Lorentzian polynomials

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SLC 91, March 2024

- Let $\mathrm{is}(\pi)$ be the length of the longest increasing sequence in the permutation π.
- Let

$$u_k(n) = |\{\pi \in \mathfrak{S}_n : \mathrm{is}(\pi) \le k\}|,$$

  and

$$U_k(x) = \sum_{n \ge 0} u_k(n) \frac{x^{2n}}{n!^2}.$$

- Theorem (Gessel, 1990).

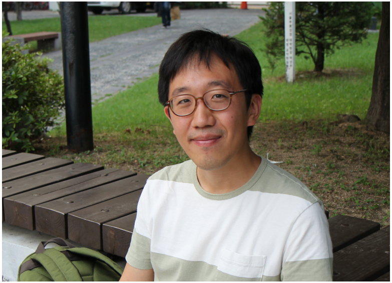
$$U_k(x) = \det(J_{i-j}(x))_{i,j=1}^{k},$$

  where

$$J_i(x) = \sum_{n \ge 0} \frac{x^{2n+i}}{n!(n+i)!}.$$

Lorentzian polynomials

Based on joint work with



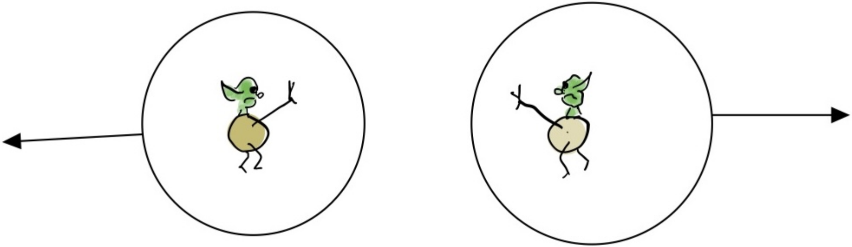
June Huh (Princeton) Jonathan Leake (Waterloo)

Outline

- Negative dependence and independence
- Two fundamental problems on independence
- Matroid Potts model and Random-cluster model
- Lorentzian polynomials
- Lorentzian polynomials and Matroid theory
- Proofs of conjectures of Pemantle and Mason
- Lorentzian polynomials on cones
- Hereditary poynomials and Chow rings of fans
- Lorentzian proof of Heron-Rota-Welsh conjecture
- Topology of spaces of Lorentzian polynomials

Negative dependence

- There are many ways of introducing Lorentzian polynomials
- Matroid theory
- Convex geometry
- Geometry of zeros of polynomials
- Hodge theory
- Negative dependence



Negative dependence

- Negative dependence traditionally models repelling particles or "repelling" random variables in statistical physics or probability theory.
- Let E be a finite set of sites, that can be either occupied by a particle or vacant.
- Let X_i, $i \in E$, be a random variable

$$X_i = \begin{cases} 0 & \text{if } i \text{ is vacant,} \\ 1 & \text{if } i \text{ is occupied.} \end{cases}$$

- If the particles are repelling, then one would expect different sites i, j to be negatively correlated:

$$\mathbb{P}[X_i = X_j = 1] \leq \mathbb{P}[X_i = 1] \cdot \mathbb{P}[X_j = 1]$$

Quest for a theory of negative dependence

"There is a natural and useful theory of positively dependent events. There is, as yet, no corresponding theory of negatively dependent events. There is, however, a need for such a theory."

Robin Pemantle, (UPenn), J. of Math. Physics, 2000.

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- Since then two successful approaches to negative dependence has been developed.
- One using the geometry of zeros of polynomials, and the other using ideas from Hodge theory.
- The theory of Lorentzian polynomials merges the two.

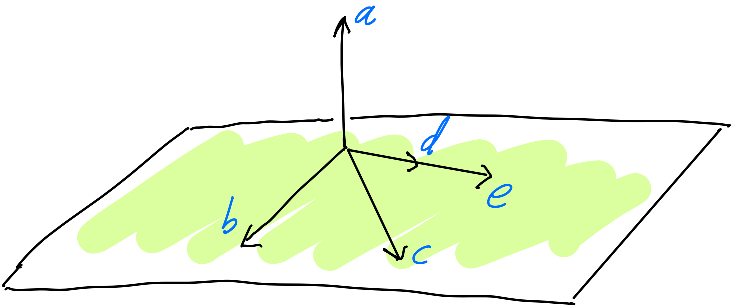
Negative dependence

Other important negative dependence inequalities are

- Log-concavity:
  $$r_k^2 \geq r_{k-1} r_{k+1},$$
  where
  $$r_k = \mathbb{P}\left[\sum X_i = k\right] = \mathbb{P}[\text{exactly } k \text{ sites are occupied}].$$
- Newton's inequalities (1707). If all the zeros of a polynomial $r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n$ are real, then
  $$\frac{r_k^2}{\binom{n}{k}^2} \geq \frac{r_{k-1}}{\binom{n}{k-1}} \cdot \frac{r_{k+1}}{\binom{n}{k+1}}, \qquad 0 < k < n.$$
- Real zeros are repelling.

Fundamental problems on independence

- Many problems on independence exhibit strong negative dependence properties.
- $E = \{a, b, c, \ldots\}$ is a finite set of vectors in a vector space.



- f_k = the number of linearly independent subsets of E of size k
- $(f_0, f_1, f_2, f_3) = (1, 5, 9, 5)$.
- What can be said about the sequence $f_0, f_1, f_2, \ldots$?

Mason's conjecture

- Mason's strong conjecture (1972).
  The sequence $f_0, f_1, \ldots, f_n$, $n = |E|$, satisfies Newton's inequalities, i.e.,

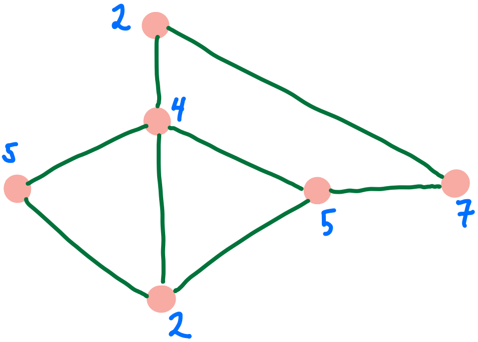
$$\frac{f_k^2}{\binom{n}{k}^2} \geq \frac{f_{k-1}}{\binom{n}{k-1}} \cdot \frac{f_{k+1}}{\binom{n}{k+1}}, \qquad 0 < k < n.$$

- The general form of the conjecture concerns independent sets in matroids.

Graph colorings

- Let $G = (V, E)$ be a graph. A proper k-coloring of G is a function $\kappa : V \to \{1, 2, \ldots, k\}$ such that

$$\{i, j\} \text{ is an edge} \implies \kappa(i) \neq \kappa(j).$$



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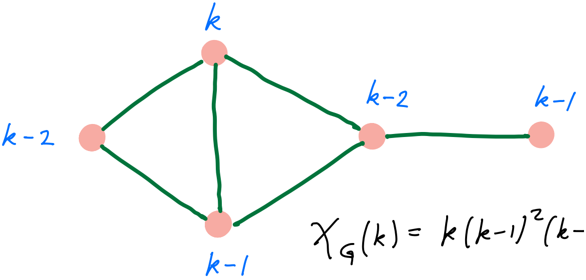
- What can be said about the chromatic polynomial?

$$\chi_G(k) = \text{the number of proper } k\text{-colorings.}$$



$$\chi_G(k) = k(k-1)^2(k-2)^2$$

Graph colorings

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- What can be said about the chromatic polynomial:

$$\chi_G(k) = \text{the number of proper } k\text{-colorings.}$$

- Introduced by George D. Birkhoff in 1912 to study the four color conjecture.

Read-Heron-Rota-Welsh conjecture

- We may write

$$\chi_G(x) = w_0 x^n - w_1 x^{n-1} + \cdots + (-1)^n w_n, \quad n = |V|,$$

where $w_0, w_1, \ldots, w_n$ are nonnegative integers called the Whitney numbers of the first kind.

- Conjecture (Read-Heron-Rota-Welsh, 1968–76).
$\{w_k\}_{k=0}^n$ is a log-concave sequence, i.e.,

$$w_k^2 \geq w_{k-1} w_{k+1}, \qquad 0 < k < n.$$

- Proved by June Huh using Hodge theory.
- In its full generality, the conjecture applies to the characteristic polynomial of a geometric lattice.
- Proved Adiprasito, Huh and Katz by developing a Hodge theory for matroids.

Matroid theory

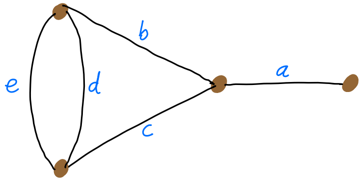
- Matroid theory is a discrete axiomatization of independence in algebra and graph theory.
- Introduced by Nakasawa and Whitney in the 1930's.

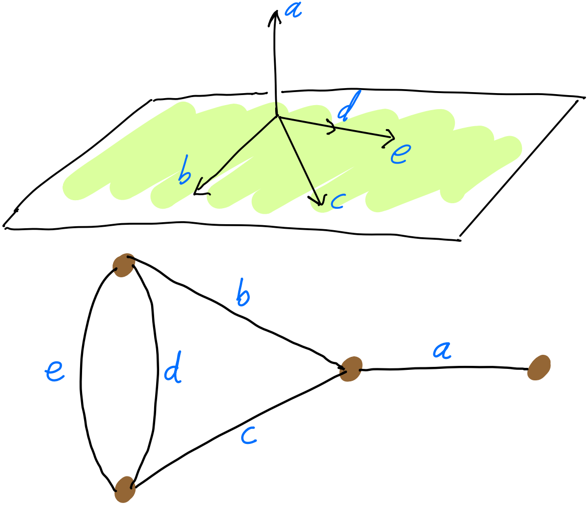

Matroid theory

- Let M be a collection of subsets of a finite set E.
- M is the set of bases of a matroid if for all $B_1, B_2 \in \mathrm{M}$:

$$i \in B_1 \setminus B_2 \Longrightarrow \exists j \in B_2 \setminus B_1 \text{ such that } (B_1 \setminus \{i\}) \cup \{j\} \in \mathrm{M}.$$

- Example. E is a finite set of vectors that span a linear space V. M is the set of bases of V drawn from E. M is called a linear matroid.
- Example. $G = (V, E)$ is a connected graph. M is the collection of spanning trees of G.





$$\mathrm{M} = \{abc, abd, abe, acd, ace\}$$

Matroid Potts model

- Let M be a matroid on E. The rank of a subset A of E is

$$r(A) = \max\{|A \cap B| : B \in \mathrm{M}\}.$$

- A subset I of E is independent if $r(I) = |I|$.
- For positive numbers q and $x_e, e \in E$, define a probability measure on $2^E = \{S : S \subseteq E\}$ by

$$\mu_q(S) = \frac{1}{Z_q} q^{-r(S)} \prod_{e \in S} x_e,$$

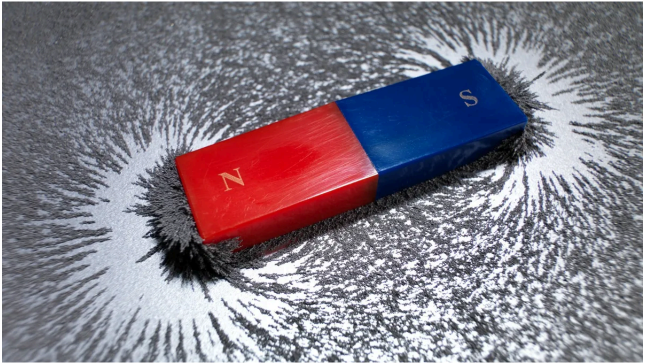
where Z_q is the normalizing factor, or the partition function.
- $q > 1$: Favors low rank. Positively dependent.
- $q < 1$: Favors high rank. Negatively dependent?

Matroid Potts model

- Recall
$$\mu_q(S) = \frac{1}{Z_q} q^{-r(S)} \prod_{e \in S} x_e,$$
- Set $x_e = q$ for all e:
$$\mu_q(S) = \frac{1}{Z_q} q^{|S|-r(S)},$$
- Let $q \to 0$:
$$\mu_q(S) \to \frac{1}{Z} \begin{cases} 1 & \text{if } S \text{ independent}, \\ 0 & \text{otherwise.} \end{cases}$$
- Hence if we can prove negative dependence for μ_q for all $0 < q \leq 1$, then we can prove negative dependence for independent sets.

Random cluster-model

- When M is a graphic matroid corresponding to a graph $G = (V, E)$, then μ_q is called the Random-cluster model (RC).
- $q = 2$: Ising model (Ferromagnet),
- $q \in \mathbb{Z}_{>0}$: q-state Potts model.



Negative dependence conjectures for Potts models

- For $q \geq 1$, the matroid Potts model is positively dependent.
- For $0 < q \leq 1$, RC is conjectured to be negatively dependent.
- Conjecture (Pemantle, Kahn, Grimmett,...). RC is negatively correlated for $0 < q \leq 1$.
- Theorem (Kirchhoff). If G is connected, and $x_e, e \in E$ are positive numbers then the measure on 2^E:

$$\mu_G(S) = \frac{1}{Z} \begin{cases} \prod_{e \in S} x_e & \text{if } S \text{ is a spanning tree}, \\ 0 & \text{otherwise}. \end{cases}$$

  is negatively correlated.
- Unknown for the random forest measure.

Negative dependence for Potts models

- Let $r_k = \mathbb{P}[k \text{ sites are occupied}]$ and $n = |E|$.
- Conjecture (Pemantle, 2000). For $0 < q \leq 1$, RC satisfies Newton's inequalities:

$$\frac{r_k^2}{\binom{n}{k}^2} \geq \frac{r_{k-1}}{\binom{n}{k-1}} \cdot \frac{r_{k+1}}{\binom{n}{k+1}}.$$

- It is natural to extend this conjecture to all matroids.
- Extended Pemantle's conjecture was proved by B. and Huh in 2020. $\Longrightarrow$ Mason's conjecture.
- Theorem (B., Huh, 2020). For $0 < q \leq 1$ and distinct sites i and j,

$$\mathbb{P}[X_i = X_j = 1] \leq 2 \cdot \mathbb{P}[X_i = 1] \cdot \mathbb{P}[X_j = 1].$$

- The proofs use Lorentzian polynomials.

Gian-Carlo Rota's idea

- Gian-Carlo Rota (1932-1999) believed that matroid negative dependence conjectures should be approached by geometric inequalities from Brunn-Minkowski theory.

Motivation: Geometric inequalities

- Brunn-Minkowski inequality (1887). For convex bodies $K_1, K_2 \subset \mathbb{R}^d$,

$$\mathrm{Vol}(K_1 + K_2)^{1/d} \geq \mathrm{Vol}(K_1)^{1/d} + \mathrm{Vol}(K_2)^{1/d},$$

  where $K_1 + K_2 = \{x_1 + x_2 : x_1 \in K_1 \text{ and } x_2 \in K_2\}$.

- Minkowski. For convex bodies $K_1, \ldots, K_m$, and $x_1, \ldots, x_m > 0$,

$$\mathrm{Vol}(x_1 K_1 + \cdots + x_m K_m) = \sum_{i_1, \ldots, i_d} V(K_{i_1}, \ldots, K_{i_d}) x_{i_1} \cdots x_{i_d},$$

  where $V(K_1, \ldots, K_d) \geq 0$ are the mixed volumes.

- Alexandrov-Fenchel inequalities (1937).

$$V(K_1, K_2, \ldots, K_n)^2 \geq V(K_1, K_1, K_3 \ldots, K_n) \cdot V(K_2, K_2, K_3, \ldots, K_n)$$

Lorentzian polynomials

- Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous degree d polynomial and $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$. Then

$$f(x_1\mathbf{v}_1 + \cdots + x_m\mathbf{v}_m) = \frac{1}{d!}\sum_{i_1,\ldots,i_d}(D_{\mathbf{v}_{i_1}} \cdots D_{\mathbf{v}_{i_d}} f)x_{i_1} \cdots x_{i_d}$$

  where $D_{\mathbf{w}} = w_1\frac{\partial}{\partial x_1} + \cdots + w_n\frac{\partial}{\partial x_n}$.
- f is called Lorentzian if f has nonnegative coefficients, and for all $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^n_{>0}$,

(AF) $(D_{\mathbf{v}_1} D_{\mathbf{v}_2} \cdots D_{\mathbf{v}_d} f)^2 \geq (D_{\mathbf{v}_1} D_{\mathbf{v}_1} \cdots D_{\mathbf{v}_d} f)(D_{\mathbf{v}_2} D_{\mathbf{v}_2} \cdots D_{\mathbf{v}_d} f)$

- If $a_k = D^k_{\mathbf{v}_1} D^{d-k}_{\mathbf{v}_2} f$, then $a_k^2 \geq a_{k-1}a_{k+1}$.

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- f is called Lorentzian if f has nonnegative coefficients, and for all $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^n_{>0}$,

(AF) The Hessian

$$\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{i,j=1}^n$$

of the quadratic polynomial $g = D_{\mathbf{v}_1} \cdots D_{\mathbf{v}_{d-2}} f$ has at most one positive eigenvalue.

Exercise

- Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be a quadratic polynomial, and write

$$f = \frac{1}{2}\mathbf{x}^T A\mathbf{x} = \frac{1}{2}\sum_{i,j} a_{ij}x_i x_j.$$

- The following are equivalent:

(a) f is Lorentzian,

(b) A has at most one positive eigenvalue,

(c) For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_{>0}$,

$$(\mathbf{u}^T A\mathbf{v})^2 \geq (\mathbf{u}^T A\mathbf{u}) \cdot (\mathbf{v}^T A\mathbf{v}).$$

Examples of Lorentzian polynomials

- Determinantal polynomials: $\det(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)$, where $A_1, \ldots, A_n$ are symmetric positive semidefinite $d \times d$ matrices.
- Stable polynomials: $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$, homogeneous of degree d, such that

$$\mathrm{Im}(x_j) > 0 \text{ for all j } \implies f(x_1, \ldots, x_n) \neq 0.$$

- Volume polynomials of convex bodies or projective varieties.
- Various matroid polynomials.
- Normalized Schur polynomials (Huh, Matherne, Mészáros, St. Dizier).

Properties of Lorentzian polynomials

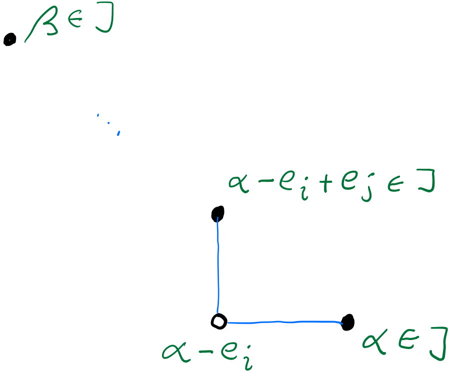
- Theorem (B., Huh, 2020). If f and g are Lorentzian, then so is fg.
- If f is Lorentzian and $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$, then $D_{\mathbf{v}} f$ is Lorentzian.
- If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is Lorentzian and A is an $m \times n$ matrix with nonnegative entries, then $f(A\mathbf{x})$ is Lorentzian.
- A bi-variate polynomial $\sum_{k=0}^{d} a_k x^k y^{d-k}$ with positive coefficients is Lorentzian iff the Newton inequalities are satisfied:
$$\frac{a_k^2}{\binom{d}{k}^2} \geq \frac{a_{k-1}}{\binom{d}{k-1}} \cdot \frac{a_{k+1}}{\binom{d}{k+1}}.$$
- Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_{\geq 0}$, and f is Lorentzian. Write
$$f(s\mathbf{u} + t\mathbf{v}) = \sum_{k=0}^{d} a_k \binom{d}{k} s^k t^{d-k}.$$
Then $\{a_k\}_{k=0}^d$ is log-concave.

Lorentzian polynomials and Matroid theory

- A finite subset J of $\mathbb{Z}^n$ is M-convex if

  $\alpha, \beta \in J$ and $\alpha_i > \beta_i \Longrightarrow$

  there is a j such that $\beta_j > \alpha_j$ and $\alpha - e_i + e_j \in J$.



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  there is a j such that $\beta_j > \alpha_j$ and $\alpha - e_i + e_j \in J$.

- Also called polymatroids or integer points of generalized permutahedra.
- If $J \subseteq \{0,1\}^n$, then J is M-convex iff J is the set of bases of a matroid.
- The support of a polynomial

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \, x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \qquad a_\alpha \in \mathbb{R},$$

is

$$\operatorname{supp}(f) = \{\alpha \in \mathbb{N}^n : a_\alpha \neq 0\}.$$

Characterization of Lorentzian polynomials

Theorem (B., Huh, 2020). Let f be a degree d homogenous polynomial with nonnegative coefficients. Then f is Lorentzian iff

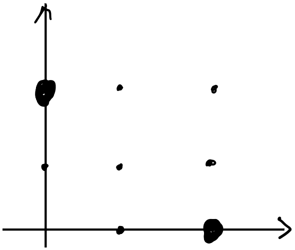
(M) $\mathrm{supp}(f)$ is M-convex, and

(L) for all $i_1, i_2, \ldots, i_{d-2}$, the Hessian of the quadratic

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f$$

has at most one positive eigenvalue.

- Non-example. $x_1^2 + x_2^2$ is not Lorentzian.



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- Non-example. $x_1^2 + x_2^2$ is not Lorentzian.
- Example. $x_1^2 + 3x_1x_2 + x_2^2$ is Lorentzian.

$$H = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

$\det(H) = -5$
so H has exactly one pos. eigenvalue

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Theorem (B., Huh, 2020). If $J \subset \mathbb{N}^n$ is M-convex, then

$$\sum_{\alpha \in J} \frac{1}{\alpha_1! \cdots \alpha_n!} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{is Lorentzian.}$$

- Hence Lorentzian polynomials characterize M-convex sets and matroids.

Bivariate polynomials

- When is a bivariate polynomial Lorentzian? Write

$$f = \frac{1}{d!}\sum_{k=0}^{d} a_k \binom{d}{k} x^k y^{d-k} = \sum_{k=0}^{d} a_k \frac{x^k}{k!}\frac{y^{d-k}}{(d-k)!}$$

- (M) says that $\{a_k\}_{k=0}^{d}$ has no internal zeros.
- (L) says that the Hessian H of

$$\frac{\partial^{k-1}}{\partial x^{k-1}}\frac{\partial^{d-k-1}}{\partial y^{d-k-1}} f = a_{k-1}\frac{y^2}{2} + a_k xy + a_{k+1}\frac{x^2}{2}x^2$$

  has at most one positive eigenvalue.

$$H = \begin{pmatrix} a_{k+1} & a_k \\ a_k & a_{k-1} \end{pmatrix}, \quad \lambda_1\lambda_2 = \det(H) = a_{k-1}a_{k+1} - a_k^2 \leq 0.$$

- Hence Lorentzian iff $\{a_k\}$ no internal zeros and log-concave.

Multivariate Tutte polynomial

- The partition function for the Potts model of M is

$$Z_{\mathrm{M}}(\mathbf{x};q) = \sum_{A \subseteq E} q^{-r(A)} \prod_{e \in A} x_e.$$

- Let

$$H_{\mathrm{M}}(\mathbf{x};q) = \sum_{A \subseteq E} q^{-r(A)} x_0^{|E \setminus A|} \prod_{e \in A} x_e.$$

- Notice

$$\frac{\partial}{\partial x_e} H_{\mathrm{M}}(\mathbf{x};q) = q^{-r(\{e\})} H_{\mathrm{M}/e}(\mathbf{x};q),$$

where M/e is the contraction of M by e:

$$\mathrm{M}/e = \{B \setminus \{e\} : B \text{ is a basis of } \mathrm{M} \text{ and } e \in B\}.$$

Matroid Potts model is Lorentzian

Theorem (B., Huh, 2020). If $0 < q \leq 1$, then $H_{\mathrm{M}}(\mathbf{x}; q)$ is Lorentzian as a polynomial in $\mathbf{x}$.

Proof.

- We should verify conditions (M) and (L) of the characterization.
- $\mathrm{supp}(H_{\mathrm{M}}(\mathbf{x}; q))$ is M-convex.
- Recall
$$\frac{\partial}{\partial x_e} H_{\mathrm{M}}(\mathbf{x}; q) = q^{-r(\{e\})} H_{\mathrm{M}/e}(\mathbf{x}; q).$$
- By induction on $r(\mathrm{M})$ (and by taking truncations if necessary) it suffices to prove for $r(\mathrm{M}) = 2$.
- The case when $r(\mathrm{M}) = 2$ is an exercise in linear algebra. □

Consequences

- Let $E = \{1, \ldots, n\}$. The previous theorem says

$$H_{\mathrm{M}}(x_0, x_1, \ldots, x_n) = \sum_{S \subseteq E} q^{-r(S)} x_0^{n-|S|} \prod_{e \in S} x_e \quad \text{is Lorentzian.}$$

- Then so is $f(s,t) = H_{\mathrm{M}}(s, x_1 t, x_2 t, \ldots, x_n t)$, where $x_j > 0$ for all j.

$$f(s,t) = \sum_{k=0}^{n} r_k s^{n-k} t^k.$$

- From this follows the extended Pemantle conjecture, and Mason's conjecture.

Motivation: Elements of Hodge theory

- Let
$$A = \mathbb{R}[x_1, \ldots, x_n]/I = \bigoplus_{k=0}^{d} A^k$$
be a graded $\mathbb{R}$-algebra.
- Suppose A^d is one-dimensional, and let
$$\deg : A^d \to \mathbb{R}$$
be a linear isomorphism.
- Suppose $\mathcal{K} \subset A^1$ is an open convex cone.

Kähler package

Desirable properties of A.

Poincaré duality (PD)

The bilinear map,

$$A^k \times A^{d-k} \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto \deg(xy),$$

is nondegenerate.

Hard Lefschetz property (HL)

For each $0 \leq k \leq d/2$, and any $\ell_1, \ell_2, \ldots, \ell_{d-2k} \in \mathcal{K}$, the linear map

$$A^k \longrightarrow A^{d-k}, \quad x \longmapsto \ell_1 \ell_2 \cdots \ell_{d-2k} x,$$

is bijective.

Kähler package

Hodge-Riemann relations (HR)

For each $0 \leq k \leq d/2$, and any $\ell_0, \ell_1, \ldots, \ell_{d-2k} \in \mathcal{K}$, the bilinear map

$$A^k \times A^k \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto (-1)^k \deg(\ell_1 \ell_2 \cdots \ell_{d-2k} x y)$$

is positive definite on $\{x \in A^k : \ell_0 \ell_1 \cdots \ell_{d-2k} x = 0\}$.

Let $\ell_1, \ldots, \ell_d \in \mathcal{K}$.

(P) For $k = 0$, (HR) says $\deg(\ell_1 \ell_2 \cdots \ell_d) > 0$.

(AF) For $k = 1$, (HR) says

$$\deg(\ell_1 \ell_2 \ell_3 \cdots \ell_d)^2 \geq \deg(\ell_1 \ell_1 \ell_3 \cdots \ell_d) \deg(\ell_2 \ell_2 \ell_3 \cdots \ell_d).$$

(LC) In particular, the sequence $a_k = \deg(\ell_1^k \ell_2^{d-k})$ is log-concave

$$a_k^2 \geq a_{k-1} a_{k+1}, \quad 0 < k < d.$$

Examples

- Classical examples of Kähler package comes from compact Kähler manifolds and projective varieties,
- Polytopes (Stanley, McMullen),
- Chow rings of matroids (Adiprasito, Huh, Katz), and similar Chow rings.

Beyond Hodge theory

- Is there a common "geometry of polynomials" setting for these examples?
- The degree map defines a homogeneous degree d polynomial in $\mathbb{R}[t_1, \ldots, t_n]$:

$$\operatorname{vol}_A(t) = \frac{1}{d!} \deg\left(\left(\sum_{i=1}^{n} t_i x_i\right)^d\right). \qquad \text{(volume polynomial)}$$

- Let $\ell = a_1 x_1 + \cdots + a_n x_n \in A^1$, $\mathbf{v} = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Then

$$D_{\mathbf{v}} \operatorname{vol}_A(t) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial t_i} \operatorname{vol}_A(t) = \frac{1}{(d-1)!} \deg\left(\ell \cdot \left(\sum_{i=1}^{n} t_i x_i\right)^{d-1}\right)$$

- Iterate: $D_{\mathbf{v}_1} D_{\mathbf{v}_2} \cdots D_{\mathbf{v}_d} \operatorname{vol}_A(t) = \deg(\ell_1 \ell_2 \cdots \ell_d)$.

Lorentzian polynomials on cones

- Let $f \in \mathbb{R}[t_1, \ldots, t_n]$ be a homogeneous degree d polynomial.
- Let $\mathcal{K}$ be an open convex cone in $\mathbb{R}^n$.
- f is called $\mathcal{K}$-Lorentzian if for all $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathcal{K}$,

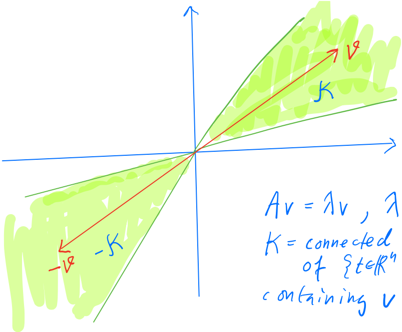
(P) $D_{\mathbf{v}_1} \cdots D_{\mathbf{v}_d} f > 0$, and

(AF) $(D_{\mathbf{v}_1} D_{\mathbf{v}_2} \cdots D_{\mathbf{v}_d} f)^2 \geq (D_{\mathbf{v}_1} D_{\mathbf{v}_1} \cdots D_{\mathbf{v}_d} f)(D_{\mathbf{v}_2} D_{\mathbf{v}_2} \cdots D_{\mathbf{v}_d} f)$

- Hence we get $\mathcal{K}$-Lorentzian polynomials from the examples from Hodge theory above.
- Example. Lorentzian polynomials are the same as $\mathbb{R}^n_{>0}$-Lorentzian polynomials.
- Example. The determinant $A \mapsto \det(A)$ is Lorentzian on the cone of positive definite matrices.
- There are $\mathcal{K}$-Lorentzian polynomials that do not come from any of the examples from Hodge-theory above.

Quadratic Lorentzian polynomials on cones

- Let $A = (a_{ij})_{i,j=1}^n$ be a (non-zero) symmetric $n \times n$ matrix.
- The polynomial
$$f(t) = \sum_{i,j} a_{ij} t_i t_j$$
is Lorentzian with respect to some cone iff A has exactly one positive eigenvalue λ.

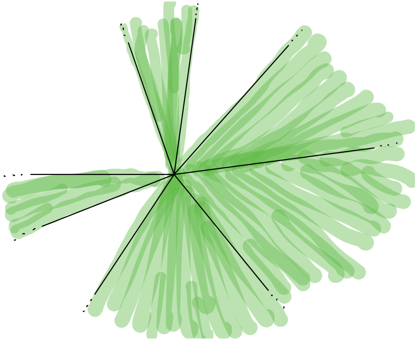


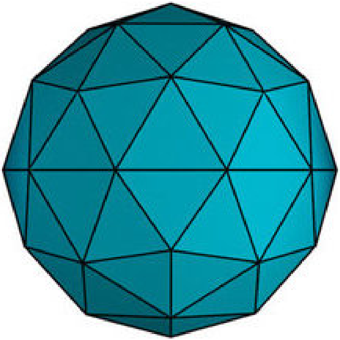
$Av = \lambda v$, $\lambda > 0$

K = connected component of $\{t \in \mathbb{R}^n : f(t) > 0\}$ containing v.

Chow rings of fans

- Let Δ be a pure abstract simplicial complex on V.
- Let $\Sigma = \{C_S\}_{S \in \Delta}$ be a collection of $|S|$-dimensional polyhedral cones such that
  - Each face of C_S is a cone in Σ, and
  - $C_S \cap C_T = C_{S \cap T}$.





Chow rings of fans

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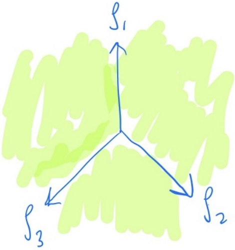
- Let $\Sigma = \{C_S\}_{S\in\Delta}$ be a collection of $|S|$-dimensional polyhedral cones such that
  - Each face of C_S is a cone in Σ, and
  - $C_S \cap C_T = C_{S\cap T}$.
- Σ is called a simplicial fan.
- Let ρ_i, $i \in V$, be specified vectors of the rays $C_{\{i\}}$.
- Let $L = L(\Sigma) = \{(\lambda(\rho_i))_{i\in V} : \lambda \in \left(\mathbb{R}^V\right)^*\}$.

Chow rings of fans

- Define two ideals in $\mathbb{R}[x_i : i \in V]$:
  - $I(\Delta)$ is generated by the monomials $\prod_{i \in T} x_i, \quad T \notin \Delta$.
  - $J(L)$ is generated by the linear forms $\sum_{i \in V} \ell_i x_i, \quad (\ell_i)_{i \in V} \in L$.
- The graded ring

$$A(\Sigma) = \bigoplus_{k=0}^{d} A^k(\Sigma) := \mathbb{R}[x_i : i \in V] \big/ (I(\Delta) + J(L))$$

  is the Chow ring of Σ.
- Important examples of Chow rings that satisfy the Kähler package are
  - The normal fan of a simple polytope (Stanley, McMullen).
  - The Chow ring of a matroid (Adiprasito, Huh and Katz), and related Chow rings.



$\rho_1 + \rho_2 + \rho_3 = 0$

$$L(\Sigma) = \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1 + \lambda_2 + \lambda_3 = 0\}$$

$$\Delta(\Sigma) = \{S \subseteq \{1,2,3\} : |S| \leq 2\}$$

$$I(\Sigma) = \langle x_1 x_2 x_3 \rangle$$

$$J(\Sigma) = \langle l_1 x_1 + l_2 x_2 + l_3 x_3 : l \in L(\Sigma) \rangle$$
$$= \langle x_1 - x_2, x_1 - x_3, x_2 - x_3 \rangle$$

$$A^1(\Sigma) \cong \mathbb{R} x_1, \quad A^2(\Sigma) \cong \mathbb{R} x_1 x_2$$
$$A^k(\Sigma) = (0), \quad k > 2$$

$$\deg : A^2(\Sigma) \to \mathbb{R}, \quad \deg(x_i x_j) = 1 \quad \forall\, i,j$$

$$\mathrm{vol}_\Sigma(t_1, t_2, t_3) = \frac{1}{2} \deg\left((x_1 t_1 + x_2 t_2 + x_3 t_3)^2 \right) = \frac{1}{2}(t_1 + t_2 + t_3)^2$$

Goals

- Try to find "polynomial proofs" of Hodge-Riemann relations of degree zero and one for Chow rings of fans.
- Would give new and elementary proofs of the Heron-Rota-Welsh conjecture and similar results.
- Characterize the Chow rings of fans that satisfy Hodge-Riemann relations of degree zero and one.
- Extend beyond fans and Hodge theory.

Volume polynomials of Chow rings of fans

- Let $\deg : A^d(\Sigma) \to \mathbb{R}$ be a linear function, and consider the volume polynomial

$$\mathrm{vol}_\Sigma(t) = \frac{1}{d!} \deg \left(\left(\sum_{i \in V} t_i x_i \right)^d \right).$$

- Properties: Let $\partial^S := \prod_{i \in S} \partial_i$, where $\partial_i := \partial / \partial t_i$.

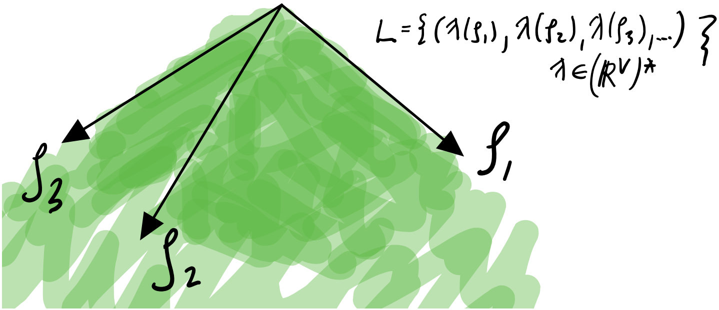
$$S \notin \Delta(\Sigma) \Longrightarrow \partial^S \, \mathrm{vol}_\Sigma = \frac{1}{(d - |S|)!} \deg \left(\prod_{i \in S} x_i \left(\sum_{i \in V} t_i x_i \right)^{d - |S|} \right) \equiv 0$$

$$\ell \in L(\Sigma) \Longrightarrow \frac{1}{(d-1)!} D_\ell \, \mathrm{vol}_\Sigma = \deg \left(\sum_{i \in V} \ell_i x_i \left(\sum_{i \in V} t_i x_i \right)^{d-1} \right) \equiv 0$$

Hereditary polynomials

- Let Δ be a pure $(d-1)$-dimensional simplicial complex on a finite set V.
- Let L be a linear subspace of $\mathbb{R}^V$.
- The pair (Δ, L) is called hereditary if for each $S \in \Delta$

$$\{(\ell_i)_{i \in S} : (\ell_i)_{i \in V} \in L\} = \mathbb{R}^S.$$



Hereditary polynomials

- Let Δ be a pure $(d-1)$-dimensional simplicial complex on a finite set V.
- Let L be a linear subspace of $\mathbb{R}^V$.
- The pair (Δ, L) is called hereditary if for each facet $S \in \Delta$

$$\{(\ell_i)_{i \in S} : (\ell_i)_{i \in V} \in L\} = \mathbb{R}^S.$$

- If Σ is a simplicial fan, then $(\Delta(\Sigma), L(\Sigma))$ is hereditary.
- Let $\mathcal{P}(\Delta, L)$ be the set of all degree d homogeneous polynomials $f \in \mathbb{R}[t_i : i \in V]$ such that

$$\begin{aligned} S \notin \Delta &\implies \partial^S f \equiv 0, \quad \text{and} \\ \mathbf{v} \in L &\implies D_{\mathbf{v}} f \equiv 0. \end{aligned}$$

- If (Δ, L) is hereditary, then $f \in \mathcal{P}(\Delta, L)$ is called hereditary.

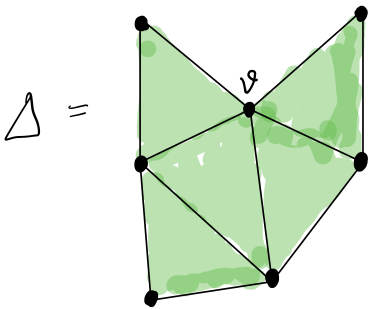
Hereditary polynomials

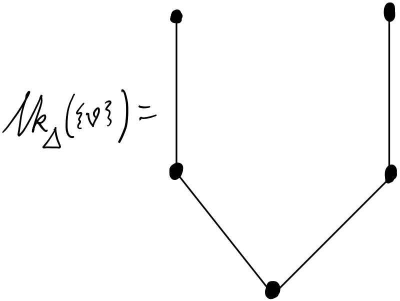
- If $S \in \Delta$, then the link of S in Δ is the simplicial complex

$$\mathrm{lk}_\Delta(S) := \{T \subseteq V \setminus S : S \cup T \in \Delta\}$$

on

$$V_S = \{i \in V \setminus S : S \cup \{i\} \in \Delta\}.$$





Hereditary polynomials

- If $S \in \Delta$, then the link of S in Δ is the simplicial complex

$$\mathrm{lk}_\Delta(S) := \{T \subseteq V \setminus S : S \cup T \in \Delta\}$$

  on

$$V_S = \{i \in V \setminus S : S \cup \{i\} \in \Delta\}.$$

- Lemma. If (Δ, L) is hereditary and $S \in \Delta$, then $(\mathrm{lk}_\Delta(S), L_S)$ is hereditary, where

$$L_S = \{(\ell_i)_{i \in V_S} : (\ell_i)_{i \in V} \in L \text{ and } \ell_j = 0 \text{ for all } i \in S\}.$$

- Lemma. If $f \in \mathcal{P}(\Delta, L)$ and $S \in \Delta$, then

$$f^S(t) := \partial^S f\big|_{t_i = 0, i \in S} \in \mathcal{P}(\mathrm{lk}_\Delta(S), L_S).$$

Hereditary polynomials

- For $i \in V$, let $\ell \in L$ be such that $\ell_i = 1$, and define a projection $\pi_i : \mathbb{R}^V \to \mathbb{R}^{V_{\{i\}}}$ by

$$\pi_i(\mathbf{v}) = (w_j)_{j \in V_{\{i\}}}, \quad \text{where } \mathbf{w} = \mathbf{v} - v_i \ell.$$

We associate an open convex cone $\mathcal{K}(\Delta, L)$ in $\mathbb{R}^V$ to any hereditary (Δ, L):

- If $d = 1$, then $\mathcal{K}(\Delta, L) = \mathbb{R}^V_{>0} + L$.
- If $d > 1$, then

$$\mathcal{K}(\Delta, L) = (\mathbb{R}^V_{>0} + L) \cap \bigcap_{i \in V} \pi_i^{-1}\left(\mathcal{K}(\mathrm{lk}_\Delta(\{i\}), L_{\{i\}})\right).$$

- $f \in \mathcal{P}(\Delta, L)$ is positive if $\partial^F f > 0$ for all facets $F \in \Delta$. Write $\mathcal{P}_+(\Delta, L)$.

Hereditary Lorentzian polynomials

- Δ is H-connected if for each $S \in \Delta$, $|S| \leq d-2$, the graph

$$\Big\{\{i,j\} : S \cap \{i,j\} = \varnothing \text{ and } S \cup \{i,j\} \in \Delta\Big\}$$

  is connected.
- Definition. $f \in \mathcal{P}_+(\Delta, L)$ is hereditary Lorentzian if f^S is $\mathcal{K}(\mathrm{lk}_\Delta(S), L_S)$-Lorentzian for each $S \in \Delta$.
- Theorem (B., Leake). Let $f \in \mathcal{P}_+(\Delta, L)$, where (Δ, L) is hereditary and $\mathcal{K}(\Delta, L) \neq \varnothing$.
  Then f is hereditary Lorentzian if and only if
  (C) Δ is H-connected, and
  (L) For each $S \in \Delta$ with $|S| = d-2$, the Hessian of f^S has at most one positive eigenvalue.

Example

- Let $\Delta = \{S \subseteq \{1, \dots, n\}$ and $|S| < n\}$ and $L = \{t_1 + t_2 + \cdots + t_n = 0\}$, and

$$f = \frac{1}{(n-1)!}(t_1 + t_2 + \cdots + t_n)^{n-1} \in \mathcal{P}_+(\Delta, L).$$

- Δ is trivially H-connected.
- $\mathcal{K}(\Delta, L) = \{t_1 + t_2 + \cdots + t_n > 0\}$.
- If $S = \{3, 4, \dots, n-1\}$, then $f^S = (t_1 + t_2)^2/2$.
- Hence f is hereditary Lorentzian.

Hereditary polynomials

- Question. For which simplicial complexes Δ is there a hereditary Lorentzian polynomial f for which

$$\{S : \partial^S f \not\equiv 0\} = \Delta?$$

Balancing condition

Theorem (B., Leake).

- Suppose (Δ, L) is hereditary, and

$$w(S), \quad S \text{ is a facet of } \Delta$$

are nonzero real numbers. Then there is at most one $f \in \mathcal{P}(\Delta, L)$ for which

$$\partial^S f = w(S) \quad \text{for all facets } S.$$

- Moreover this polynomial exists iff for each $S \in \Delta$, $|S| = d - 1$, the linear form

$$\sum_{i \notin S} w(S \cup \{i\}) t_i$$

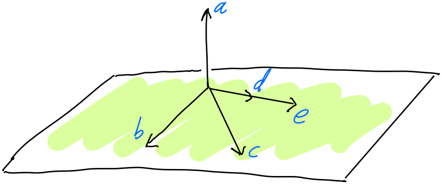
is identically zero on L_S.

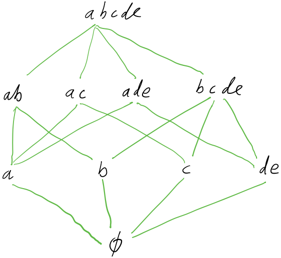
Lorentzian polynomials for geometric lattices

- A flat of a matroid M on E is a subset F of E for which

$$e \in E \setminus F \implies r(F \cup \{e\}) > r(F).$$

- The set of all flats is a geometric lattice $\mathcal{L} = \mathcal{L}(\mathrm{M})$.





$$\mathrm{M} = \{abc, abd, abe, acd, ace\}$$

Lorentzian polynomials for geometric lattices

- Let $\mathcal{L}$ be the lattice of flats of a matroid M on E, with set of loops K, and let $\underline{\mathcal{L}} = \mathcal{L} \setminus \{K, E\}$.
- The faces of the order complex, $\Delta(\underline{\mathcal{L}})$, are $\{F_1 < F_2 < \cdots < F_k\}$, where $F_i \in \underline{\mathcal{L}}$ for all i.
- Let $\mathcal{M}$ be the subspace $\mathbb{R}^{\underline{\mathcal{L}}}$ of all $(y_F)_{F \in \underline{\mathcal{L}}}$ for which there are real numbers w_i, $i \in E \setminus K$, such that

$$\sum_{i \in E \setminus K} w_i = 0 \quad \text{and} \quad y_F = \sum_{i \in F \setminus K} w_i \text{ for all } F \in \underline{\mathcal{L}}.$$

- Lemma. $(\Delta(\underline{\mathcal{L}}), \mathcal{M})$ is hereditary.
- By using the theorem on the previous slide, there is a unique polynomial $\mathrm{pol}_{\mathcal{L}} \in \mathcal{P}(\Delta(\underline{\mathcal{L}}), \mathcal{M})$ for which

$$\partial^S \mathrm{pol}_{\mathcal{L}} = 1, \qquad \text{for all facets } S \text{ of } \Delta(\underline{\mathcal{L}}).$$

Lorentzian polynomials for geometric lattices

- If $r(\mathcal{L}) = 2$, then

$$\mathrm{pol}_{\mathcal{L}} = \sum_{K<F<E} t_F.$$

- If $r(\mathcal{L}) = 3$, then

$$2\,\mathrm{pol}_{\mathcal{L}} = \left(\sum_{K \prec F} t_F\right)^2 - \sum_{G \prec E}\left(t_G - \sum_{K \prec F \prec G} t_F\right)^2.$$

- $\mathcal{K}(\Delta(\underline{\mathcal{L}}), \mathcal{M})$ is nonempty and contains all strictly submodular vectors:

$$y_S + y_T > y_{S\cup T} + y_{S\cap T}, \quad y_K = y_E = 0,$$

whenever S and T are incomparable.

Lorentzian polynomials for geometric lattices

- Theorem (B., Leake, after Adiprasito, Huh, Katz). $\mathrm{pol}_{\mathcal{L}}$ is hereditary Lorentzian.

Proof. According to the characterization we need to verify properties (C) and (L).

- (C) follows from semimodularity of $\mathcal{L}$.
- Notice that $\mathrm{lk}_{\Delta(\underline{\mathcal{L}})}(\{F\}) = \Delta((K,F)) \times \Delta((F,E))$.
- By the uniqueness in the characterization of hereditary polynomials it follows that

$$\mathrm{pol}_{\mathcal{L}}^{\{F\}} = \frac{\partial}{\partial t_F} \mathrm{pol}_{\mathcal{L}} \Big|_{t_F=0} = \mathrm{pol}_{[K,F]} \cdot \mathrm{pol}_{[F,E]} .$$

- Hence if $S \in \Delta(\underline{\mathcal{L}})$, $|S| = r-3$, then either $\mathrm{pol}_{\mathcal{L}}^S$ is a product of two linear polynomials, or of the form

$$\left(\sum_{K \prec F} t_F\right)^2 - \sum_{G \prec E} \left(t_G - \sum_{K \prec F \prec G} t_F\right)^2 . \quad \square$$

Heron-Rota-Welsh

- Recall the characteristic polynomial of a geometric lattice

$$\chi_{\mathcal{L}}(t) = \sum_{F \in \mathcal{L}} \mu(\hat{0}, F) t^{r - r(F)} = w_0 t^r - w_1 t^{r-1} + \cdots ,$$

  where $w_i \geq 0$ are the Whitney numbers of the first kind.
- Conjecture (Read-Heron-Rota-Welsh, 1968–76).
  $\{w_k\}_{k=0}^{n}$ is a log-concave sequence, i.e.,

$$w_k^2 \geq w_{k-1} w_{k+1}, \qquad 0 < k < n.$$

- Proved by Adiprasito, Huh and Katz (2018) by developing a Hodge theory for matroids.
- In fact they proved log-concavity of the coefficients of the reduced characteristic polynomial

$$\overline{\chi}_{\mathcal{L}}(t) = (-1)^r \chi_{\mathcal{L}}(-t)/(t+1) = \overline{w}_0 t^{r-1} + \overline{w}_1 t^{r-2} + \cdots .$$

Heron-Rota-Welsh conjecture

- Recall that if f is $\mathcal{K}$-Lorentzian of degree d and $\alpha, \beta \in \overline{\mathcal{K}}$, then the sequence

$$D_\alpha^k D_\beta^{d-k} f, \quad 0 \leq k \leq d,$$

  is log-concave.
- Let $\alpha, \beta \in \overline{\mathcal{K}(\Delta(\underline{\mathcal{L}}), \mathcal{M})}$ be

$$\alpha = \left(\frac{|F \setminus K|}{|E \setminus K|}\right)_{F \in \underline{\mathcal{L}}} \quad \text{and} \quad \beta = \left(\frac{|E \setminus F|}{|E \setminus K|}\right)_{F \in \underline{\mathcal{L}}}.$$

- Then $\overline{w}_k = D_\alpha^k D_\beta^{r-1-k} \operatorname{vol}_{\mathcal{L}}$, for $0 \leq k \leq r-1$.
- Heron-Rota-Welsh conjecture now follows from the Alexandrov-Fenchel inequalities for $\operatorname{pol}_{\mathcal{L}}$.

Lorentzian Chow rings

- Let $A(\Sigma)$ be a Chow ring of a simplicial fan.
- If $S \in \Delta(\Sigma)$, then $\mathrm{star}(S, \Sigma)$ is the simplicial fan with $\Delta(\mathrm{star}(S, \Sigma)) = \mathrm{lk}_\Delta(S)$, and the cones of $\mathrm{star}(S, \Sigma)$ are

$$C_{S \cup T}/\mathbb{R}C_S, \quad T \in \mathrm{lk}_\Delta(S).$$

- If $\deg : A^d(\Sigma) \to \mathbb{R}$ is a linear map and $S \in \Delta(\Sigma)$, then $\deg_S : A^{d-|S|}(\mathrm{star}(S, \Sigma)) \to \mathbb{R}$

$$\deg_S(y) = \deg\left(y \prod_{i \in S} x_i\right)$$

  is linear.
- It follows that the volume polynomials of Σ and $\mathrm{star}(S, \Sigma)$ are related by

$$\mathrm{vol}_{\mathrm{star}(S,\Sigma)} = \partial^S \mathrm{vol}_\Sigma \Big|_{t_i=0, i \in S} = \mathrm{vol}_\Sigma^S .$$

Lorentzian Chow rings

- A functional $\deg : A^d(\Sigma) \to \mathbb{R}$ is positive if
$$\deg\left(\prod_{j\in S} x_j\right) > 0$$
for all facets S of $\Delta(\Sigma)$.
- Let $\mathcal{K}(\Sigma) = \mathcal{K}(\Delta(\Sigma), L(\Sigma))$.
- The pair $(\Sigma, \deg)$, where $\deg$ is positive, is called hereditary Lorentzian if vol_Σ is hereditary Lorentzian (w.r.t. $\mathcal{K}(\Sigma)$).
- This is equivalent to that for all $S \in \Delta(\Sigma)$, the Chow ring
$$A(\mathrm{star}(S, \Sigma))$$
satisfies the Hodge-Riemann relations of degree 0 and 1.

Lorentzian Chow rings

Theorem (B., Leake). Let $A(\Sigma)$ be a Chow ring of a simplicial fan, and $\deg : A^d(\Sigma) \to \mathbb{R}$ positive.
If $\mathcal{K}(\Sigma) \neq \varnothing$, then $A(\Sigma)$ and all its stars satisfy the Hodge-Riemann relations of degree 0 and 1 if and only if

(C) $\Delta(\Sigma)$ is H-connected, and

(L) For each $S \in \Delta(\Sigma)$ with $|S| = d - 2$, the Chow ring of the star of S in Σ satisfies the Hodge-Riemann relations of degree zero and one.

Applications

- The Chow ring of the normal fan of a simple polytope (Stanley-McMullen).
- This implies the Alexandrov-Fenchel inequalities for convex bodies.
- The Chow ring of a matroid (Adiprasito-Huh-Katz).
- This implies the Heron-Rota-Welsh conjecture on the characteristic polynomial of a matroid.

Working with Lorentzian polynomials

- What operations preserve the Lorentzian property?
- Which linear operators preserve the Lorentzian property?
- Let $\kappa \in \mathbb{N}^n$, and let $\mathbb{R}_\kappa[\mathbf{x}]$ be the linear space of all polynomials in $\mathbb{R}[x_1, \dots, x_n]$ that have degree at most κ_i in the variable x_i for all i.
- The symbol of a linear operator $T : \mathbb{R}_\kappa[\mathbf{x}] \to \mathbb{R}[\mathbf{x}']$ is the polynomial

$$G_T(\mathbf{x}', \mathbf{y}) = T\left((\mathbf{x} + \mathbf{y})^\kappa)\right) = \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} T(\mathbf{x}^\alpha) \mathbf{y}^{\kappa - \alpha}.$$

- Example.

$$G_{\frac{\partial}{\partial x_i}} = \kappa_i (\mathbf{x} + \mathbf{y})^{\kappa - e_i}.$$

Working with Lorentzian polynomials

- Theorem (B., Huh). If the symbol G_T is Lorentzian, then T preserves the Lorentzian property.
- Example. If f and g are Lorentzian, then so is fg.

Proof. First fix $g(\mathbf{x})$, and consider $T(f) = fg$. We want to prove that the symbol $(\mathbf{x}+\mathbf{y})^\kappa g$ is Lorentzian.

- To do this, consider the linear operator $S(g) = (\mathbf{x}+\mathbf{y})^\kappa g$.
- The symbol of S is $(\mathbf{x}+\mathbf{y})^\kappa(\mathbf{x}+\mathbf{z})^\kappa$
- This polynomial is stable, and hence Lorentzian. □
- The special case for bivariate polynomials is: If $\{a_k\}_{k=0}^n$ and $\{b_k\}_{k=0}^m$ satisfy Newton's inequalities, then so does the convolution $\{c_k\}_{k=0}^{m+n}$

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

A non-linear operator

- Theorem (B., Huh). Suppose

$$\sum_{\alpha \in J} a(\alpha) \frac{\mathbf{x}^\alpha}{\alpha!},$$

where $a(\alpha) > 0$ for all $\alpha \in J$, is Lorentzian. Then so is

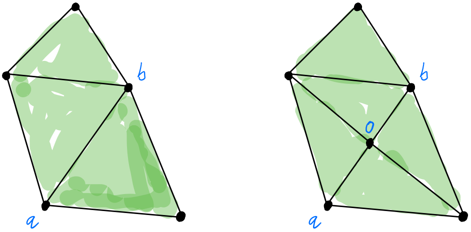
$$\sum_{\alpha \in J} a(\alpha)^s \frac{\mathbf{x}^\alpha}{\alpha!},$$

for all $0 \le s \le 1$.

- Hence this defines a contraction of any Lorentzian polynomial to the exponential generating polynomial of its support.

Stellar subdivisions

- Let $S \in \Delta$, where $|S| \geq 2$. The stellar subdivision, Δ_S, of Δ on S is the simplicial complex on $V \cup \{0\}$, where $0 \notin V$, obtained by
  - removing all faces containing S, and
  - adding all faces $R \cup \{0\}$, where $S \not\subseteq R$ and $R \cup S \in \Delta$.



Δ $\Delta_{\{a,b\}}$

Stellar subdivisions

- Let $S \in \Delta$, where $|S| \geq 2$. The stellar subdivision, Δ_S, of Δ on S is the simplicial complex on $V \cup \{0\}$, where $0 \notin V$, obtained by
  - removing all faces containing S, and
  - adding all faces $R \cup \{0\}$, where $S \not\subseteq R$ and $R \cup S \in \Delta$.
- The stellar subdivision of a fan Σ is defined analogously. Add a ray $\rho = \sum_{i \in S} c_i \rho_i$ in the interior of C_S.
- For positive real numbers $\mathbf{c} = (c_i)_{i \in S}$, let

$$L^{\mathbf{c}} = \left\{ (\ell_0, \ell) \in \mathbb{R} \times \mathbb{R}^V : \ell \in L \text{ and } \ell_0 = \sum_{i \in S} c_i \ell_i \right\}.$$

Stellar subdivisions

- If (Δ, L) is hereditary, then so is $(\Delta_S, L^{\mathbf{c}})$.
- Let $z = t_0 - \sum_{i \in S} c_i t_i$ and define a linear operator by

$$\mathrm{sub}_S^{\mathbf{c}}(f) = f - (-1)^s \sum_{n=s}^{\infty} \frac{z^n}{n!} \cdot h_{n-s}(\bar{\partial})\, \bar{\partial}^S f, \qquad \text{where } s = |S|,$$

where $h_k(\bar{\partial})$ is the complete homogeneous symmetric polynomial of degree k in the variables $\bar{\partial}_i = \partial_i / c_i$, $i \in S$.

Stellar subdivisions

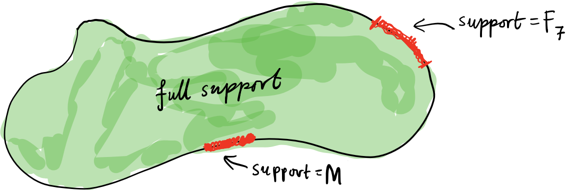
- Theorem (B., Leake). Let (Δ, L) be hereditary. Then $\mathrm{sub}_S^{\mathbf{c}} : \mathcal{P}^d(\Delta, L) \longrightarrow \mathcal{P}^d(\Delta_S, L^{\mathbf{c}})$ is bijective.
- Theorem (B., Leake). Let $f \in \mathcal{P}^d(\Delta, L)$ and $g = \mathrm{sub}_S^{\mathbf{c}}(f)$. If $\mathcal{K}(\Delta, L)$ and $\mathcal{K}(\Delta_S, L^{\mathbf{c}})$ are nonempty, then f is hereditary Lorentzian iff g is.
- The support of a fan is the union of its cones.
- Fact. Two fans have the same support iff one can be derived from the other by a sequence of stellar and inverse stellar subdivisions.
- Corollary (B., Leake). Suppose Σ and Σ' are fans with the same support, and that $\mathcal{K}(\Sigma)$ and $\mathcal{K}(\Sigma')$ are nonempty. If $\mathcal{A}^d(\Sigma)$ is one-dimensional, then vol_Σ is hereditary Lorentzian iff $\mathrm{vol}_{\Sigma'}$ is.
- An analogous theorem was proved for the Kähler package by Ardila, Denham and Huh.

Topology of spaces of Lorentzian polynomials

- Topological spaces defined in terms of zeros of univariate or multivariate polynomials have been studied by e.g. Arnold, Nui, Shapiro-Welker.
- Many combinatorially defined spaces are (conjectured to be) homeomorphic to closed Euclidean balls, and sometimes admit a division into cells so as to form a regular CW-complex.
- For example the totally positive Grassmannian (Galashin, Karp, Lam).

Topology of spaces of Lorentzian polynomials

- Let $\mathcal{L}_n^d$ be the space of all Lorentzian polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d for which $f(\mathbf{1}) = 1$, where $\mathbf{1} = (1, 1, \ldots, 1)$.
- The topology is taken from the linear (Euclidean) space of all homogeneous degree polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ of degree d.
- Let $\underline{\mathcal{L}}_n^d$ be the intersection of $\mathcal{L}_n^d$ with the space of multiaffine polynomials (degree at most one in each variable).
- Theorem (B., Huh). $\mathcal{L}_n^d$ and $\underline{\mathcal{L}}_n^d$ are compact contractable sets, which are equal to the closures of their interiors.



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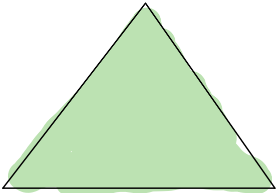
- Conjecture (B., Huh). $\mathcal{L}_n^d$ and $\underline{\mathcal{L}}_n^d$ are homeomorphic to closed Euclidean balls.

Topology of spaces of Lorentzian polynomials

- Example.

$$\mathcal{L}_n^1 = \left\{ \sum_{i=1}^{n} a_i x_i : a_i \geq 0 \text{ and } \sum_{i=1}^{n} a_i = 1 \right\}.$$

A simplex

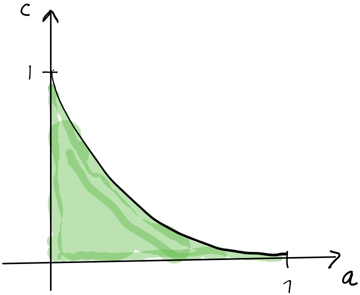


Topology of spaces of Lorentzian polynomials

- Example.

$$\mathcal{L}_2^2 = \left\{ ax^2 + bxy + cy^2 : a, b, c \geq 0, a + b + c = 1 \text{ and } b^2 \geq 4ac \right\}.$$

Two parameters a, c.



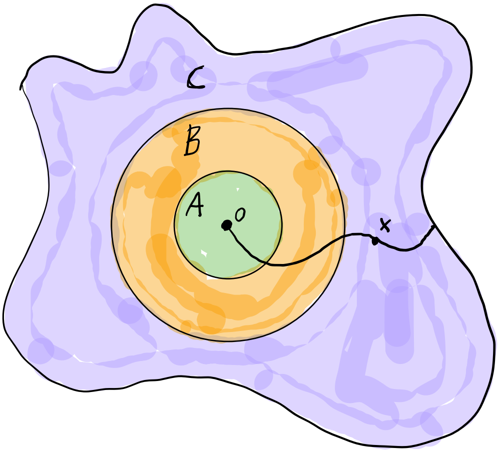
Contractive flows

- Let V be a Euclidean space and $T : \mathbb{R} \times V \to V$ a continuous map.
- Write $T_s(\mathbf{x})$ for $T(s, \mathbf{x})$.
- T is a contractive flow if for all $\mathbf{x} \in V$,
  (a) $T_{s+t}(\mathbf{x}) = T_s(T_t(\mathbf{x}))$ for all $s, t \in \mathbb{R}$, and
  (b) $T_0(\mathbf{x}) = \mathbf{x}$, and
  (c) $\|T_s(\mathbf{x})\| < \|\mathbf{x}\|$, for all $\mathbf{x} \neq 0$ and $s > 0$.
- Lemma (Galashin, Karp, Lam). If U is an open and bounded set in V and T is a contractive flow such that

$$T_s(\overline{U}) \subset U, \qquad \text{for all } s > 0,$$

then $\overline{U}$ is homeomorphic to a closed Euclidean ball.

Contractive flows



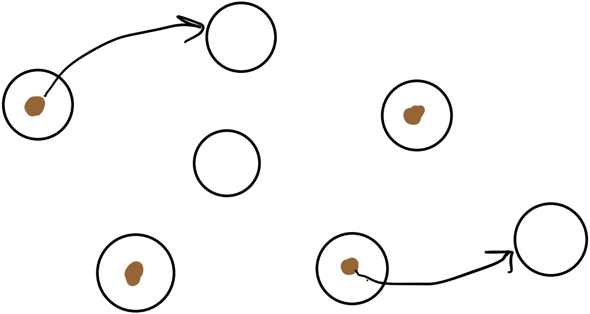
- Map C to B,
- Map $A \cup B$ to A.

Symmetric exclusion process

- Can we find a constructive flow for multiaffine Lorentzian polynomials?
- The symmetric exclusion process (SEP) is one of the most studied models in Interacting particle systems.
- It models particles moving on a finite or countable set in a continuous way.

Symmetric exclusion process

- Let $E = [n]$ be a set of sites that can either be vacant or occupied by one particle.
- At each time t a particle at site i jumps to site j (if vacant) at rate $q_{ij} \geq 0$, where $q_{ij} = q_{ji}$ for all i, j.



Symmetric exclusion process

- Let $E = [n]$ be a set of sites that can either be vacant or occupied by one particle.
- At each time t a particle at site i jumps to site j (if vacant) at rate $q_{ij} \geq 0$, where $q_{ij} = q_{ji}$ for all i, j.
- A discrete probability measure μ on 2^E may be represented by its multivariate partition function

$$f_\mu(\mathbf{x}) = \sum_{S \subseteq E} \mu(S) \prod_{i \in E} x_i, \qquad \text{where } f_\mu(\mathbf{1}) = 1.$$

- The symmetric group on $E = [n]$ acts on polynomials f by $\sigma(f) = f(x_{\sigma(1)}, \dots, x_{\sigma(1)})$.

Symmetric exclusion process

- Particles jump between sites i and j at rate q corresponds to

$$f_\mu \longrightarrow (1-q)f_\mu + q\tau(f_\mu), \qquad \tau = (ij).$$

- For each transposition τ associate a rate $q_\tau \geq 0$ so that

$$\sum_\tau q_\tau = 1.$$

- In terms of polynomials, SEP (with rates $\{q_\tau\}$) is the flow on multiaffine polynomials:

$$T_s(f) = e^{s(L-I)}f, \qquad \text{where } L = \sum_\tau q_\tau \tau \quad \text{and} \quad I = \text{ identity.}$$

Symmetric exclusion process

- Theorem (Borcea, B., Liggett, 2009, B. Huh, 2020). If $s > 0$, then T_s preserves stability and the Lorentzian property.
- Assume from now that $q_\tau = 1/\binom{n}{2}$ for all τ.
- T_s is a flow on $\mathcal{M}_n^d$, the linear space of multiaffine polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ of degree d.
- Notice that $L = \sum_\tau q_\tau \tau : \mathcal{M}_n^d \to \mathcal{M}_n^d$ is symmetric when viewed as matrix.

Symmetric exclusion process

Lemma. Suppose A is a symmetric $n \times n$ matrix with nonnegative entries.

- Suppose A^N has positive entries for N sufficiently large.
- Let $\mathbf{w}$ and λ be the Perron eigenvector and eigenvalue of A.

Then

$$e^{s(A-\lambda I)}$$

is a contractive flow on $\mathbf{w}^\perp$, the orthogonal complement of $\mathbf{w}$.

- Let $f_0 = e_d(\mathbf{x})/\binom{n}{d}$, the normalized elementary symmetric polynomial of degree d. Then $L(f_0) = f_0$.
- Since the set of transpositions generate $\mathfrak{S}_n$, L^N has positive entries for N sufficiently large.
- Corollary. T_s is a contractive flow on the orthogonal complement $f_0^\perp$ of f_0 in $\mathcal{M}_n^d$.

Symmetric exclusion process

- We may view $\underline{\mathcal{L}}_n^d$ as a topological space in $f_0^\perp$.
- Theorem(B., 2021). $\underline{\mathcal{L}}_n^d$ and $\mathcal{L}_n^d$ are homeomorphic to closed Euclidean balls.
- A similar proof applies to prove that the projective space of homogeneous degree d stable polynomials in n variables is homeomorphic to a Euclidean ball.
- Let J be an M-convex set, and $\mathcal{L}_n^d(J)$ the space of polynomials in $\mathcal{L}_n^d$ with support contained in J.
- Conjecture. $\mathcal{L}_n^d(J)$ is homemorphic to a closed Euclidean ball.
- Problem. Can we decompose $\mathcal{L}_n^d$ into cells so as to make it into a regular CW-complex?