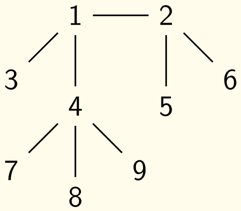
On the determinant of the distance matrix of a tree

Álvaro Gutiérrez (University of Sevilla, University of Bristol)
Joint with E. Briand, L. Esquivias, A. Lillo, M. Rosas

The Graham–Pollak formula

$T =$



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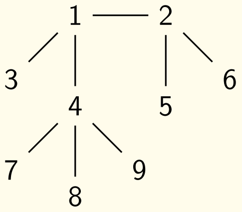

$$M(T) = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 1 & 1 & 3 & 3 & 3 \\ 1 & 2 & 0 & 2 & 3 & 3 & 3 & 3 & 3 \\ 1 & 2 & 2 & 0 & 3 & 3 & 1 & 1 & 1 \\ 2 & 1 & 3 & 3 & 0 & 2 & 4 & 4 & 4 \\ 2 & 1 & 3 & 3 & 2 & 0 & 4 & 4 & 4 \\ 2 & 3 & 3 & 1 & 4 & 4 & 0 & 2 & 2 \\ 2 & 3 & 3 & 1 & 4 & 4 & 2 & 0 & 2 \\ 2 & 3 & 3 & 1 & 4 & 4 & 2 & 2 & 0 \end{pmatrix}$$

$$M(T)_{ij} = d(i,j)$$

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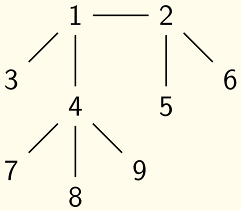

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Graham–Pollak '71

$\det M(T) = (-1)^{n-1}(n-1)2^{n-2}$

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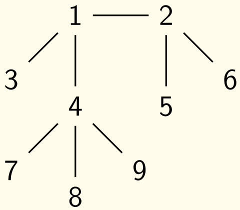

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↑ # ways of choosing an edge e

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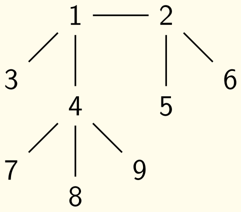

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$(n-1)$: # ways of choosing an edge e

2^{n-2}: # orientations of $T - e$

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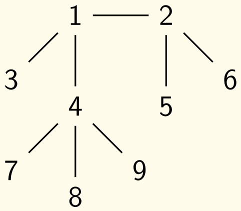
$T =$ 

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sign of a permutation? → $(-1)^{n-1}$

ways of choosing an edge e → $(n-1)$

orientations of $T - e$ → 2^{n-2}

Enumeration of catalysts

First step:

$$\det M(T) = \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma) d(1, \sigma(1)) \cdots d(n, \sigma(n)).$$

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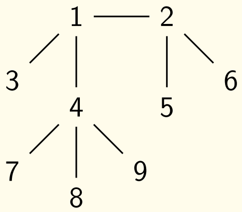

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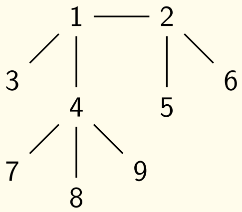

$\sigma : V$	$\to V$	$f : V$	$\to E$
1	$\mapsto 6$	1	$\mapsto 12$
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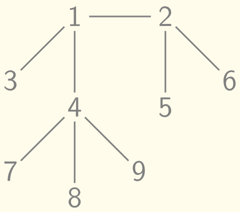

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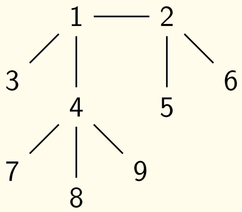

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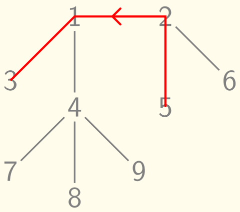

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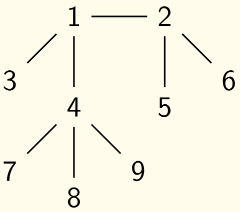

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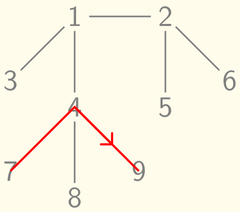

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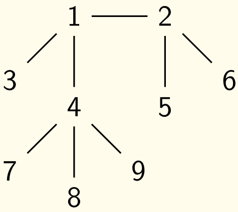

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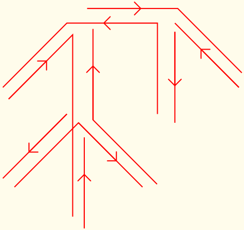

Note (for later): can define equivalently $(\sigma, \vec{f})$ with $\vec{f} : V \to \vec{E}$.

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Goal:

Define involution on the set of catalysts such that

$$\det M(T) = (-1)^{n-1} \cdot \#\text{fixed points of the involution.}$$

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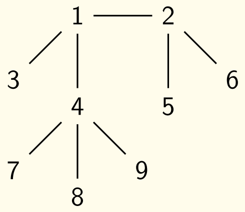
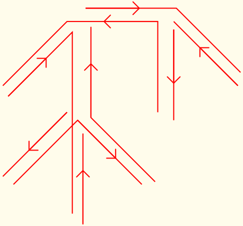
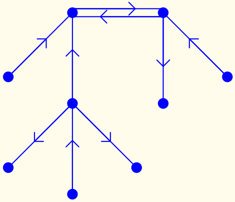
We will define one involution per *arrowflow*.





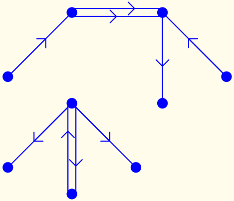
catalyst



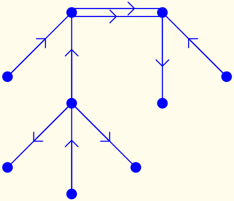
arrowflow

Arrowflows

Def. An arrowflow is a multiset of n directed edges of $\vec{E}$.



zero-sum



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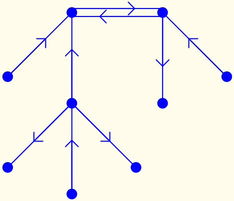

unital

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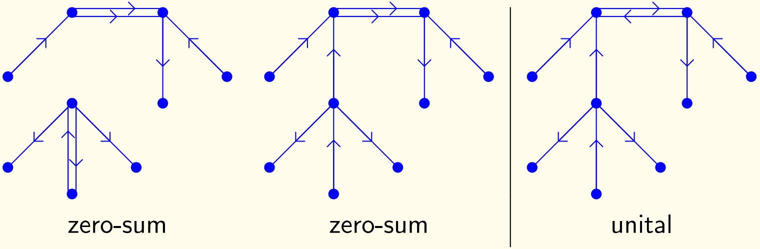

$$\det M(T) = \sum_{A \text{ arrowflow}} \sum_{(\sigma,f) \text{ catalyst of } A} \operatorname{sgn}(\sigma).$$

Theorem

$$\sum_{(\sigma,f) \text{ catalyst of } A} \operatorname{sgn}(\sigma) = \begin{cases} 0 & \text{if } A \text{ is zero-sum,} \\ (-1)^{n-1} & \text{if } A \text{ is unital.} \end{cases}$$

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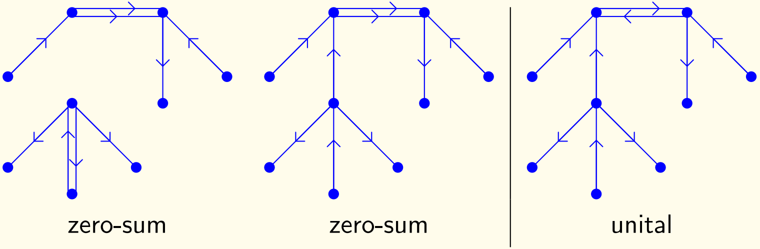

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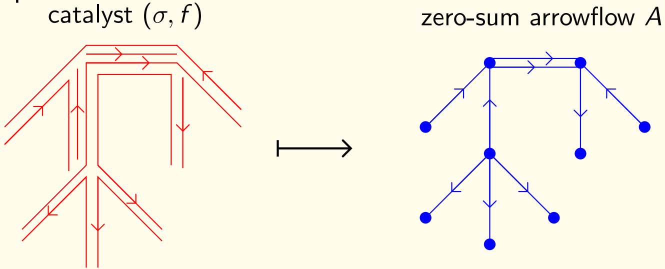
The zero-sum involution

Second step:



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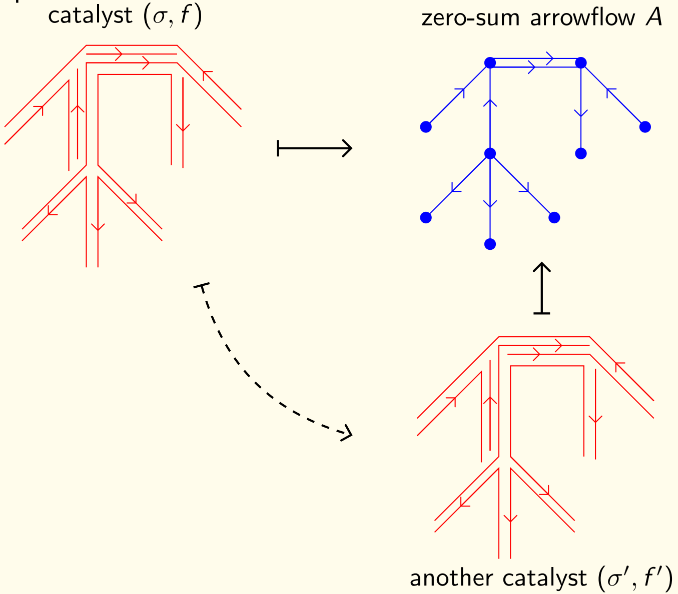
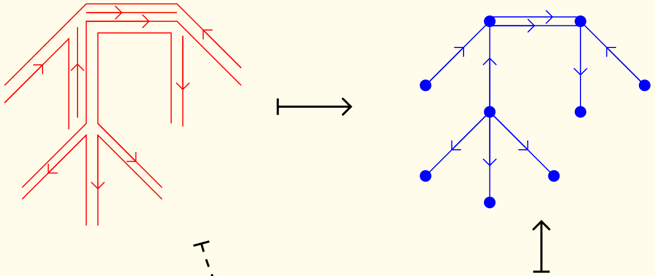
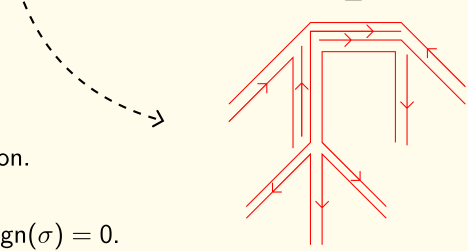


The zero-sum involution

Second step:

catalyst (σ, f) zero-sum arrowflow A





another catalyst (σ', f')

Changes σ by a transposition.

It's an involution.

This shows $\sum_{(\sigma,f) \text{ catalyst of } A} \operatorname{sgn}(\sigma) = 0.$

The zero-sum involution

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The unital involution

Third and final step:

Lindström '73, Gessel–Viennot '85

G acyclic digraph. Two sets $\{v(1), \ldots, v(n)\}$ and $\{v'(1), \ldots, v'(n)\}$ of distinguished points. Then,

$$\sum_{P=(P_1,\ldots,P_n)} \operatorname{sgn}(\sigma_P) = \sum_{\substack{P=(P_1,\ldots,P_n)\\ \text{non-intersecting}}} \operatorname{sgn}(\sigma_P)$$

where P_i is a path $v(i) \to v'(\sigma_P(i))$.

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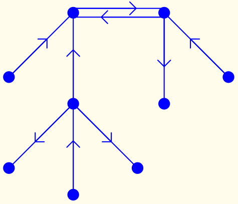
Strategy: given A unital arrowflow, construct G_A such that

$$\{\text{catalysts of } A\} \longleftrightarrow \{n\text{-paths in } G_A\}$$
$$(\sigma, f) \mapsto P_{(\sigma,f)} = (P_1, \dots, P_n)$$

and such $\exists!$ non-intersecting P.

A glimpse at G_A

$A =$



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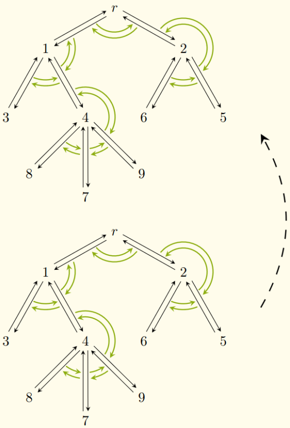

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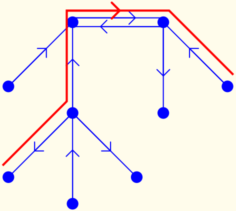

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