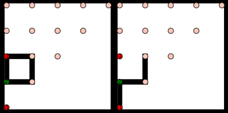

Vector partition functions

SLC 91

Universidad de Sevilla

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March 18, 2024

(partially based on joint work with Emmanuel Briand, Mercedes Rosas, and Marni Mishna)

Vector partition functions

Coin exchange problem

Definition

Let $a_1, \ldots, a_n$ be positive integers. The problem of computing the number of solutions $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$ of

$$a_1x_1 + \cdots + a_nx_n = b$$

for a non-negative integer b is called the *coin exchange problem*.

Example: How many ways can one pay for an item worth 6 dollars (4.08 euros) with 1 dollar (0.68 euro) coins, 2 dollar (1.36 euro) coins and 5 dollar (3.4 euro) bills?

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Example: How many ways can one pay for an item worth 6 dollars (4.08 euros) with 1 dollar (0.68 euro) coins, 2 dollar (1.36 euro) coins and 5 dollar (3.4 euro) bills?

Answer: five ways

1) 6 loonies **2)** 4 loonies, 1 toonie **3)** 2 loonies, 2 toonies
4) 3 toonies **5)** 1 fiver, 1 loonie

Definition

Let A be a $d \times n$ matrix with integer entries, of rank d, and satisfying $\ker(A) \cap \mathbb{R}^d_{\geq 0} = \{\mathbf{0}\}$. The *vector partition function* of A,

$$p_A : \mathbb{Z}^d \to \mathbb{N}$$

is defined by

$$p_A(b) = \#\{\mathbf{x} \in \mathbb{N}^n : A\mathbf{x} = \mathbf{b}\}.$$

$d = 1 \iff$ coin exchange

Example (I)

Compute $p_A(\mathbf{b})$ for

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Equivalently, find the number of solutions $(x_1, x_2, x_3, x_4) \in \mathbb{N}^4$ to

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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Observation: for any "valid" choice of x_4, there is a single choice for x_1, x_2, x_3

$$\implies p_A(b_1, b_2, b_3) = \min(b_1, b_2, b_3) + 1$$

Example (II)

Region I. $0 \le b_1 \le b_2, b_3$

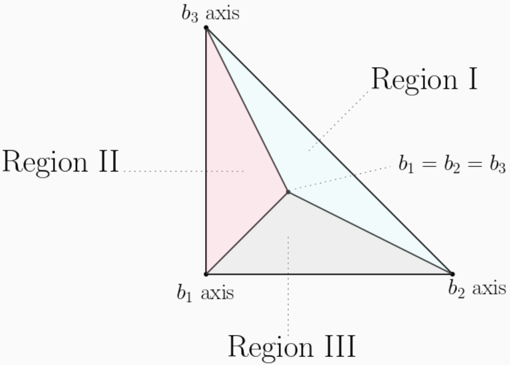
$p_A(\mathbf{b}) = b_1 + 1$

Region II. $0 \le b_2 \le b_1, b_3$

$p_A(\mathbf{b}) = b_2 + 1$

Region III. $0 \le b_3 \le b_1, b_2$

$p_A(\mathbf{b}) = b_1 + 1$



Vector partition function background

- Sturmfels (1994): Vector partition function p_A can be expressed as a piecewise quasi-polynomial (essentially periodic polynomial) of degree $n - d$
- Pieces of polynomiality are *chambers* (maximal cones) of a fan called the *chamber complex* of A
- Vector partition function can be computed using *Barvinok* developed by Koeppe, Verdoolaege, and Woods (2008) (time is polynomial for fixed dimension)
- $p_A(\mathbf{b})$ counts number of integer points in polytope $\{x \in \mathbb{N}^n : A\mathbf{x} = \mathbf{b}\}$ for any $\mathbf{b} \in \mathbb{Z}^d \cap \mathrm{pos}_{\mathbb{R}}(A)$

Our contribution

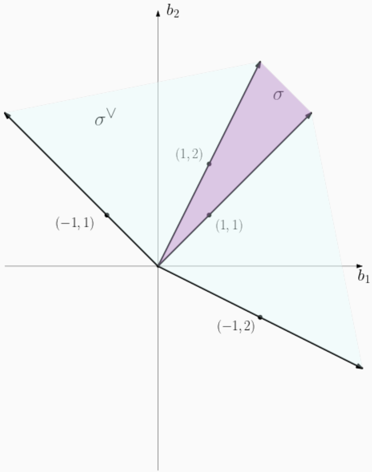
- reduction of dimension via *external columns*
- *external chambers* and coin exchange problem, determinantal formula
- negative binomial coefficient formula
- application to multigraph counting
- examples in computation of (quasi)-polynomials associated to Littlewood-Richardson coefficients and Kronecker coefficients

- symmetry and stability results for Littlewood-Richardson coefficients
- bounds on Kronecker coefficients

External columns and chambers

Cones

- *Cone:* set σ of form $\mathrm{pos}_{\mathbb{R}}(\mathbf{v}_1, \dots, \mathbf{v}_d) := \{\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n : \lambda_1, \dots, \lambda_n \geq 0\}$ for vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{Q}^d$ called *ray generators of* σ
- *Faces:* Intersection of cone with supporting hyperplane (dimension 1 faces are *rays*, co-dimension 1 faces are *facets*)
- *Dual cone:*
  $$\sigma^\vee := \{\mathbf{m} \in \mathbb{R}^d : \mathbf{m} \cdot \mathbf{u} \geq 0 \text{ for all } \mathbf{u} \in \sigma\}$$
  (ray generators of $\sigma^\vee$ are inner facet normals of σ)
- *Fan:* set Σ of cones such that intersection of a pair is a face of both, and face of any cone in Σ is also in Σ.



Some chambers are “nicer” than others

$$A_3 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix} \end{array}$$

I. $\frac{1}{6}(2b_1 + 3b_2 + 3b_3 + 3)(b_1 + 2)(b_1 + 1)$

II. $\frac{1}{6}b_1^3 + \frac{1}{2}b_1^2 b_2 - \frac{1}{2}b_1 b_3^2 - \frac{1}{6}b_3^3 + b_1^2 + \frac{3}{2}b_1 b_2 + \frac{1}{2}b_1 b_3 - \frac{1}{2}b_3^2 + \frac{11}{6}b_1 + b_2 + \frac{2}{3}b_3 + 1$

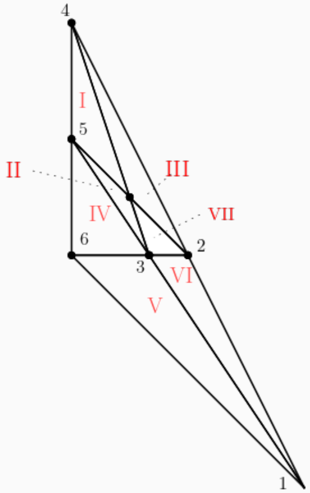
III. $\frac{1}{6}(2b_1 - b_2 - b_3 + 3)(b_1 + b_2 + b_3 + 2)(b_1 + b_2 + b_3 + 1)$

IV. $\frac{1}{6}(b_1 + 3b_2 + 3)(b_1 + 2)(b_1 + 1)$

V. $\frac{1}{6}(b_1 + b_2 + 3)(b_1 + b_2 + 2)(b_1 + b_2 + 1)$

VI. $\frac{1}{6}b_1^3 + \frac{1}{2}b_1^2 b_2 - \frac{1}{6}b_2^3 - \frac{1}{2}b_2^2 b_3 - \frac{1}{2}b_1 b_3^2 - \frac{1}{2}b_2 b_3^2 - \frac{1}{3}b_3^3 + b_1^2 + \frac{3}{2}b_1 b_2 + \frac{1}{2}b_1 b_3 - \frac{1}{2}b_3^2 + \frac{11}{6}b_1 + \frac{7}{6}b_2 + \frac{5}{6}b_3 + 1$

VII. $\frac{1}{6}(b_1 + b_2 + b_3 + 2)(b_1 + b_2 + b_3 + 1)(b_1 + b_2 - 2b_3 + 3)$



Chamber V is very “nice” - formula given by negative binomial coefficient.

External columns & chambers

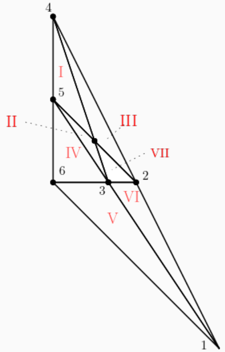
Definitions

External ***column***: column $\mathbf{a}_j$ not contained in $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$

External ***ray***: 1-d cone generated by a single external column

External ***chamber***: all but one ray of γ generated by external column of A

External ***facet***: cone generated by external columns of external chamber



- external columns: 1, 4, 6
- external chambers: V
- external facets: $\mathrm{pos}_{\mathbb{R}}(1, 6)$

Dimension drop via external columns

Main idea: dimension drops by number of external columns in chamber

$$A = \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}, \gamma = \mathrm{pos}_{\mathbb{R}}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

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$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \gamma = \mathbf{pos}_{\mathbb{R}}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

- Apply invertible linear transformation preserving the VPF

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$$MA = \begin{pmatrix} 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \ M\gamma = \mathrm{pos}_{\mathbb{R}}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

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- Apply invertible linear transformation preserving the VPF
- View external column variables as slack variables

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$$A = \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}, \gamma = \mathrm{pos}_{\mathbb{R}}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

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$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \gamma'' = \mathrm{pos}_{\mathbb{R}}\left(\begin{bmatrix} 1 \end{bmatrix} \right)$$

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- Apply invertible linear transformation preserving the VPF
- View external column variables as slack variables

$p_A^{\gamma}(\mathbf{b}) = p_B((Mb)_d)$, so $p_A^{\gamma}(\mathbf{b})$ arises from coin exchange!

Main result and determinantal formula

Theorem (T. 2023)

Let A be a $d \times n$ matrix of rank d with integer entries, and let γ be a chamber of A that is simplicial. Without loss of generality assume that $\mathbf{a}_1, \dots, \mathbf{a}_\ell$ are the external columns of γ. Assume additionally that the semigroup $\mathrm{pos}_{\mathbb{N}}(()\{\mathbf{a}_1, \dots, \mathbf{a}_\ell\})$ is saturated in $\mathcal{L}(A)$. Let B be the matrix obtained by removing the first ℓ rows and columns from $M_{\gamma^\vee}A$. Then

$$p_A^\gamma(\mathbf{b}) = p_B^{\gamma'}\left((M_{\gamma^\vee}\mathbf{b})_{\ell+1}, \dots, (M_{\gamma^\vee}\mathbf{b})_d\right)$$

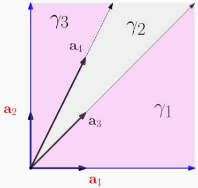
for all $\mathbf{b} \in \gamma \cap \mathrm{pos}_{\mathbb{N}(A)}$. Moreover, γ' is the positive orthant in $\mathbb{R}^{d-\ell}$.

Determinantal formula (T. 2023)

$p_A^\gamma(\mathbf{b}) = f\left(\frac{\det(\mathbf{a}_1,\dots,\mathbf{a}_{d-1},\mathbf{b})}{\det(\mathbf{a}_1,\dots,\mathbf{a}_{d-1},\mathbf{v})}\right)$, $\mathbf{v}$ - internal ray generator of γ, $f(t) = p_A(t\mathbf{v})$

Example

$$A^{2,2} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$



$$p^{\gamma_1}_{A^{2,2}}(\mathbf{b}) = f\left(\frac{\det(\mathbf{a}_1, \mathbf{b})}{\det(\mathbf{a}_1, \mathbf{a}_3)}\right) = f\left(\frac{det\begin{bmatrix} 1 & b_1 \\ 0 & b_2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}\right) = f(b_2) = \begin{cases} \frac{(b_2+1)^2}{4} & \text{if } b_2 \equiv 0 \mod 2 \\ \frac{(b_2+1)(b_2+3)}{4} & \text{if } b_2 \equiv 1 \mod 2. \end{cases}$$

Here $f(t) := p^{2,2}_A(t\mathbf{a}_3)$ (Ehrhart quasi-polynomial along internal ray) computed using *Latte*.

Polynomiality and unimodularity

Observation: coin exchange p_B is polynomial iff each entry of B is equal (say β). In this case, p_B is negative binomial coefficient.

Under saturation, and dot product condition we find:

Theorem (T. 2023)

$$p_A^\gamma(\mathbf{b}) = \binom{\frac{\iota\cdot\mathbf{b}}{\beta} + n - d}{n-d} = \binom{\frac{\det(\mathbf{a}_1,\ldots,\mathbf{a}_{d-1},\mathbf{b})}{\det(\mathbf{a}_1,\ldots,\mathbf{a}_{d-1},\mathbf{a}_{d+\ell})} + n - d}{n-d}$$

Unimodular: determinant of $d \times d$ submatrices $\in \{0, \pm 1\}$
Unimodular $\iff$ p_A^γ polynomial (de Loera, Sturmfels 2003)
If A unimodular, $\beta = 1$

Constructing external chambers

- **Issue:** it can be very time intensive to compute the entire chamber complex (for example, number of chambers for only first 7 cases are known in much studied Kostant's partition function), so we would like to avoid this.
- **Lemma:** external facet $\iff$ facet of $\mathrm{pos}_{\mathbb{R}}(A)$ containing exactly $d-1$ columns
- Compute external facet: this can be done by computing dual cone, and then computing dot products to check number of columns on facet. Call generators of external facet $\mathbf{a}_1, \dots, \mathbf{a}_{d-1}$.
- Then external chamber is

$$\bigcap_{j=d}^{n} \mathrm{pos}_{\mathbb{R}}(\mathbf{a}_1, \dots, \mathbf{a}_{d-1}, \mathbf{a}_j). \tag{1}$$

- Note: external chambers don't exist for all vector partition functions

An application in the enumeration of mulitgraphs

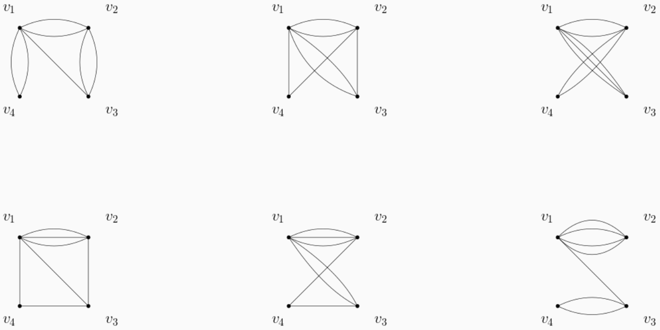
Multigraph counting (I)

Goal: count number $M_m(d_1, \dots, d_m)$ of loopless multigraphs on vertices $v_1, \dots, v_m$ with degree sequence $(d_1, \dots, d_m)$

Multigraph counting (I)

Goal: count number $M_m(d_1, \dots, d_m)$ of loopless multigraphs on vertices $v_1, \dots, v_m$ with degree sequence $(d_1, \dots, d_m)$

Example: There are 6 multigraphs on v_1, v_2, v_3, v_4 with degree sequence $(5, 4, 3, 2)$:



Multigraph counting (II)

- G_m - incidence matrix of complete graph on m vertices
- Connection to vector partition functions:

$$M_m(d_1, \ldots, d_m) = p_{G_m}(d_1, \ldots, d_m)$$

- Idea: study the vector partition function p_{G_m} - for example, does it have any external chambers for general m?

Multigraph counting (II)

- G_m - incidence matrix of complete graph on m vertices
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- Idea: study the vector partition function p_{G_m} - for example, does it have any external chambers for general m?
- Answer: yes! External chamber γ is defined by monotonicity inequalities $d_1 \geq d_2 \geq \cdots \geq d_m$, as well as $d_1 + d_m \geq d_2 + \cdots + d_{m-1}$

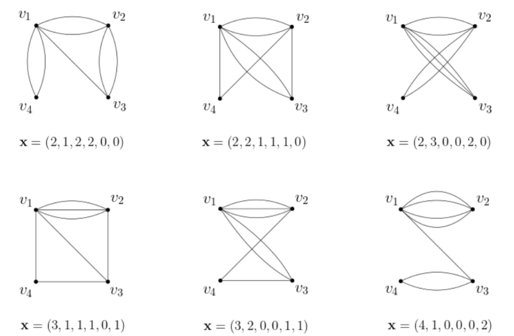
$$\boxed{p^{\gamma}_{G_m}(d_1, \dots, d_m) = \binom{|E| - d_1 + \binom{m-1}{2} - 1}{\binom{m-1}{2} - 1}}$$

Multigraph counting (III)

- $G_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$
- $(5, 4, 3, 2)$ is in external chamber!
- $m = 4, d_1 = 5, |E| = \frac{5+4+3+2}{2} = 7$, so

$$p_{G_m}^{\gamma}(5, 4, 3, 2) = \binom{7 - 5 + \binom{3}{2} - 1}{\binom{3}{2} - 1} = 6$$

Solutions $\mathbf{x} \in \mathbb{N}^6$ to $G_4\mathbf{x} = (5, 4, 3, 2)$



More general setting

Vector partition-like functions

- **Briand, Rosas, Orellana:** *vector partition-like function:* piecewise quasi-polynomial F whose pieces are chambers of a fan
- analogue of external column: $F(t\mathbf{v}) = 1$ for all t (we call this an *F-external ray*)
- analogue of external chamber: all (but one) rays generating chamber are external rays (we call this an F-external chamber).
- we consider two vector partition-like functions (Littlewood-Richardson function and Kronecker function) which arise more indirectly from vector partition function
- note: vector partition-like function is a little too general for our purposes - question: what restrictions should we impose?

Littlewood-Richardson coefficients

Littlewood-Richardson coefficients $c_{\lambda,\mu}^{\nu}$ describe how to express a product of Schur functions in the basis of Schur functions:

$$s_\lambda s_\mu = \sum_\nu c_{\lambda,\mu}^{\nu} s_\nu$$

- Rassart (2004): proved *Littlewood-Richardson function*

  $$\Phi_k(\lambda, \mu, \nu) = c_{\lambda,\mu}^{\nu}$$

  (for $\ell(\lambda), \ell(\mu), \ell(\nu) \leq k$) is a piecewise-polynomial, "pieces" are maximal cones of a fan $\mathcal{LR}_k$
- "Unfortunately the Littlewood–Richardson rule is much harder to prove than was at first suspected. The author was once told that the Littlewood–Richardson rule helped to get men on the moon but was not proved until after they got there."
  - Gordon James

The piecewise polynomial Φ_3

Minimal ray generators

$\mathbf{a}_1 = (1,1,1 \mid 0,0,0 \mid 1,1,1)$ $\quad$ $\mathbf{a}_2 = (0,0,0 \mid 1,1,1 \mid 1,1,1)$

$\mathbf{b} = (2,1,0 \mid 2,1,0 \mid 3,2,1)$ $\quad$ $\mathbf{c} = (1,1,0 \mid 1,1,0 \mid 2,1,1)$

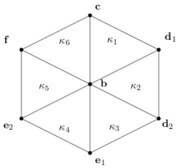
$\mathbf{d}_1 = (1,1,0 \mid 1,0,0 \mid 1,1,1)$ $\quad$ $\mathbf{d}_2 = (1,0,0 \mid 1,1,0 \mid 1,1,1)$

$\mathbf{e}_1 = (1,1,0 \mid 0,0,0 \mid 1,1,0)$ $\quad$ $\mathbf{e}_2 = (0,0,0 \mid 1,1,0 \mid 1,1,0)$

$\mathbf{f} = (1,0,0 \mid 1,0,0 \mid 1,1,0)$

$\mathbf{g}_1 = (1,0,0 \mid 0,0,0 \mid 1,0,0)$ $\quad$ $\mathbf{g}_2 = (0,0,0 \mid 1,0,0 \mid 1,0,0)$

$\mathcal{LR}_3$ (in my mind)



Piecewise polynomial Φ_3

Chamber	Ray generators	Polynomial
κ_1	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2$	$1 - \lambda_2 - \mu_2 + \nu_1$
κ_2	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{g}_1, \mathbf{g}_2$	$1 + \nu_2 - \nu_3$
κ_3	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{g}_1, \mathbf{g}_2$	$1 + \lambda_1 + \mu_1 - \nu_1$
κ_4	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}$	$1 + \nu_1 - \nu_2$
κ_5	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$	$1 + \lambda_2 + \mu_2 - \nu_3$
κ_6	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$	$1 - \lambda_3 - \mu_3 + \nu_3$
κ_7	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{g}_1$	$1 + \lambda_3 + \mu_1 - \nu_3$
κ_8	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_2, \mathbf{g}_2$	$1 + \lambda_1 + \mu_3 - \nu_3$
κ_9	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{g}_2$	$1 + \lambda_1 - \lambda_2$
κ_{10}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{g}_1$	$1 + \mu_1 - \mu_2$
κ_{11}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{e}_1, \mathbf{g}_1, \mathbf{g}_2$	$1 - \lambda_2 - \mu_3 + \nu_2$
κ_{12}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_2, \mathbf{e}_2, \mathbf{g}_1, \mathbf{g}_2$	$1 - \lambda_3 - \mu_2 + \nu_2$
κ_{13}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{f}, \mathbf{g}_1$	$1 - \lambda_1 - \mu_3 + \nu_1$
κ_{14}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_2$	$1 - \lambda_3 - \mu_1 + \nu_1$
κ_{15}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{e}_1, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$	$1 + \mu_2 - \mu_3$
κ_{16}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_2, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$	$1 + \lambda_2 - \lambda_3$
κ_{17}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_2$	$1 + \lambda_1 + \mu_2 - \nu_2$
κ_{18}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1$	$1 + \lambda_2 + \mu_1 - \nu_2$

Determinantal formula

Observation: each ray is Φ_3-external except for the one generated by $\mathbf{b}$, therefore each chamber is Φ_3-external

Briand, Rosas, T., 2023:

Theorem

Let $\mathbf{p} := (\lambda, \mu, \nu) \in \kappa$ for some chamber κ of $\mathcal{LR}_3$ with minimal ray generators $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$. Then

$$c_{\lambda,\mu}^{\nu} = \left| \det(\tilde{\mathbf{p}}, \tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_5) \right| + 1$$

NOTE: LR coefficient represents continuous volume of paralleliped

$c_{\lambda,\mu}^{\nu} = 1 \iff$ volume of paralliped $= 0$ (dimension drops)

The $k = 4$ case

Φ_4-external chamber γ generated by the following rays:

$v_1 := (0,0,0,0,1,0,0,0,1,0,0)$ $\quad v_2 := (0,0,0,0,1,1,1,0,1,1,1)$ $\quad v_3 := (0,0,0,0,1,1,1,1,1,1,1)$

$v_4 := (1,0,0,0,0,0,0,0,1,0,0)$ $\quad v_5 := (1,1,0,0,0,0,0,0,1,1,0)$ $\quad v_6 := (1,1,0,0,1,0,0,0,1,1,1)$

$v_7 := (1,1,0,0,1,1,0,0,2,1,1)$ $\quad v_8 := (1,1,0,0,1,1,1,0,2,1,1)$ $\quad v_9 := (1,1,1,0,0,0,0,0,1,1,1)$

$v_{10} := (1,1,1,1,0,0,0,0,1,1,1)$ $\quad v_{11} := (4,3,1,0,3,2,1,0,6,4,3)$

Each ray except for v_{11} is Φ_4-external.

Determinant formula holds here:

$$\Phi_4^{\gamma} = \binom{\frac{\det(v_1,\ldots,v_{10},b)}{\det(v_1,v_2,\ldots,v_{11})} + 3}{3}$$

21 such chambers in this case, each of where determinant formula holds.

Unfortunately, no Φ_k-external chambers for larger k

Kronecker coefficients (I)

- *Kronecker coefficients* are the structure constants in the decomposition of a tensor product of irreducible representations of the symmetric group into irreducible representations:

$$V_\mu \otimes V_\nu = \bigoplus_\lambda g_{\lambda,\mu,\nu} V_\lambda.$$

- Schur functions:

$$s_\lambda[XY] = \sum_{\mu,\nu} g_{\lambda,\mu,\nu} s_\mu[X] s_\nu[Y]$$

$X = (x_1, \dots, x_m), Y = (y_1, \dots, y_n), XY = (x_1y_1, x_1y_2 \dots, x_my_n)$

"In part due to the fact that they lack a combinatorial interpretation, even the most basic questions present seemingly insurmountable challenges, while even the simplest examples are already hard to compute. Yet, this should not prevent us from pursuing both." - Pak, Panova

Kronecker coefficients (II)

- **Briand, Rosas, Orellana:** full description of the Kronecker function $G_{2,2}$ described by auxiliary function $G^*_{2,2} : \mathbb{Z}^5 \to \mathbb{Z}_{\geq 0}$.
- The following chamber is $G^*_{2,2}$-external:

$$\gamma = \mathrm{pos}_{\mathbb{R}} \begin{pmatrix} \mathbf{v}_1 := (3,1,1,1,1), & \mathbf{v}_2 := (1,0,0,0,0), & \mathbf{v}_3 := (2,1,0,1,0) \\ \mathbf{v}_4 := (2,0,1,1,0), & \mathbf{v}_5 := (6,2,2,2,1) & \end{pmatrix}.$$

$G^*_{2,2}(tv_i) = 1$ for each $i = 1,2,3,4$

$$G^*_{2,2}(tv_5) = \begin{cases} (t+1)^2 & \text{if } t \equiv 0 \mod 2 \\ \frac{(t+1)(t+3)}{4} & \text{if } t \equiv 1 \mod 2 \end{cases}$$

- $$(G^*_{2,2})^{\gamma_{62}} = \begin{cases} (r+s-g_1-g_2+1)^2 & \text{if } r+s-g_1-g_2 \equiv 0 \mod 2 \\ \frac{(r+s-g_1-g_2+1)(r+s-g_1-g_2+3)}{4} & \text{if } r+s-g1-g2 \equiv 1 \mod 2 \end{cases}$$
- determinant formula seems to hold!

Kronecker coefficients (III)

- The determinant formula holds for each of the $G^*_{2,2}$-external chambers.
- For some of these cases, we have a polynomial, not a quasi-polynomial. In each of these cases, we obtain a negative binomial coefficient (analogous to the vector partition function case).
- Example: the chamber

$$\gamma = \mathrm{pos}_{\mathbb{R}} \begin{pmatrix} \mathbf{v}_1 := (3,1,1,1,1), & \mathbf{v}_2 := (1,0,0,0,0), & \mathbf{v}_3 := (4,1,2,1,1), \\ \mathbf{v}_4 := (2,0,1,1,0), & \mathbf{v}_5 := (10,3,4,3,2) & \end{pmatrix}$$

  is $G^*_{2,2}$.
- $(G^*_{2,2})^\gamma = \binom{r-g_2+2}{2}$.

Future work

Future work

- Generalization beyond vector partition functions (does dimension reduction hold as well here?)
- “Persistent” chambers
- Nearly external chambers?

Thanks!

"Thank you for attending!"
-Me

Articles

Emmanuel Briand, Mercedes Rosas, and Stefan Trandafir.
The chamber complex for the Littlewood-Richardson coefficients of GL_4.
EAMD 2020: XI Encuentros Andaluces de Matemática Discreta, 2020.

Emmanuel Briand, Mercedes Helena Rosas Celis, and Stefan Trandafir.
All linear symmetries of the su (3) tensor multiplicities.
Journal of Physics A: Mathematical and Theoretical, 57 (1), 015205-1., 2023.

Stefan Trandafir.
External columns and chambers of vector partition functions.
arXiv preprint arXiv:2307.13112, 2023.