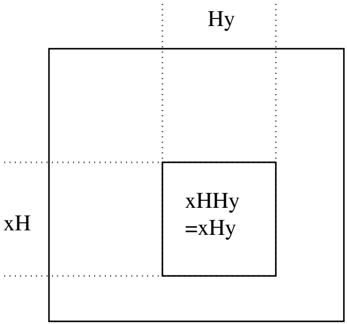

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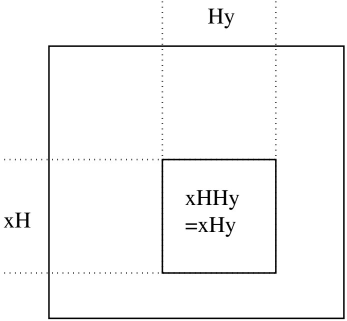

...produces a subsquare of order k in the Cayley table of G.

In fact, this is the only way that subsquares arise in group tables.

Corollary: The number of subsquares of order k in G is $(n/k)^2$ times the number of subgroups of order k.

Minimum number of subsquares

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The two most studied problems are constructions for

- N_2 latin squares; i.e. ones without intercalates, and
- N_∞ latin squares; i.e. ones without proper subsquares

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This was proved by a sequence of papers including:

- [Kotzig/Lindner/Rosa'75] Orders that aren't powers of 2.
- [McLeish'75] Powers of 2 that are > 32.
- [Kotzig/Turgeon'76] 16 and 32.
- [Denniston'78] catalogues all examples of order 8.
- [McLeish'80] (corrected in [W'01]) constructs examples for $n > 30$.

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This conjecture has been confirmed as follows:

- [Denniston'78] Order 8.
- [Heinrich'80] Orders $pq \neq 6$ for primes p, q.
- [Andersen/Mendelsohn'82] Orders divisible by a prime $\geqslant 5$.
- [Gibbons/Mendelsohn'91] Order 12.
- [Elliot/Gibbons'92] Order 16,18.
- [W.'97] Orders < 256.
- [Maenhaut/W./Webb'07] Odd orders.
- [Allsop/W.'24+] All remaining orders.

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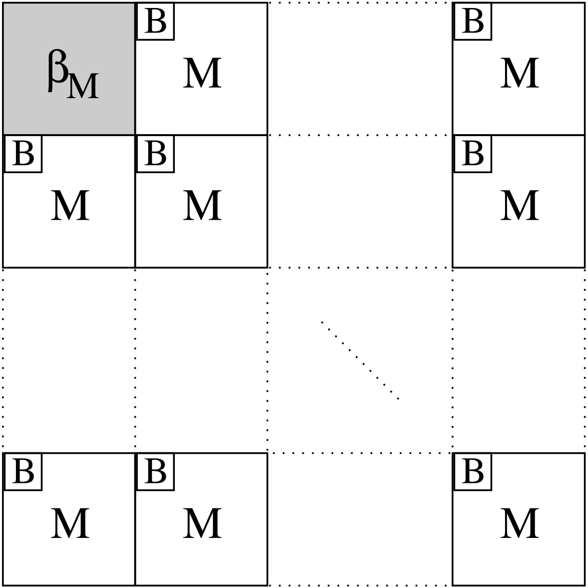
The *corrupted product* $P = (A, B) *_s M$ of shift $s \not\equiv 0 \bmod m$ is defined by

$$P\big[(i,j),(k,l)\big] = \begin{cases} \big(A[i,k], (M[j,l]+s)_m\big) & i = k = 1, \\ \big(B[i,k], M[j,l]\big) & (i,k) \neq (1,1) = (j,l), \\ \big(A[i,k], M[j,l]\big) & \text{otherwise;} \end{cases}$$

Corrupted products



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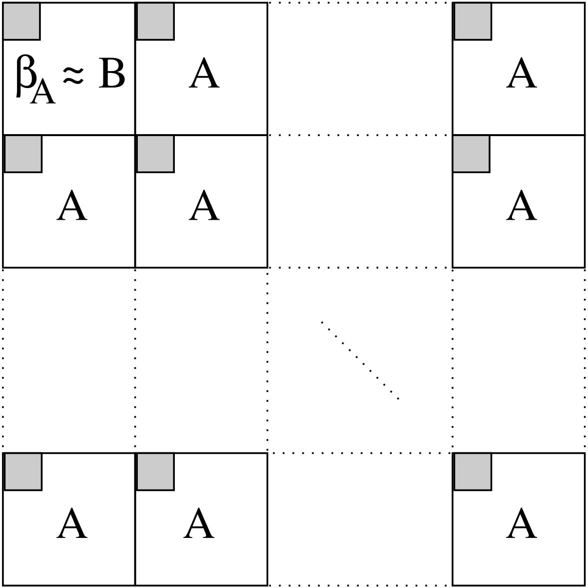

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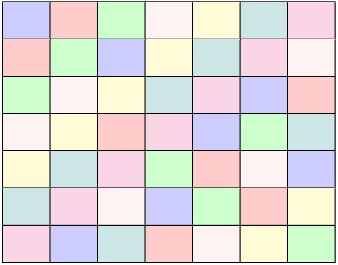
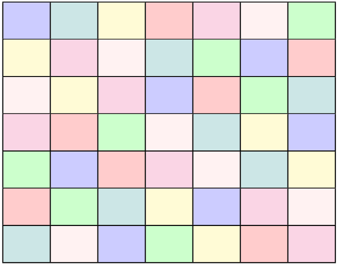
$A =$  , $B =$  .

$$M = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array}.$$

The corrupted product $(A, B) *_1 M$ is ...

2	3	1	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1
3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2
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3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2
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Once we have that in place, we just need base cases of sizes $\{12, 16, 18, 24, 32, 36, 48, 54, 64, 72\}$.

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The latter paper also showed that elementary abelian 2-groups uniquely maximise the number of subsquares of order $k = 2^t$.

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- A quasigroup achieves equality iff every loop-isotope has exponent 3.
- There is a Steiner triple system associated with every row, column and symbol in any example that achieves equality.

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In fact you can't have more than cubically many copies of any subsquare that contains a cycle of length more than $k/2$.

Open problem: Is there a family of latin squares with more than cubically many subsquares of order p?

Other small orders

Let $\psi(k)$ be the "correct exponent" for $S_k(n)$ as $n \to \infty$.

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3	3
4	4
5	3
6	4
7	$3\ldots4$
8	5
9	4
10	4

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1	2	
2	3	
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4	4	
5	3	
6	4	$\longleftarrow \quad \mathbb{Z}_3^t \rtimes \mathbb{Z}_2$
7	$3 \ldots 4$	
8	5	
9	4	
10	4	$\longleftarrow \quad \mathbb{Z}_5^t \rtimes \mathbb{Z}_2$

General bounds

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N.B. Elementary abelian 2 groups have $S_k(n) = \Theta(n^{2+\log_2 k})$ when k is a power of 2.

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It follows that Latin square isomorphism can be tested in average-case polynomial time.

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- Can isomorphism be solved in average case polynomial time for STS and 1-factorisations?

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That's all folks!!

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$\otimes$	$(y,0)$	$(y,1)$
$(x,0)$	$(xy,0)$	$(yx,1)$
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Where the first coordinate is calculated in a subgroup G of index 2.

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Take G to be of exponent p and consider any subgroup H of order p. Then

$\otimes$	$(Hb, 0)$	$(cHa^{-1}, 1)$
$(aH, 0)$	$(aHb, 0)$	$(cH, 1)$
$(cHb, 1)$	$(cH, 1)$	$(aHb, 0)$

gives us a subsquare of order $2p$.

van Rees loops of order 27

- Elementary abelian group.
- Non-abelian group of exponent 3.
- A Bol loop with trivial center, discovered by [Keedwell'63].
- Two power-associative conjugacy closed loops, described in [Kinyon/Kunen'06].
- A universal left conjugacy closed loop (which is not conjugacy closed) with the left inverse property.
- A commutative, weak inverse property loop.
- A (noncommutative) weak inverse property loop such that each inner mapping of the form $L_x^{-1}R_x$ is an automorphism.

The Bol loop is the only one where each loop in the species has trivial center.

There are no other examples of order 27 with at least one nontrivial nucleus.