On Some Cases of the Directed Uniform Hamilton-Waterloo Problem

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Join work with Sibel Özkan and Uğur Odabaşı

Outline

- A *decomposition* of a graph G is a set $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ of edge-disjoint subgraphs of G such that $\bigcup_{i=1}^{k} E(H_i) = E(G)$. It is called an $\{H_1, H_2, \ldots, H_k\}$ -decomposition of G.

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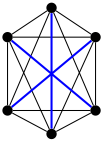

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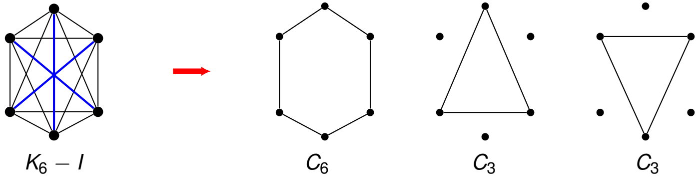

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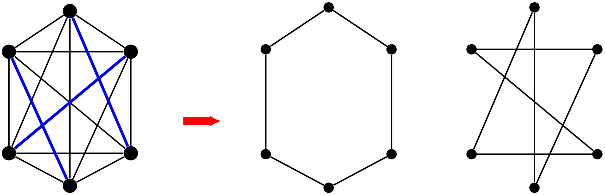

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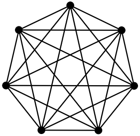

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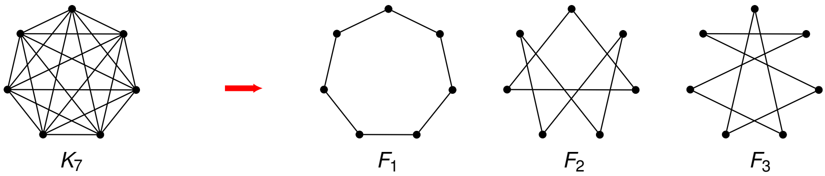

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- If F_1 is an m-cycle factor and F_2 is an n-cycle factor, then the corresponding Hamilton-Waterloo problem is denoted by $\mathrm{HWP}(v; C_m^r, C_n^s)$.

Necessary Conditions

Lemma (Necessary Conditions)

Let v, m, n, r and s be non-negative integers with $m, n \geq 3$. If there exists a solution to $\mathrm{HWP}(v; C_m^r, C_n^s)$, then

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3) $r + s = \left\lfloor \frac{v-1}{2} \right\rfloor$.

- For a simple graph G, we use G^* to denote symmetric digraph with vertex set $V(G^*) = V(G)$ and arc set $A(G^*) = \bigcup_{\{x,y\} \in E(G)} \{(x, y), (y, x)\}$. Hence, K_v^* is the *complete symmetric digraph* of order v.

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- We use $(x, y)^*$ to denote the double arc which consists of (x, y) and (y, x).

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- Similarly, HWP$^*(v; m^r, n^s)$ denotes the uniform directed Hamilton-Waterloo Problem with directed cycle sizes m and n.
- Any of its solutions will be referred to as a $\{\vec{C}_m^r, \vec{C}_n^s\}$-factorization of K_v^*.

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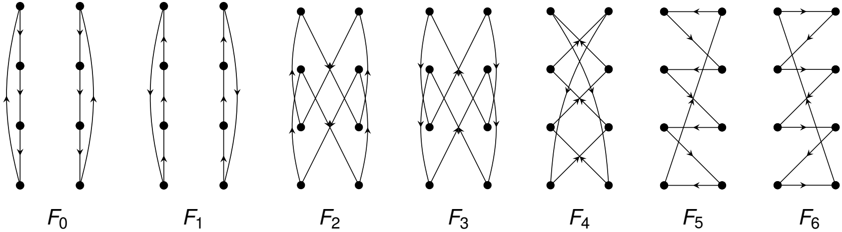

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Observation (F.Yetgin et al. (2023))

If $\mathrm{HWP}(v; m^r, n^s)$ has a solution for some r and s and v is odd, then $\mathrm{HWP}^*(v; m^{2r}, n^{2s})$ has a solution for the same r and s.

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Proposition (F.Yetgin et al. (2023))

Let G be a graph and H be a subgraph of G. If G has an H-factorization then, G^* has an H^*-factorization.

Known Results

The following theorem summarizes the previous results on the uniform version of the Directed Oberwolfach Problem.[**By R. Abel (2002) P. Adams and D. Bryant, J. C. Bermond et al. (1979), F. E. Bennett and Zhang (1990), Burgess and Sajna (2014), A. C. Burgess et al. (2018), A. Lacaze (2023)**]

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Theorem (F.Yetgin et al. (2023))

For nonnegative integers r and s, $\mathrm{HWP}^*(v; m^r, n^s)$ has a solution for

1. $(m, n) \in \{(4, 6), (4, 8), (4, 12), (4, 16), (6, 12), (8, 16)\}$ *when v is even,*
2. $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$ *when v is odd*

if and only if $r + s = v - 1$ and $\mathrm{lcm}(m, n) | v$ except possibly $s \in \{1, 2, 3\}$ when $(m, n) = (3, 5)$ and $s = 1$ when $(m, n) = (3, 15)$.

Main Result

Theorem (F.Yetgin et al. (2023))

Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, $\mathrm{HWP}^*(v; m^r, (2m)^s)$ has a solution if and only if $m|v$, $r + s = v - 1$ and $v \geq 4$ except for $(s, v, m) \in \{(0,4,4),(0,6,3),(0,6,6)\}$, and except possibly when $s \in \{1,3\}$.

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Lemma (F.Yetgin et al. (2023))

K^*_{2x} has a K^*_2-factorization for every integer $x \geq 1$.

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Lemma (Burgess and Sajna (2014))

Let $m \geq 4$ be an even integer and x be a positive integer. Then $K^*_{(\frac{mx}{2}:2)}$ has a $\vec{C}_m$-factorization.

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- If G_1 and G_2 are two edge (arc)-disjoint graphs (digraphs) with $V(G_1) = V(G_2)$, then $G_1 \oplus G_2$ is used to denote the graph on the same vertex set with $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$ $(A(G_1 \oplus G_2) = A(G_1) \cup A(G_2))$.

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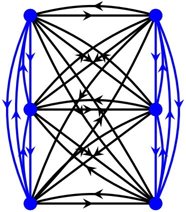

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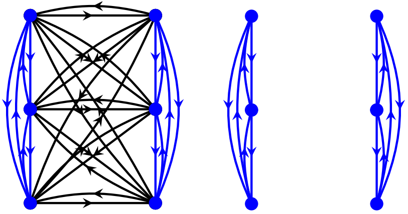

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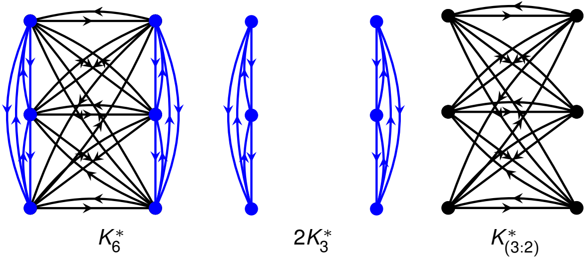

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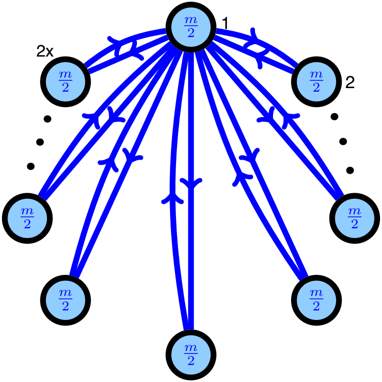

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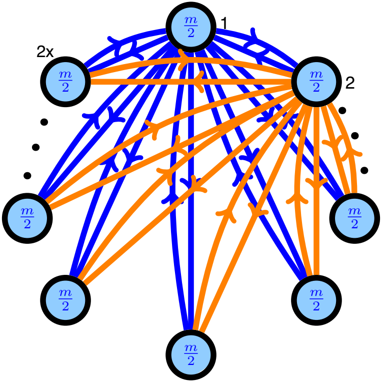

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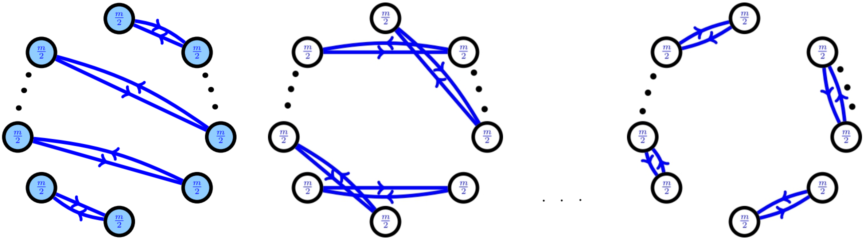

F_1 F_2 $\cdots$ F_{2x-1}

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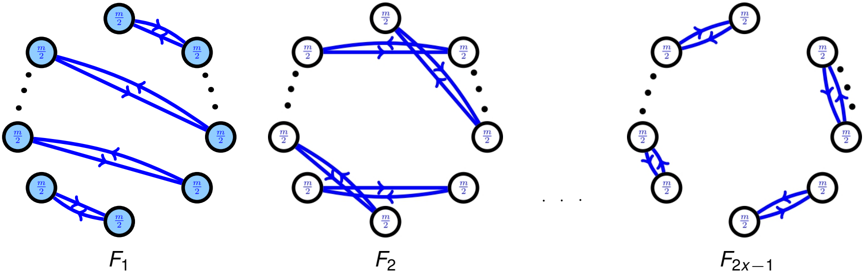

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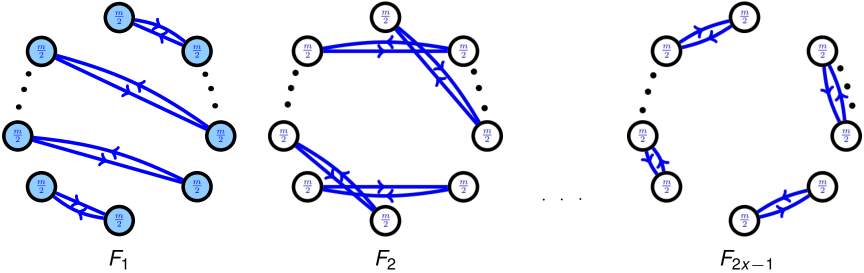

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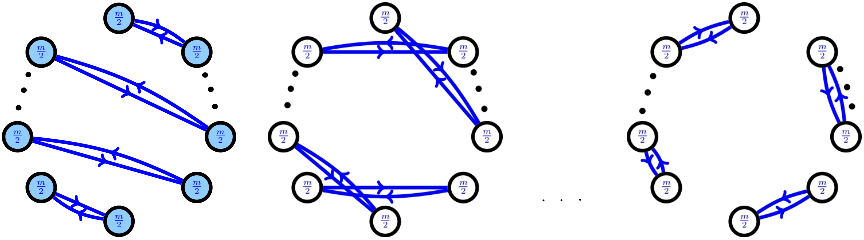

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- Let D be a digraph and $D_0, D_1, \ldots, D_{k-1}$ be k vertex disjoint copies of D with $v_i \in V(D_i)$ for each $v \in V(D)$. Then, $D[k]$ has the vertex set $V(D[k]) = V(D_0) \cup V(D_1) \cup \cdots \cup V(D_{k-1})$ and arc set $A(D[k]) = \{(u_i, v_j) : (u, v) \in A(D) \text{ and } 0 \leq i, j \leq k-1\}$. (Note that $K_m^*[2] \cong K_{2m}^* - mK_2^*$ and $K_y^*[x] \cong K_{(x:y)}^*$)

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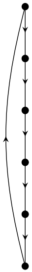

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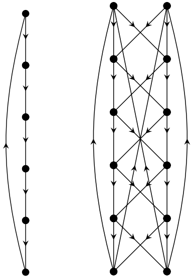

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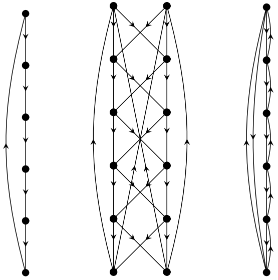

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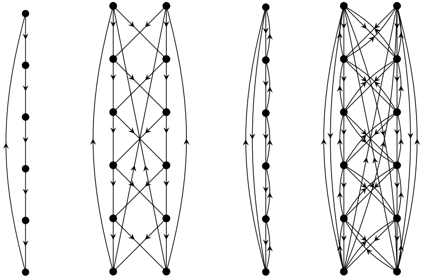

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We use Γ_m^* to denote $C^*[2] \oplus F_m^*[2]$, for the rest of the presentation.

Cayley Graphs

Definition

Let B be a finite additive group and let S be a subset of B, where S does not contain the identity of B. The Directed Cayley graph $\vec{X}(B; S)$ on B with connection set S is a digraph with $V(\vec{X}(B; S)) = B$ and $A(\vec{X}(B; S)) = \{(x, y) : x, y \in B, y - x \in S\}$.

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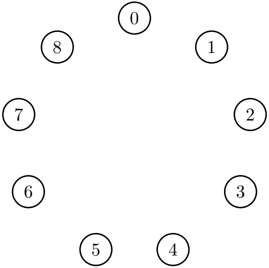

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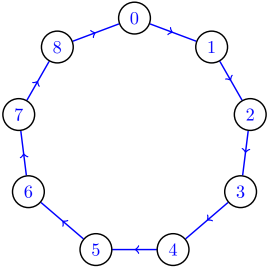

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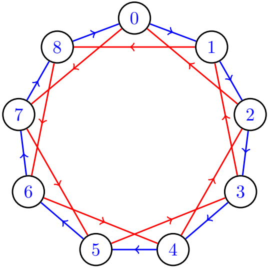

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- We will represent $C_m^*[2]$ and $C_m^*[2] \oplus I_{2m}^*$ as the directed Cayley graphs $\overrightarrow{X}\left(\mathbb{Z}_2 \times \mathbb{Z}_m, S\right)$ and $\overrightarrow{X}\left(\mathbb{Z}_2 \times \mathbb{Z}_m, S \cup \{(1, 0)\}\right)$ where $S = \{(0, 1), (1, 1), (0, -1), (1, -1)\}$.

- Also, a factor F_m^* is defined as a K_2^*-factor of K_m^* with $A(F_m^*) = \{(0, m/2)^*, (i, m-i)^* : 1 \leq i \leq (m/2) - 1\}$. The arc set of F_m^* which is denoted by $A(F_m^*)$, can be expressed as $\{((0,0),(0,m/2))^*, ((0,i),(0,m-i))^* : 1 \leq i \leq (m/2) - 1\}$ using above bijective function.

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- Thus, the vertex set and the arc set of Γ_m^* can be represented as $V(\Gamma_m^*) = \mathbb{Z}_2 \times \mathbb{Z}_m$ and $A(\Gamma_m^*) = \bigcup_{j=0}^{m-1} \left\{\left((i,j),(i,j+1)\right)^*, \left((i,j),(i+1,j+1)\right)^*\right\} \cup A(F_m^*)$ for $i = 0, 1$, respectively.

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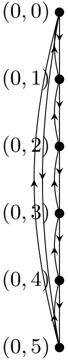

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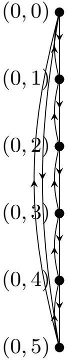

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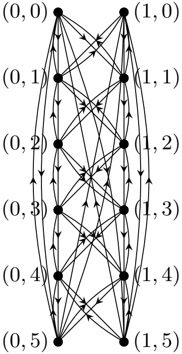

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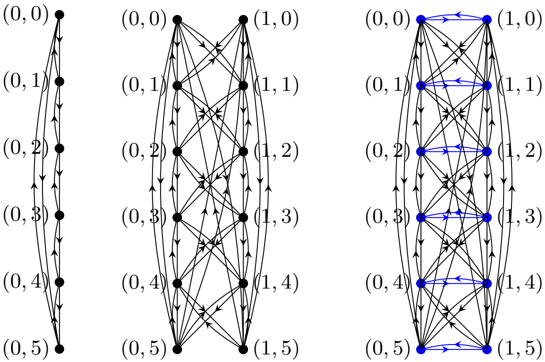

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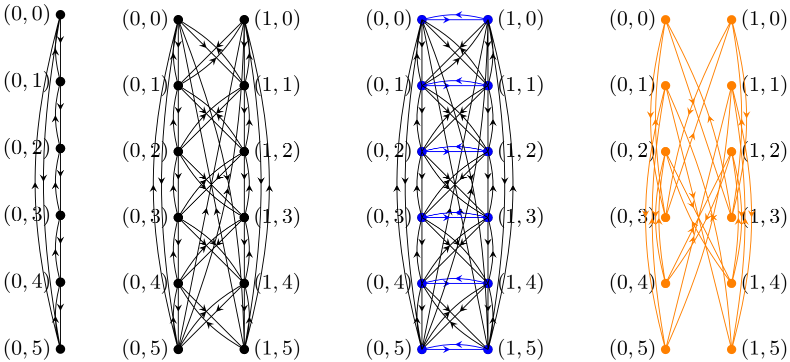

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Solutions to HWP$^*(v; m^r, (2m)^s)$

Lemma (F.Yetgin et al. (2023))

Let $m \geq 4$ be even integer, then Γ_m^* has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$ -factorization for $r \in \{0, 6\}$ and $r + s = 6$.

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Proof : ($r = 0$) Γ_m^* has a $\vec{C}_{2m}$-factorization.

Solutions to HWP$^*(v; m^r, (2m)^s)$

Lemma (F.Yetgin et al. (2023))

Let $m \geq 4$ be even integer, then Γ_m^* has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$ -factorization for $r \in \{0, 6\}$ and $r + s = 6$.

Proof : ($r = 0$) Γ_m^* has a $\vec{C}_{2m}$-factorization.
($r = 6$) Γ_m^* has a $\vec{C}_m$-factorization for $m \equiv 0 \pmod 4$.

When $m \equiv 2 \pmod 4$, define the following directed m-cycles.

$$\vec{C}_m^{(0)} = (v_0, v_1, \ldots - v_{m-1}) \quad \text{where } v_i = (0, i) \quad \text{for } 0 \leqslant i \leqslant m-1.$$

When $m \equiv 2 \pmod 4$, define the following directed m-cycles.

$$\vec{C}_m^{(0)} = (v_0, v_1, \ldots - v_{m-1}) \quad \text{where } v_i = (0, i) \quad \text{for } 0 \leqslant i \leqslant m-1.$$

$$\vec{C}_m^{(1)} = (x_0, x_1, \ldots x_{m-1}) \text{ where } x_0 = (0,0) \text{ and for } 1 \le i \le m-1$$

$$x_{i=} \begin{cases} \left(\frac{1-(-1)^i}{2}, \frac{m}{2} - \lfloor \frac{i}{2} \rfloor\right), \text{ for } i \equiv 1,2 \pmod 4 \\ \left(\frac{1-(-1)^i}{2}, \frac{m}{2} + \lfloor \frac{i}{2} \rfloor\right), \text{ for } i = 0,3 \pmod 4 \end{cases}$$

$$\vec{C}_m^{(2)} = (u_0, u_1, \ldots, u_{m-1}) \text{ where } u_i = \begin{cases} (1, m-1-i) & \text{if } 0 \le i \le \frac{m}{2}, \\ (0, m-1-i) & \text{if } \frac{m}{2} + 1 \le i \le m-1. \end{cases}$$

$\vec{C}_m^{(3)} = (y_0, y_1, \ldots y_{m-1})$ where $y_0 = (0,0)$, $y_1 = (0, \frac{m}{2})$, $y_2 = (1, \frac{m}{2}+1)$, $y_3 = (1, \frac{m}{2}-1)$ and

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$\vec{C}_m^{(4)} = (z_0, z_1, \ldots, z_{m-1})$ where

$$z_{i=} \begin{cases} y_{m-i} + (1,0) & \textit{if } \ 1 \le i \le m-3 \\ y_{m-i} & \textit{if } \ m-2 \le i \le m \end{cases}.$$

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$F_0 = \vec{C}_m^{(0)} \cup (\vec{C}_m^{(0)} + (1,0))$, $F_1 = \vec{C}_m^{(1)} \cup R(\vec{C}_m^{(1)} + (1,0))$,

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$$y_i = \begin{cases} \left(1, \frac{m}{2} + (-1)^{i+1} \lfloor \frac{i}{2} \rfloor\right) \text{ if } i \equiv 0,1 \pmod 4 \\ \left(0, \frac{m}{2} + (-1)^{i} \lfloor \frac{i}{2} \rfloor\right) \text{ if } i = 2,3 \pmod 4 \end{cases} \text{ for } 4 \leq i \leq m-1.$$

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Then, $\{F_0, F_1, F_2, F_3, F_4, F_5\}$ is a $\vec{C}_m$-factorization of Γ_m^*.

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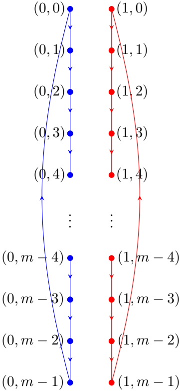

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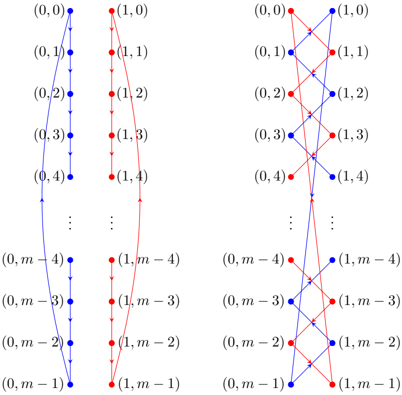

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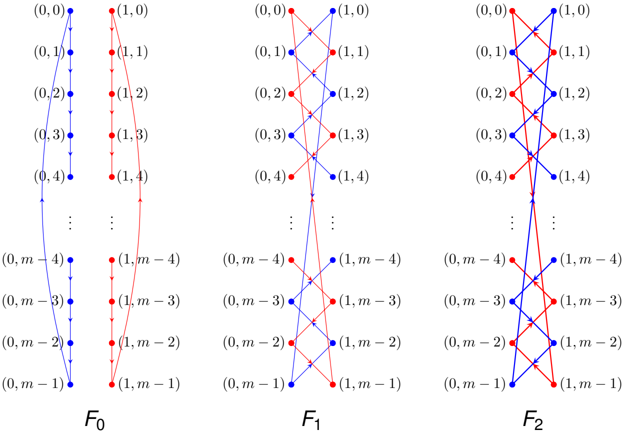
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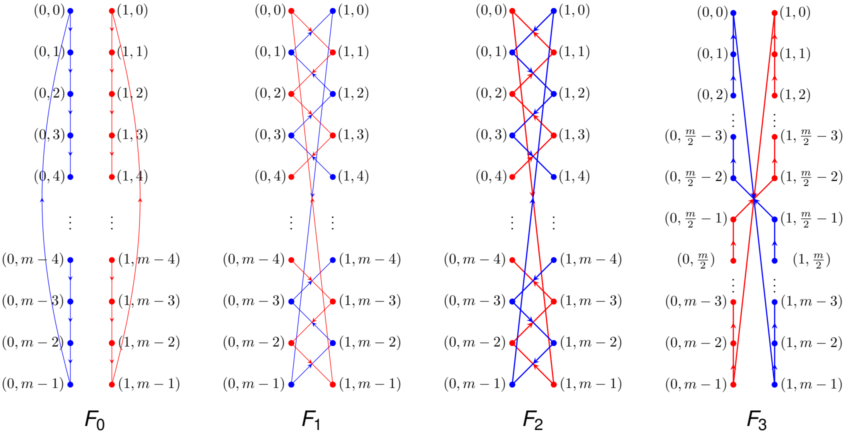
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F_0 F_1 F_2 F_3

Lemma (F.Yetgin et al. (2023))

Let $m \geq 4$ be even integer, then $C_m^*[2]$ has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $r \in \{0, 2, 4\}$ and $r + s = 4$.

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Lemma (F.Yetgin et al. (2023))

Let $m \geq 4$ be even integer, then $C_m^*[2] \oplus I_{2m}^*$ has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$- factorization for $r \in \{0, 1, 3\}$ and $r + s = 5$.

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Theorem (F.Yetgin et al. (2023))

Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, $\mathrm{HWP}^*\ (v; m^r, (2m)^s)$ has a solution if and only if $m \mid v$, $r + s = v - 1$ and $v \geq 4$ except for $(s, v, m) \in \{(0, 4, 4), (0, 6, 3)\}$ and $(r, v, m) \in \{(0, 6, 6)\}$, and except possibly for $s \in \{1, 3\}$.

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Proof : We can factorize K^*_{2mx} as follows :

$$K^*_{2mx} \cong xK^*_{2m} \oplus \underbrace{xK^*_{(m:2)} \oplus xK^*_{(m:2)} \oplus \ldots \oplus xK^*_{(m:2)}}_{2x-2} \tag{3}$$

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Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, HWP^* $(v; m^r, (2m)^s)$ *has a solution if and only if* $m \mid v$, $r + s = v - 1$ *and* $v \geq 4$ *except for* $(s, v, m) \in \{(0,4,4),(0,6,3)\}$ *and* $(r, v, m) \in \{(0,6,6)\}$, *and except possibly for* $s \in \{1,3\}$.

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$$K^*_{2m} \cong K^*_m[2] \oplus I^*_{2m} \cong C^*_m[2] \oplus C^*_m[2] \oplus \ldots \oplus C^*_m[2] \oplus F^*_m[2] \oplus I^*_{2m} \tag{4}$$

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So, K^*_{2m} has a $\{(C^*_m[2])^{\frac{m-6}{2}}, C^*_m[2] \oplus I^*_{2m}, \Gamma^*_m\}$-factorization.

Conclusion

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Conclusion

- Case which remain to solve $\mathrm{HWP}^*(v; m^r, (2m)^s)$ is $s \in \{1, 3\}$,
  1. $C_m^*[2]$ must have a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $s = 1$
  2. $C_m^*[2] \oplus I_{2m}^*$ must have a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $s = 1$
  3. Γ_m^* must have a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $s = 1$
  4. $K_{(m:2)}^*$ must have a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $s = 1$

Thank You for Your Attention

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