Graphs of Latin Regular Hexahedra

Akihiro Yamamura

Department of of Mathematical Science and Electrical-Electronic-Computer Engineering, Akita University, Japan

Combinatorial Designs and Codes
July 8-12, 2024

Institute of Mathematics of the University of Seville, Spain.

System of Latin rectangles

- System of Latin rectangles is a finite set of Latin rectangles
- It has a solution if we can obtain Latin rectangles by substituting a certain Latin rectangle for each variable

Example 1

- System of Latin rectangles where A and B are Latin rectangles and X is a variable

$$\boxed{A \mid X} \qquad \boxed{B \mid X}$$

- It has a solution if there exists a Latin rectangle C so that $\boxed{A \mid C}$ and $\boxed{B \mid C}$ are Latin rectangles

Example 1

- Suppose A and B are arrays $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array}$
- Let C be an array $\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}$
- C makes both $\begin{array}{|c|c|}\hline A & C \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline B & C \\ \hline\end{array}$ Latin rectangles and so C is a solution

Example 1

- Suppose A and B are arrays

1	2
3	4

and

4	3
2	1

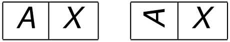
- A system of Latin rectangles

A	X	B	X

has no solution.

Example 2

- A system of Latin rectangles



  - ∢ is obtained from A by rotating $\frac{\pi}{2}$ counterclockwise

Example 2

- Suppose A is an array

1	2
3	1

- $\rotatebox{90}{$A$}$ is the array

2	1
1	3

Example 2

- Let C be an array

3	4
4	2

  is a Latin rectangle
- C makes both $\boxed{A \mid C}$ and $\boxed{A^{T} \mid C}$ Latin rectangles
1	2	3	4
3	1	4	2

    is a Latin rectangle
2	1	3	4
1	3	4	2

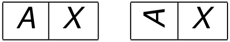
    is a Latin rectangle
- C is a solution

Example 2

- If A is an array

1	2
3	4

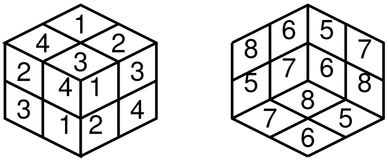
, then



has no solution

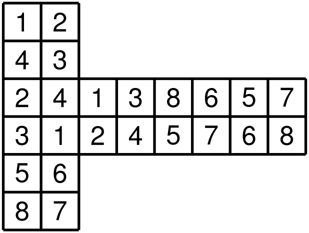
Latin regular hexahedron

- Regular hexahedron of order n is a polyhedron consisting of six faces, each of which forms an $n \times n$ matrix filled with integers in $\{1, 2, 3, \ldots, 4n\}$



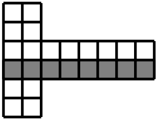
Net

- A net is an arrangement of a non-overlapping edge-joined polygon which can be folded along edges to become faces of the hexahedron



Circuit

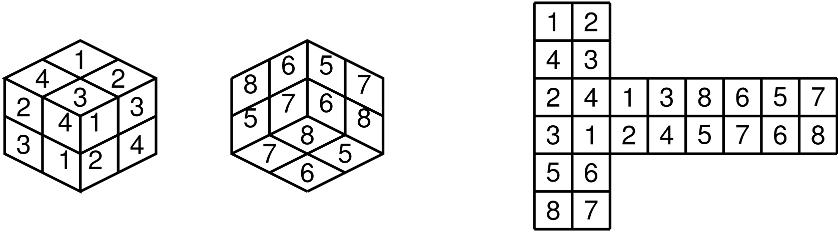
- A circuit of a regular hexahedron of order n is a $1 \times 4n$ subarray in one of its nets



- There exist $3n$ circuits on a regular hexahedron of order n

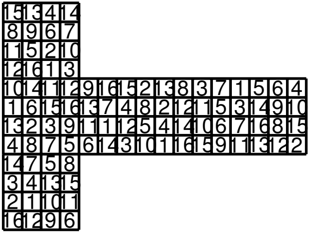
Latin regular hexahedron

- Regular hexahedron of order n is called Latin if every integer in $\{1, 2, 3, \ldots, 4n\}$ appears exactly once in every circuit



Sudoku Latin square

- If each column, each row, and each sub-matrices contain all of the integers from 1 to 9, it is called a Sudoku Latin square



Latin quasi-sudoku regular hexahedron

- If every integer in $\{1, 2, 3, \ldots, 4n\}$ appears exactly m times on each face, it is called a Latin quasi-sudoku regular hexahedron with multiplicity m

System of Latin rectangles

Theorem

A system of Latin rectangles with variables (R, T, U, V, W, Y)

U	V	W	Y

R	U	T	W

Y	R	V	T

(1)

has a solution
If (A, B, C, D, E, F) is a solution of (1), then

A			
C	D	E	F
B			

(2)

provides a net of a Latin regular hexahedron of order $2n$.
Conversely, if (2) is a net of a Latin regular hexahedron of order $2n$, then the sextuple (A, B, C, D, E, F) is a solution of (1).

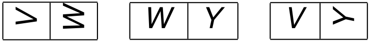
Existence

Theorem

(1) There exists a Latin quasi-sudoku regular hexahedron of order $4n$ with multiplicity n
(2) There exists a Latin sudoku regular hexahedron of order 4

Latin three-axis design

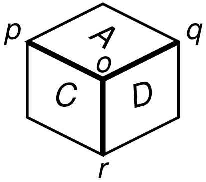
- A system of Latin rectangles with variables (V, W, Y)

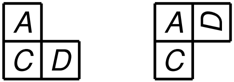


- A solution (A, C, D) is called a Latin three-axis design of order n

Latin three-axis design and its net

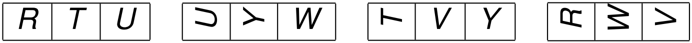
- Latin three-axis design (A, C, D) is considered as a complex with three axes op, oq and or



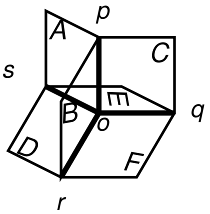


Latin four-axis design

- System of Latin rectangles with variables (R, T, U, V, W, Y)



- A solution (A, B, C, D, E, F) is called a Latin four-axis design



Existence

Theorem

(1) There exists a Latin three-axis design of order $2n$
(2) There exists a Latin four-axis design of order n

1-factor of a graph

- A subgraph of $G = (V, E)$ is called a factor if it includes V
- 1-factor of G is a spanning 1-regular subgraph of G
- G is called 1-factorizable if E can be partitioned into disjoint 1-factors

1-factorization

Lemma

(1) $K_{2n,2n,2n}$ is 1-factorizable if and only if there exists a Latin three-axis design of order $2n$
(2) $K_{n,n,n,n}$ is 1-factorizable if and only if there exists a Latin four-axis design of order n

1-factorization

Theorem

$K_{2n,2n,2n}$ and $K_{n,n,n,n}$ are 1-factorizable for every positive integer n

Perfect 1-factorization

- A 1-factorization of a graph G is called *perfect* if the union of any two distinct 1-factors forms a Hamilton cycle of G
- A complete bipartite graph has a perfect 1-factorization (I.Wanless)
  - $K_{p,p}$ has a perfect 1-factorization for every prime p
  - $K_{2p-1,2p-1}$ has a perfect 1-factorization for every prime p
  - K_{p^2,p^2} have a perfect 1-factorization for every odd prime p

1-factorization of $K_{2,2,2}$

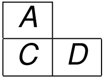
Theorem

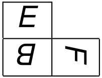
Every 1-factorization of $K_{2,2,2}$ is perfect

- We also observed that $K_{2,2,2,2}$ and $K_{3,3,3,3}$ have perfect 1-factorizations
- Can we characterize integers n for which $K_{n,n,n}$ (or $K_{n,n,n,n}$) has a perfect 1-factorization?
- There exists a pan-Hamiltonian square of order n if and only if $K_{n,n}$ has a perfect 1-factorization (Wanless)
  Can we characterize a similar property for $K_{n,n,n}$ and $K_{n,n,n,n}$?

Construction of a Latin regular hexahedron using Latin three-axis designs

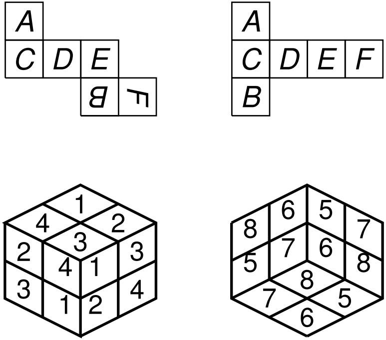
- Suppose L_1 and L_2 are Latin three-axis designs such that
  $|L_1| = \{1, 2, 3, \ldots, 4n\}$ and
  $|L_2| = \{4n+1, 4n+2, 4n+3, \ldots, 8n\}$

L_1 :


L_2 :


- We obtain a Latin regular hexahedron pasting L_1 and L_2

Nets of Latin regular hexahedron obtained by pasting L_1 and L_2



Separable Latin regular hexahedron

- A Latin regular hexahedron is called *separable* if it can be constructed from two three-axis designs D_1 and D_2 with $|D_1| \cap |D_2| = \emptyset$, and *inseparable* otherwise.

Separable Latin regular hexahedron

Theorem

Every Latin regular hexahedron of order 2 is separable.

Example of an inseparable Latin regular hexahedron

- Latin regular hexahedron below is inseparable

15	13	4	14												
8	9	6	7												
11	5	2	10												
12	16	1	3												
10	14	11	12	9	16	15	2	13	8	3	7	1	5	6	4
1	6	15	16	13	7	4	8	2	12	11	5	3	14	9	10
13	2	3	9	11	1	12	5	4	14	10	6	7	16	8	15
4	8	7	5	6	14	3	10	1	16	15	9	11	13	12	2
14	7	5	8												
3	4	13	15												
2	1	10	11												
16	12	9	6												

To check separability

- Let L be a Latin regular hexahedron of order $2n$. For every integer i in $\{1, 2, 3, \ldots, 8n\}$, we define $L(i)$ to be the set of integers that are placed on the cells of L in contrapositions of the cells on which i is placed.
- If L_2 is obtained from L_1 by transposing integers on a cell and its contraposition, then $L_1(i) = L_2(i)$ holds for every $i \in \{1, 2, 3 \ldots, 8n\}$