# IMMANANT VARIETIES

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$V := \mathbb{C}^n \qquad \{e_1, \dots, e_n\}$

$k \in \mathbb{N} \setminus \{0\}$

$V^{\otimes k} \qquad \{e_x : x \in [n]^k\}$

$e_x := e_{x_1} \otimes \dots \otimes e_{x_k} \qquad x = (x_1, \dots, x_k) \in [n]^k$

$\mathrm{Sep}(k,n) := \{[v_1 \otimes \dots \otimes v_k] : v_i \in V \setminus \{0\}\} \subseteq \mathbb{P}(V^{\otimes k})$

$\cong$

$\mathbb{P}(V)^{\times k}$

$S_k$ ACTS ON $[n]^k$ BY

$\sigma x = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)})$

$S_n$ ACTS ON $x \in [n]^k$ BY

$$\sigma * x = (\sigma(x_1), \dots, \sigma(x_k))$$

$G \leq S_k$ , $\chi : G \to \mathbb{C}$

SIMPLE CHAR.

$$P_\chi := \frac{\chi(e)}{|G|} \sum_{g \in G} \chi(g^{-1}) g \in \mathbb{C}[S_k]$$

$$P_\chi \in \mathrm{END}(V^{\otimes k})$$

$$\hat{P}_\chi : \mathrm{Seg}(k, n) \setminus \mathbb{P}(\mathrm{Ker}(P_\chi)) \longrightarrow \mathbb{P}(V_\chi^{\otimes k})$$

$$V_\chi^{\otimes k} := \mathrm{Im}(P_\chi) \subseteq V^{\otimes k}$$

$$Gr_\chi(k,n) := \overline{\Sigma_n(\hat{P}_\chi)}$$

EX: $k = 2 = n$ $\qquad \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

$$(a_1, a_2) \neq (0,0) \neq (b_1, b_2)$$

$$(a_1 e_1 + a_2 e_2) \otimes (b_1 e_1 + b_2 e_2)$$

$$\chi = 1_{S_k} \longrightarrow a_1 b_1 e_1 \otimes e_1 + \frac{1}{2}(a_1 b_2 + a_2 b_1)[e_1 \otimes e_2 + e_2 \otimes e_1] + a_2 b_2 e_2 \otimes e_2$$

$$\chi = (-1)^{\ell(\sigma)} \longrightarrow \frac{1}{2}(a_1 b_1 - a_2 b_1)[e_1 \otimes e_2 - e_2 \otimes e_1]$$

$\chi$

ONE-DIMENSIONAL $\qquad G \subseteq S_k$

LET $x \in [n]^k$ $\qquad \mathcal{O}_x = \{gx : g \in G\}$

$\overline{x} = \mathrm{MIN}_{\leq_{LEX}} O_x \quad \cancel{x \in} \quad [n]^k$

$B_\chi(k,n) := \{ \overline{x} ; \ x \in [n]^k \}$

$\cancel{x \leq y} \quad x \leq y \iff x \leq g y \ \exists g \in G$

$V_\chi^{\otimes k}$ $\cancel{=}$ HAS BASIS $\{ P_\chi(e_x) : x \in B_{\chi,G}(k,n) \}$

$M \subseteq B_\chi(k,n)$ IS A $\chi$-MATROID

IF $\{ \overline{\sigma^*(x)} : x \in M \}$ HAS MAXIMUM

$\forall \sigma \in S_n$

EX: $(e_1 + e_2 + e_3) \wedge (e_1 + e_4)$

$= e_1 \wedge e_4 + e_2 \wedge e_1 + e_2 \wedge e_4 + e_3 \wedge e_1$

$+ e_3 \wedge e_4 \longrightarrow \{ (1,4), (1,2), (1,3), (2,4), (3,4) \}$

TRIVIAL CH.

$$\mathbb{1}_G : G \to \mathbb{C} \setminus \{0\}$$

$Gr_{\mathbb{1}_G}(k,n)$ HAS THE MAX PROP.

$$B_{\mathbb{1}_G}(k,n) \qquad \sum_{x \in B_{\mathbb{1}_G}(k,n)} q^{p(x)} =$$

$$\frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{k} (1 + q + \dots + q^{i-1})^{c_i(g)}$$

CONJ: THE ORDER COMPLEX OF $B_{\mathbb{1}_G}(k,n)$ IS SHELLABLE

CONJ: THE COORDINATE RING OF THE INCIDENCE STRAT OF $Gr_{\mathbb{1}_G}(k,n)$ IS COHEN-MACAULAY.