On the unimodality consequence of the Neggers- Stanley conjecture

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Joint work with

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Preliminaries

1. P is a poset on $[n] = \{1, .., n\}$ ordered by $<_P$.
2. P is naturally labeled if

$$p <_P q \implies p < q$$

   We will consider naturally labeled poset in our work.
3. A linear extension of P is a total order $\prec$ on $[n]$ such that

$$p <_P q \implies p \prec q$$

   If $p_1 \prec \cdots \prec p_n$ then we write $\pi = p_1 ... p_n \in S_n$ for the total order
4. The length of P is

$$l(P) = \max\{r \mid \exists \ \ p_1 < ... < p_{i-1} < p_r \text{ in } P\}$$

Example

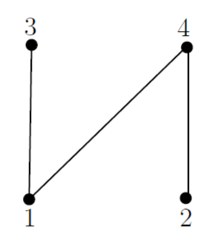


Figure: Naturally labeled poset P

The linear extensions of P are

$$1234, 1324, 1243, 2134, 2143$$

Example

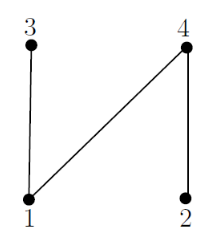


Figure: Naturally labeled poset P

The linear extensions of P are

$$\underbrace{1234}_{0},\underbrace{1324}_{1},\underbrace{1243}_{1},\underbrace{2134}_{1},\underbrace{2143}_{2}$$

Define

$$h_P(t) = \sum_{\pi} t^{des\pi}$$

- Neggers-Stanley conjectured that $h_P(t)$ is real rooted. This was shown to be false by Branden and Stembridge.
- **Well known:** Real rooted implies unimodality.
- Unimodality implication of Neggers-Stanley (still open)
- **This talk:** Extend a known approach to unimodality from graded to non-graded posets (no proof of unimodality yet)

1. An order ideal I in a poset P is a subset $I \subseteq P$ such that if $x \in I$ and $y <_P x$ implies $y \in I$.
2. The distributive lattice of P is

$$L(P) = \{I \mid I \text{ order ideal in } P\}$$

ordered by inclusion.
3. A P-partition is a map $\sigma : P \to \mathbb{N} \in \mathbb{R}^P$ satisfy the following condition

If $s <_P t$ in P, then $\sigma(t) \leq \sigma(s)$.

For an order ideal I its indicator function $\sigma_I : P \to \mathbb{N} \in \mathbb{R}^P$ is a P-partition.
4. The order polytope $O(P)$ is defined as

$$O(P) = \text{Conv}\{\sigma_I \mid I \text{ order ideals in } P\}$$

Triangulations of order polytope

Triangulation of $O(P)$: Geometric simplicial complex Δ with realization $O(P)$.

1. A triangulation of a polytope Γ in $\mathbb{R}^m$ is called regular if it can be obtained by projecting the lower envelope of a lifting of Γ to $\mathbb{R}^{m+1}$.
2. A triangulation Δ' of Γ is unimodular if normalized volume $Vol(\gamma) = 1$ for every maximal simplex γ in Δ'.

Example

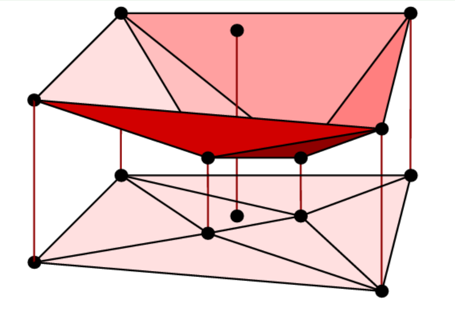


Figure: Regular Triangulation

Theorem (Stanley)

$$\Delta_S = \{\mathrm{Conv}\{\sigma_{I_1}, \ldots, \sigma_{I_r}\} \mid I_1 \subset \ldots \subset I_r \in L(P)\}$$

is a regular unimodular triangulation of $O(P)$.

Lemma

If Δ is a regular unimodular triangulation of $O(P)$ then

$$h_P(t) = h_\Delta(t) \text{ the } h\text{-polynomial of } \Delta).$$

Together with Ehrhart's theorem these results imply:

Theorem (well known)

$$\sum_{n\geq 0} \left| nO(P) \cap \mathbb{N}^P \right| \cdot t^n = \frac{\sum_{i=0}^{r} h_i t^i}{(1-t)^{|P|+1}} = \frac{h_P(t)}{(1-t)^{|P|+1}}$$

where $r = |P| - l(P)$.

- If P is graded then by [Reiner,Welker] in 2005 there is a regular unimodular triangulation Δ of $O(P)$ such that
$$\Delta = 2^{\Omega} * \Delta'$$
where Δ' is a simplicial polytope.
$$\implies h_P(t) = h_\Delta(t) = h_{\Delta'}(t)$$
- g-theorem for simplicial polytope $\implies$ $h_P(t)$ is unimodular.

Theorem (Khan,Welker)

Let P be a poset then there is a triangulation Δ of $O(P)$ such that where

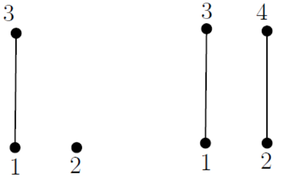
$$\Delta = 2^{\{1,\dots,l(P)\}} * \Delta'$$

$$\Delta' = \begin{cases} \text{ball of dim } |P| - l(P) & \text{if } P \text{ is not graded} \\ \text{sphere of dim } |P| - l(P) - 1 & \text{if } P \text{ is graded } [Reiner, Welker] \end{cases}$$

Corollary

If P is not graded then $h_P(t) = \sum_{i=0}^{r} h_i t^i$ where $r = |P| - l(P)$ and $(h_0, \dots, h_r, 0)$ is the h-vector of a triangulated ball of dimension $|P| - l(P)$.

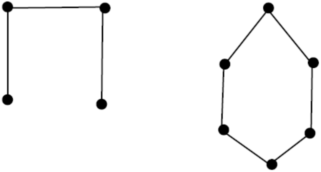
Bad news: No g-theorem for triangulated balls
Still hope: Find g-theorem for "special" balls



Q P

$1 + 2t + 0t^2$ $1 + 4t + t^2$

$\dim = |P| - 2$ $\dim = |P| - 2 - 1$



Approach

Recall definitions for graded posets: P with k rank sets $P_1, ..., P_k$.

Definition (Equatorial P-partition)

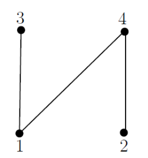
A P-partition σ is called equatorial if

1. $\min_{p \in P} \sigma(p) = 0$.
2. For every $j \in [2, k]$ $\exists$ $p_{j-1} <_p p_j$ with $p_{j-1} \in P_{j-1}, p_j \in P_j$ and $\sigma(p_{j-1}) = \sigma(p_j)$.

Definition (Rank constant P-partition)

A P-partition σ is called rank-constant if it is constant along ranks i.e $\sigma(p) = \sigma(q)$ whenever $p, q \in P_j$ for some j.

Example



For the poset the indicator function of ideals which are equatorial P-partitions are

$$(1) = (1,0,0,0),\ (2) = (0,1,0,0),\ (13) = (1,0,1,0),$$

$$(123) = (1,1,1,0),\ (124) = (1,1,0,1)$$

The indicator functions of ideals which are rank-constant P-partitions are $(12) = (1,1,0,0)$ and $(1234) = (1,1,1,1)$.

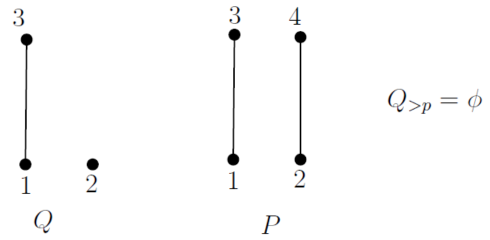
Extension to non-graded posets:

Definition

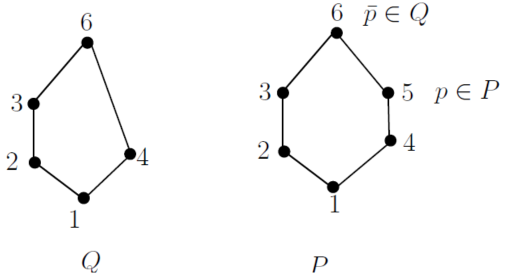
Let $Q \subseteq P$ be posets. We call Q unique (in P) if for all $p \in P \backslash Q$

- $$Q_{>p} = \{y \in Q \mid y > p\}$$
  is either empty or has a unique minimal element $\bar{p}$ and
- $I(P) = I(Q)$.

1)



2)



Lemma

Let Q be a poset then there exists a graded poset $Q \subseteq P$ with Q unique in P.

Definition

Assume $Q \subseteq P$ is unique. We define map i which maps a Q-partition to a P-partition. If f be a Q-partition then $i(f)$ be defined as

$$i(f)(p) = \begin{cases} f(p) & \text{if } p \in Q \\ 0 & \text{if } p \notin Q, Q_{>p} = \phi \\ f(\bar{p}) & \text{if } p \notin Q, \bar{p} \in Q,\ \bar{p} \geq p \end{cases}$$

Lemma

If f is a Q-partition then $i(f)$ is a P-partition.

Lemma

If Q is unique in P and g is a P-partition then there exists a Q-partition f with $i(f) = g$ iff the following holds.

1. $g(p) = 0$ *for* $p \notin Q$, $Q_{>p} = \phi$.
2. $g(p) = f(\bar{p})$ *for* $p \notin Q$, *and* $\bar{p} \in Q$ *the unique minimal element of* $Q_{>p}$.

Definition

A Q-partition f is called equatorial (resp., rank-constant) if $i(f)$ is equatorial (rank-constant).

Definition

A chain of ideals $I_1 \subset \cdots \subset I_d$ in $L(P)$ is called equatorial (resp., rank-constant) if the

$$\sum_{j=1}^{d} i(I_j)$$

is an equatorial (resp., rank-constant) P-partition.

Lemma

Let $Q \subseteq P$ posets and Q is unique in P.
For every Q-partition f there exists a unique decomposition

$$f = f^{eq} + f^{rc}$$

where f^{eq} is an equatorial Q-partition and f^{rc} is a rank-constant Q-partition.

Proposition

Let P be a non-graded poset then there is a triangulation Δ of $O(P)$ such that

$$\Delta = 2^{\{1,\dots,l(P)\}} * \Delta'$$

where

- $2^{\{1,\dots,l(P)\}}$ *is the simplex of rank constant chains in $L(P)$ and*
- Δ' *is the simplicial complex of equatorial chains $I_1 \subset \cdots \subset I_d$ in $L(P)$.*

Additional work:

Reiner and Welker sketch an idea by Dennis White of a *jeu-de-taquin* like bijection between

$$\left\{\text{linear extensions of } P\right\} \leftrightarrow \left\{\text{ maximal equatorial chains }\right\}.$$

Our work: Make this idea rigorous

Thank You

Definition

A map $\pi : P \to [m]$ is i-unambiguous if it satisfies the following conditions.

1. $\pi(p) = \pi(p') \implies p \leq p'$ or $p' \leq p$
2. $|\{\pi(p) \mid l(p) \geq i\}| = |\{p \mid l(p) \geq i\}|$
3. For all $j \leq i-1$ there is exactly one pair p, p', $l(p') = j$, p covers p' with $\pi(p) = \pi(p')$.

Theorem

The map $\phi : \pi \to I_{|P|-n}$ where $n = l(P)$ is a bijection between linear extensions and equatorial chains of P.