

On Some Cases of the Directed Uniform Hamilton-Waterloo Problem

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Join work with Sibel Özkan and Uğur Odabaşı

Outline

- 1 Introduction
- 2 The Directed Hamilton-Waterloo Problem
- 3 Preliminary Results
- 4 Solutions to HWP* ($v; m^r, (2m)^s$)

- A *decomposition* of a graph G is a set $\mathcal{H} = \{H_1, H_2, \dots, H_k\}$ of edge-disjoint subgraphs of G such that $\bigcup_{i=1}^k E(H_i) = E(G)$. It is called an $\{H_1, H_2, \dots, H_k\}$ -decomposition of G .

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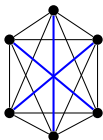
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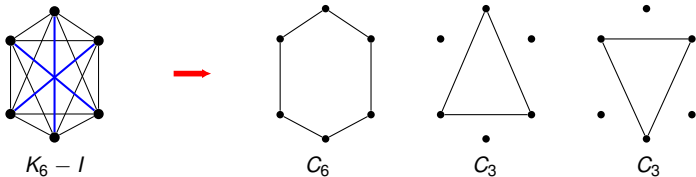


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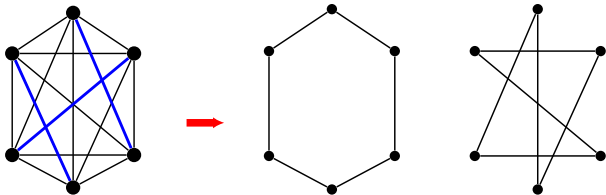
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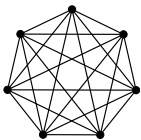
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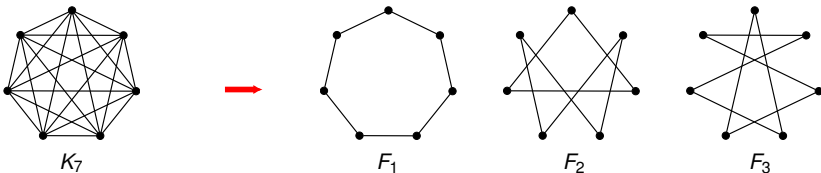


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- If F_1 is an m -cycle factor and F_2 is an n -cycle factor, then the corresponding Hamilton-Waterloo problem is denoted by $\text{HWP}(v; C_m^r, C_n^s)$.

Necessary Conditions

Lemma (Necessary Conditions)

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- For a simple graph G , we use G^* to denote symmetric digraph with vertex set $V(G^*) = V(G)$ and arc set $A(G^*) = \bigcup_{\{x,y\} \in E(G)} \{(x,y), (y,x)\}$. Hence, K_v^* is the *complete symmetric digraph* of order v .

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- We use $(x,y)^*$ to denote the double arc which consists of (x,y) and (y,x) .

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- Similarly, $HWP^*(v; m^f, n^S)$ denotes the uniform directed Hamilton-Waterloo Problem with directed cycle sizes m and n .

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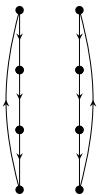
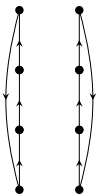
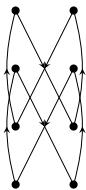
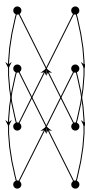
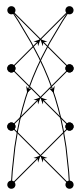
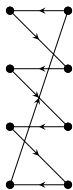
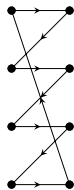
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- Any of its solutions will be referred to as a $\{\overset{\#}{C}_m^r, \overset{\#}{C}_n^S\}$ -factorization of K_V^* .

Example

HWP*(8; 4⁵, 8²) has a solution. ({ C₄⁵, C₈² }-factorization of K₈^{*})

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Observation (F.Yetgin et al. (2023))

If $\text{HWP}(v; m^r, n^s)$ has a solution for some r and s and v is odd, then $\text{HWP}^*(v; m^{2r}, n^{2s})$ has a solution for the same r and s .

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Proposition (F.Yetgin et al. (2023))

Let G be a graph and H be a subgraph of G . If G has an H -factorization then, G^* has an H^* -factorization.

Known Results

The following theorem summarizes the previous results on the uniform version of the Directed Oberwolfach Problem. [By **R. Abel (2002)**, **P. Adams and D. Bryant, J. C. Bermond et al. (1979)**, **F. E. Bennett and Zhang (1990)**, **Burgess and Sajna (2014)**, **A. C. Burgess et al. (2018)**, **A. Lacaze (2023)**]

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Let m and k be nonnegative integers. Then, $\text{OP}^(m^k)$ has a solution if and only if $(m, k) \notin \{(3, 2), (4, 1), (6, 1)\}$.*

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Theorem (F.Yetgin et al. (2023))

For nonnegative integers r and s , $\text{HWP}^(v; m^r, n^s)$ has a solution for*

- $(m, n) \in \{(4, 6), (4, 8), (4, 12), (4, 16), (6, 12), (8, 16)\}$ when v is even,*
- $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$ when v is odd*

if and only if $r + s = v - 1$ and $\text{lcm}(m, n) \mid v$ except possibly $s \in \{1, 2, 3\}$ when $(m, n) = (3, 5)$ and $s = 1$ when $(m, n) = (3, 15)$.

Main Result

Theorem (F.Yetgin et al. (2023))

Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, $\text{HWP}^(v; m^r, (2m)^s)$ has a solution if and only if $m|v$, $r + s = v - 1$ and $v \geq 4$ except for $(s, v, m) \in \{(0, 4, 4), (0, 6, 3), (0, 6, 6)\}$, and except possibly when $s \in \{1, 3\}$.*

Preliminary Results

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The complete symmetric equipartite digraph $K_{(x;y)}^*$ has a $C_m^{\#}$ -factorization for $m \geq 3$ and $x \geq 2$ if $m|xy$, $x(y-1)$ is even, m is even when $y = 2$.

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Lemma (Burgess and Sajna (2014))

Let $m \geq 4$ be an even integer and x be a positive integer. Then $K_{(\frac{mx}{2}:2)}^*$ has a $\#C_m$ -factorization.

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- If G_1 and G_2 are two edge (arc)-disjoint graphs (digraphs) with $V(G_1) = V(G_2)$, then $G_1 \oplus G_2$ is used to denote the graph on the same vertex set with $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$ ($A(G_1 \oplus G_2) = A(G_1) \cup A(G_2)$).

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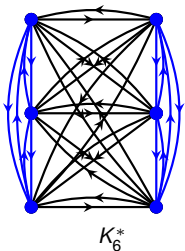
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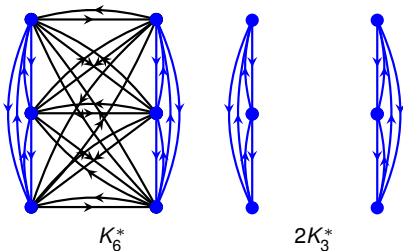


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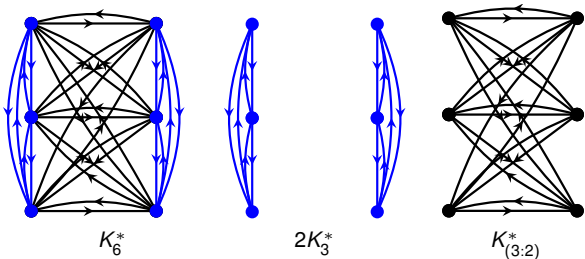


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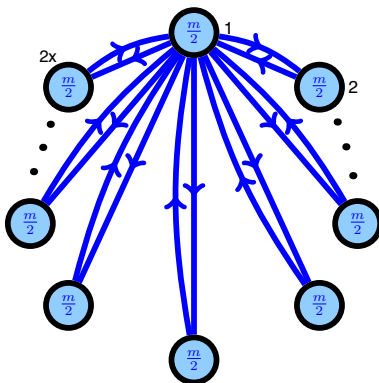
- If G_1 and G_2 are two edge (arc)-disjoint graphs (digraphs) with $V(G_1) = V(G_2)$, then $G_1 \oplus G_2$ is used to denote the graph on the same vertex set with $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$ ($A(G_1 \oplus G_2) = A(G_1) \cup A(G_2)$).
- αG will denote the vertex disjoint union of the α copies of G .

Example

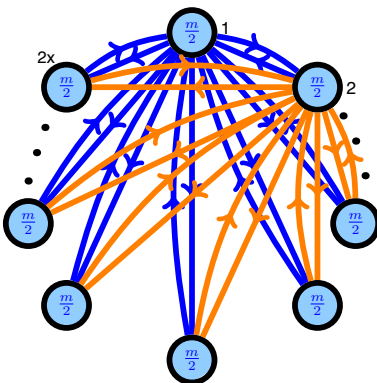
K_6^* is isomorphic to $2K_3^* \oplus K_{(3:2)}^*$



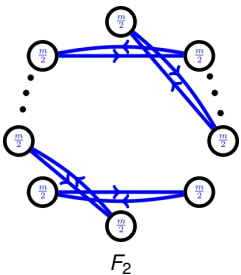
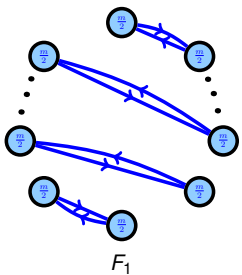
The Blow-Up Construction



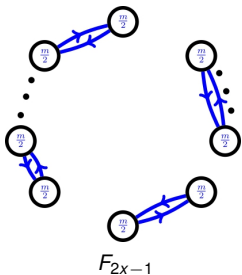
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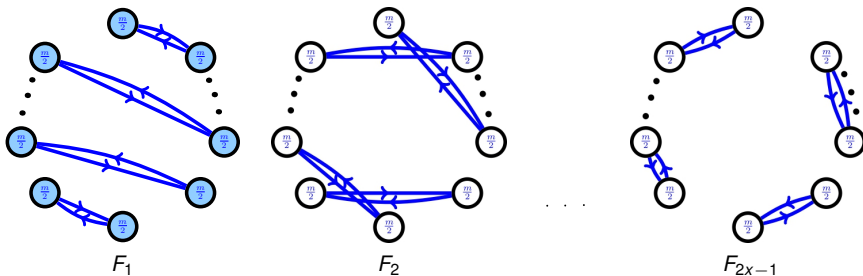
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...

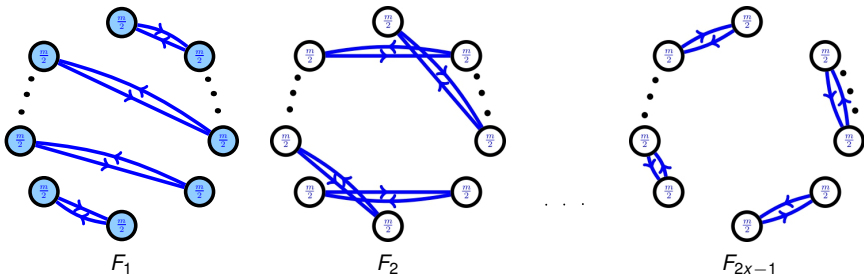


The Blow-Up Construction



- $K_{m \times}^*$ has a $\{F_1, F_2, \dots, F_{2x-1}\}$ -factorization. (K_{2x}^* has a K_2^* -factorization)

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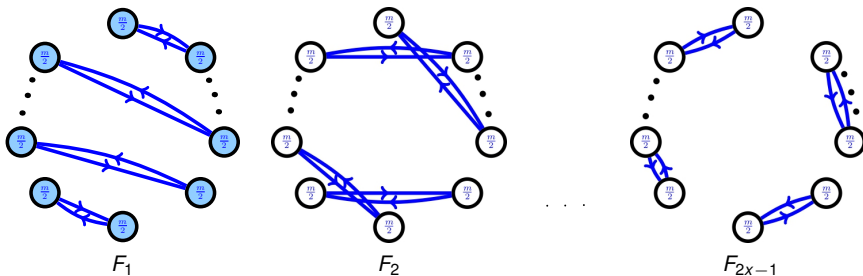


- $K_{m^x}^*$ has a $\{F_1, F_2, \dots, F_{2x-1}\}$ -factorization. ($K_{2^x}^*$ has a K_2^* -factorization)



$$K_{m^x}^* \cong xK_m^* \oplus \underbrace{xK_{\left(\frac{m}{2}; 2\right)}^* \oplus xK_{\left(\frac{m}{2}; 2\right)}^* \oplus \dots \oplus xK_{\left(\frac{m}{2}; 2\right)}^*}_{2x-2}$$

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The Power of a Graph

- Let D be a digraph and D_0, D_1, \dots, D_{k-1} be k vertex disjoint copies of D with $v_i \in V(D_i)$ for each $v \in V(D)$. Then, $D[k]$ has the vertex set $V(D[k]) = V(D_0) \cup V(D_1) \cup \dots \cup V(D_{k-1})$ and arc set $A(D[k]) = \{(u_i, v_j) : (u, v) \in A(D) \text{ and } 0 \leq i, j \leq k-1\}$. (Note that $K_m^*[2] \cong K_{2m}^* - mK_2^*$ and $K_y^*[x] \cong K_{(x \cdot y)}^*$)

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$\overset{\#}{C}_6, \overset{\#}{C}_6[2], C_6^*$ and $C_6^*[2]$ can be graphed as follows.

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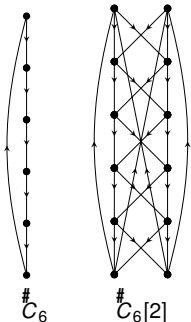


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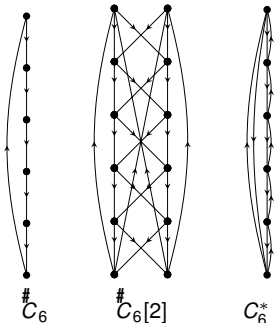


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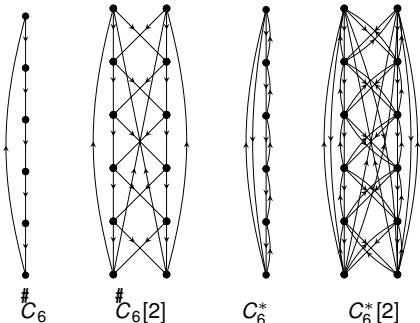


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For all odd $m \geq 3$, K_m decomposes into $\left(\frac{m-1}{2}\right)$ Hamilton cycles.

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We use Γ_m^* to denote $C^*[2] \oplus F_m^*[2]$, for the rest of the presentation.

Cayley Graphs

Definition

Let B be a finite additive group and let S be a subset of B , where S does not contain the identity of B . The Directed Cayley graph $X(B; S)$ on B with connection set S is a digraph with $V(X(B; S)) = B$ and $A(X(B; S)) = \{(x, y) : x, y \in B, y - x \in S\}$.

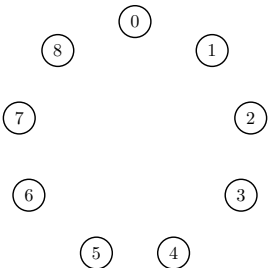
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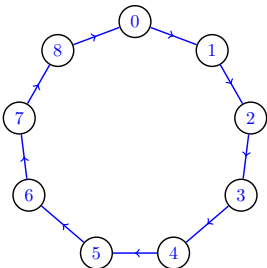
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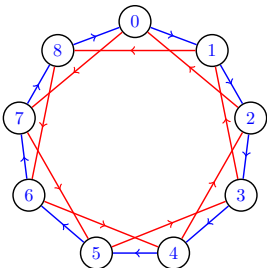
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- We will represent $C_{m\#}^*[2]$ and $C_m^*[2] \oplus I_{2m}^*$ as the directed Cayley graphs $\overset{\#}{X}(\mathbb{Z}_2 \times \mathbb{Z}_m, S)$ and $\overset{\#}{X}(\mathbb{Z}_2 \times \mathbb{Z}_m, S \cup \{(1, 0)\})$ where $S = \{(0, 1), (1, 1), (0, -1), (1, -1)\}$.

- Also, a factor F_m^* is defined as a K_2^* -factor of K_m^* with $A(F_m^*) = \{(0, m/2)^*, (i, m-i)^* : 1 \leq i \leq (m/2) - 1\}$. The arc set of F_m^* which is denoted by $A(F_m^*)$, can be expressed as $\{(0, 0), (0, m/2)\}^*, \{(0, i), (0, m-i)\}^* : 1 \leq i \leq (m/2) - 1\}$ using above bijective function.

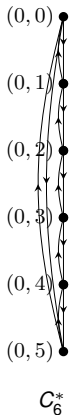
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- Thus, the vertex set and the arc set of Γ_m^* can be represented as $V(\Gamma_m^*) = \mathbb{Z}_2 \times \mathbb{Z}_m$ and $A(\Gamma_m^*) = \bigcup_{j=0}^{m-1} \left\{ ((i, j), (i, j+1))^*, ((i, j), (i+1, j+1))^* \right\} \cup A(F_m^*)$ for $i = 0, 1$, respectively.

Example

C_6^* , $C_6^*[2]$, $C_6^*[2] \oplus I_{12}^*$ and $F_m^*[2]$ can be graphed as follows.

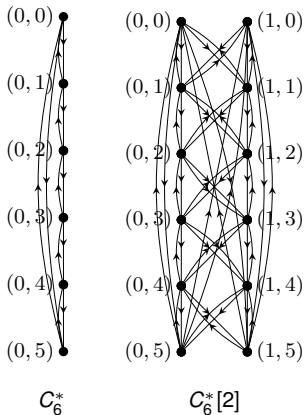
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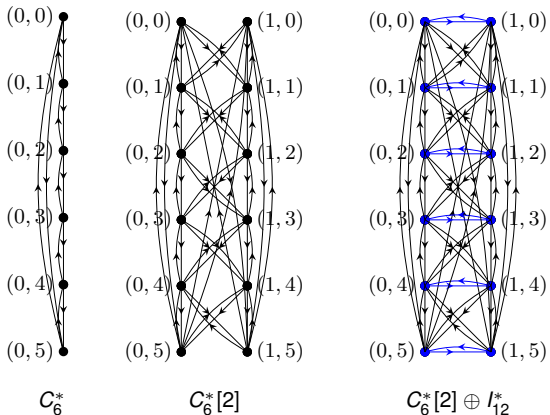
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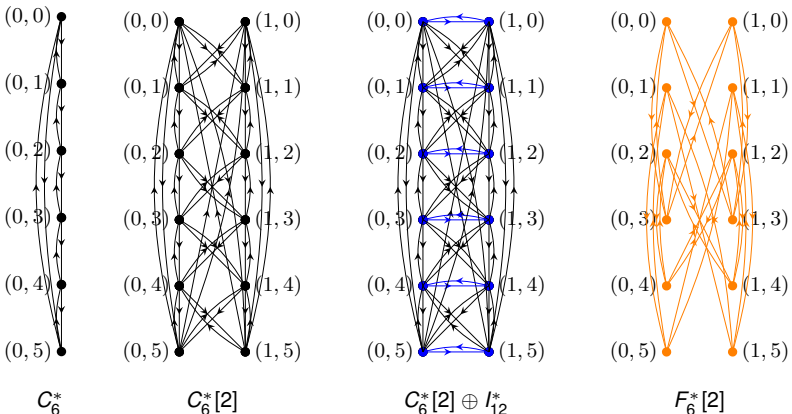
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Solutions to HWP* ($v; m^r, (2m)^s$)

Lemma (F.Yetgin et al. (2023))

Let $m \geq 4$ be even integer, then Γ_m^* has a $\{\overset{\#}{C}_m^r, \overset{\#}{C}_{2m}^s\}$ -factorization for $r \in \{0, 6\}$ and $r + s = 6$.

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Proof : ($r = 0$) Γ_m^* has a $\overset{\#}{C}_{2m}$ -factorization.

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($r = 6$) Γ_m^* has a C_m -factorization for $m \equiv 0 \pmod{4}$.

When $m \equiv 2 \pmod{4}$, define the following directed m -cycles.

$$\#C_m^{(0)} = (v_0, v_1, \dots, v_{m-1}) \quad \text{where } v_i = (0, i) \quad \text{for } 0 \leq i \leq m-1.$$

When $m \equiv 2 \pmod{4}$, define the following directed m -cycles.

$$\#C_m^{(0)} = (v_0, v_1, \dots, v_{m-1}) \quad \text{where } v_i = (0, i) \quad \text{for } 0 \leq i \leq m-1.$$

$$\#C_m^{(1)} = (x_0, x_1, \dots, x_{m-1}) \quad \text{where } x_0 = (0, 0) \quad \text{and for } 1 \leq i \leq m-1$$

$$x_i = \begin{cases} \left(\frac{1-(-1)^i}{2}, \frac{m}{2} - \lfloor \frac{i}{2} \rfloor \right), & \text{for } i \equiv 1, 2 \pmod{4} \\ \left(\frac{1-(-1)^i}{2}, \frac{m}{2} + \lfloor \frac{i}{2} \rfloor \right), & \text{for } i \equiv 0, 3 \pmod{4} \end{cases}$$

$$\#C_m^{(2)} = (u_0, u_1, \dots, u_{m-1}) \quad \text{where } u_i = \begin{cases} (1, m-1-i) & \text{if } 0 \leq i \leq \frac{m}{2}, \\ (0, m-1-i) & \text{if } \frac{m}{2} + 1 \leq i \leq m-1. \end{cases}$$

$C_m^{(3)} = (y_0, y_1, \dots, y_{m-1})$ where $y_0 = (0, 0)$, $y_1 = (0, \frac{m}{2})$, $y_2 = (1, \frac{m}{2} + 1)$,
 $y_3 = (1, \frac{m}{2} - 1)$ and

$$y_i = \begin{cases} \left(1, \frac{m}{2} + (-1)^{i+1} \lfloor \frac{i}{2} \rfloor \right) & \text{if } i \equiv 0, 1 \pmod{4} \\ \left(0, \frac{m}{2} + (-1)^i \lfloor \frac{i}{2} \rfloor \right) & \text{if } i \equiv 2, 3 \pmod{4} \end{cases} \quad \text{for } 4 \leq i \leq m-1.$$

$\#C_m^{(3)} = (y_0, y_1, \dots, y_{m-1})$ where $y_0 = (0, 0)$, $y_1 = (0, \frac{m}{2})$, $y_2 = (1, \frac{m}{2} + 1)$,
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$\#C_m^{(4)} = (z_0, z_1, \dots, z_{m-1})$ where

$$z_i = \begin{cases} y_{m-i} + (1, 0) & \text{if } 1 \leq i \leq m-3 \\ y_{m-i} & \text{if } m-2 \leq i \leq m \end{cases}.$$

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$$F_0 = \vec{C}_m^{(0)} \cup (\#C_m^{(0)} + (1, 0)), \quad F_1 = \#C_m^{(1)} \cup R(\#C_m^{(1)} + (1, 0)),$$

$\#C_m^{(3)} = (y_0, y_1, \dots, y_{m-1})$ where $y_0 = (0, 0)$, $y_1 = (0, \frac{m}{2})$, $y_2 = (1, \frac{m}{2} + 1)$,
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$$F_5 = \#C_m^{(4)} \cup (\#C_m^{(4)} + (1, 0)).$$

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Then, $\{F_0, F_1, F_2, F_3, F_4, F_5\}$ is a $\#C_m$ -factorization of Γ_m^* .

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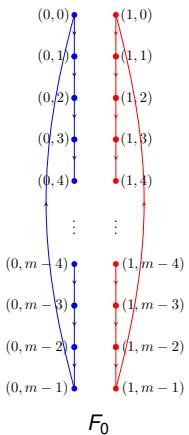
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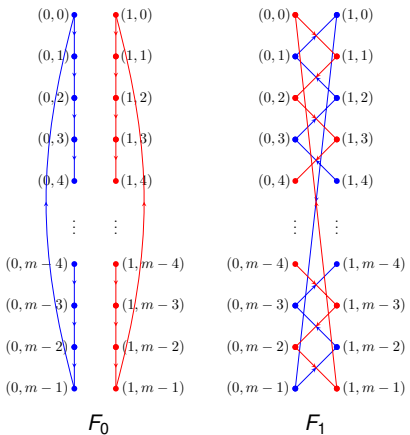
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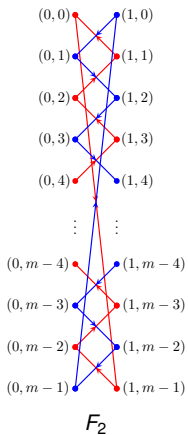
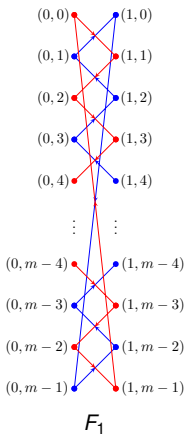
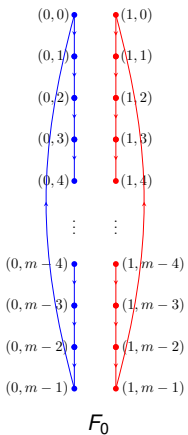
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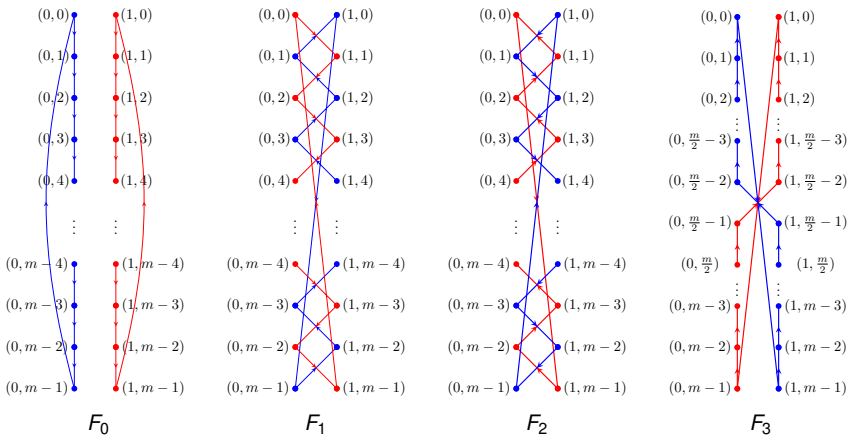
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Then, $\{F_0, F_1, F_2, F_3, F_4, F_5\}$ is a $\#C_m$ -factorization of Γ_m^* .









Lemma (F.Yetgin et al. (2023))

Let $m \geq 4$ be even integer, then $C_m^*[2]$ has a $\{\overset{\#}{C}_m^r, \overset{\#}{C}_{2m}^s\}$ -factorization for $r \in \{0, 2, 4\}$ and $r + s = 4$.

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Theorem (F.Yetgin et al. (2023))

Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, $\text{HWP}^(v; m^r, (2m)^s)$ has a solution if and only if $m \mid v$, $r + s = v - 1$ and $v \geq 4$ except for $(s, v, m) \in \{(0, 4, 4), (0, 6, 3)\}$ and $(r, v, m) \in \{(0, 6, 6)\}$, and except possibly for $s \in \{1, 3\}$.*

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Proof : We can factorize K_{2mx}^* as follows :

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So, K_{2m}^* has a $\{(C_m^*[2])^{\frac{m-6}{2}}, C_m^*[2] \oplus I_{2m}^*, \Gamma_m^*\}$ -factorization.

Conclusion

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Conclusion

- Case which remain to solve HWP* $(v, m^f, (2m)^s)$ is $s \in \{1, 3\}$,
 - 1 $C_m^*[2]$ must have a $\{C_m^r, C_{2m}^s\}$ -factorization for $s = 1$
 - 2 $C_m^*[2] \oplus I_{2m}^*$ must have a $\{C_m^r, C_{2m}^s\}$ -factorization for $s = 1$
 - 3 Γ_m^* must have a $\{C_m^r, C_{2m}^s\}$ -factorization for $s = 1$
 - 4 $K_{(m,2)}^*$ must have a $\{C_m^r, C_{2m}^s\}$ -factorization for $s = 1$

Thank You for Your Attention

References

- [1] B. Alspach, H. Gavlas, M. Sajna, and H. Verrall, *Cycle decompositions IV : complete directed graphs and fixed length directed cycles*, J. Comb. Theory Ser. A. **103(1)** (2003), 165-208.
- [2] P. Adams, E. J. Billington, D. E. Bryant, and S. I. El-Zanati, *On the Hamilton-Waterloo problem*, Graphs Combin. **18** (2002), 31-51.
- [3] D. Bryant, P. Danziger, *On bipartite 2-factorizations of $K_n - I$ and the Oberwolfach problem*, J. Graph Theory **68(1)** (2011), 22-37.
- [4] D. Bryant, P. Danziger, and M. Dean, *On the Hamilton-Waterloo Problem for Bipartite 2-Factors*, J. Comb. Des. **21(2)** (2013), 60-80.

References

- [5] J. C. Bermond, A. Germa, and D. Sotteau, *Resolvable decomposition of K_n^** , J. Comb. Theory Ser. A. **26(2)** (1979), 179-185.
- [6] F. E. Bennett, X. Zhang, *Resolvable Mendelsohn designs with block size 4*, Aequationes Math. **40(1)** (1990), 248-260.
- [7] A. Burgess, N. Francetic, and M. Sajna, *On the directed Oberwolfach Problem with equal cycle lengths : the odd case*, Australas. J. Comb. **71(2)** (2018), 272-292.
- [8] A. Burgess, M. Sajna, *On the directed Oberwolfach Problem with equal cycle lengths*, Electron. J. Comb. **21(1)** (2014), 1-15.

- [9] A. Burgess, P. Danziger, and T. Traetta, *On the Hamilton-Waterloo problem with odd orders*, J. Comb. Des. **25(6)** (2017), 258-287.
- [10] A. Burgess, P. Danziger, and T. Traetta, *On the Hamilton-Waterloo problem with odd cycle lengths*, J. Comb. Des. **26(2)** (2018), 51-83.
- [11] S. Bonvicini, M. Buratti, *Octahedral, dicyclic and special linear solutions of some Hamilton-Waterloo problems*, Ars Math. Contemp. **14(1)** (2017), 1-14
- [12] P. Danziger, G. Quattrocchi, and B. Stevens, *The Hamilton-Waterloo problem for cycle sizes 3 and 4*, J. Comb. Des., **17(4)** (2009), 342-352.

- [13] R. K. Guy, *Unsolved combinatorial problems*, In : *Proceedings of the Conference on Combinatorial Mathematics and Its Applications* , Oxford, 1967 (D. J. A. Welsh, Ed.), Academic Press, New York, 1971.
- [14] R. Haggkvist, *A lemma on cycle decompositions*, North-Holland Mathematics Studies **115** (1985), 227-23
- [15] W. Imrich, S. Klavzar *Product graphs : Structure and Recognition*, John Wiley and Sons Incorporated, New York, 2000.
- [16] M. Keranen, S. Özkan, *The Hamilton-Waterloo problem with 4-cycles and a single factor of n-cycles*, Graphs Combin. **29** (2013), 1827–1837.
- [17] J. Liu, *The equipartite Oberwolfach problem with uniform tables*, J. Comb. Theory Ser. A. **101** (2003), 20–34.

E. Shabani, M. Sajna, *On the Directed Oberwolfach Problem with variable cycle lengths*, 2020, arXiv preprint arXiv :2009.08731.

[19] U. Odabasi, S. Özkan, *The Hamilton-Waterloo problem with C_4 and C_m factors*, Discrete Math. **339(1)** (2016), 263-269.

[20] F. Yetgin, U. Odabasi and S. Özkan, *On the Directed Hamilton-Waterloo Problem with Two Cycle Sizes*, Contributions to Discrete Mathematics, (Accepted).

[21] F. Yetgin, U. Odabasi and S. Özkan, *The Directed Uniform Hamilton-Waterloo Problem Involving Even Cycle Sizes*, Discussiones Mathematicae Graph Theory, (Accepted).

[22] U. Odabasi, *Factorizations of complete graphs into cycles and 1-factors*, Contributions to Discrete Mathematics **15(1)** (2020), 80-89.

[23] E. Lucas, *Recreations mathematiques*, vol. 2, Gauthier-Villars, Paris, 1892.