The monopole-dimer model on Cartesian products of plane graphs

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Definitions and Background

A *graph* G is a pair (V, E), where V is a collection whose elements are called *vertices* and E is a collection of two-element subsets of V and the elements of E are called as *edges*.

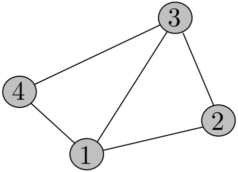

Figure: A graph on 4 vertices.

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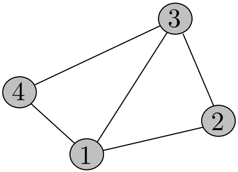

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If there are no multiple edges we will say the graph G is *simple*. If there is an edge between two vertices $v, v' \in V$ then we will write $v \sim v'$.

Definition

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The *Pfaffian of* of a $2n \times 2n$ skew-symmetric matrix A is defined as

$$Pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} sgn(\sigma)\, A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \cdots A_{\sigma(2n-1)\sigma(2n)}.$$

Dimer model

Let G be an edge-weighted graph on $2n$ vertices with edge-weight w_e for $e \in E(G)$.

Dimer model:- is the collection of all dimer cover together with the weight of each dimer cover given by

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The partition function of the dimer model on G,

$$Z_G := \sum_{M \in \mathcal{M}(G)} w(M).$$

Plane graph

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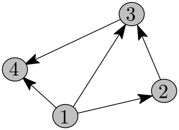
Pfaffian orientation

An orientation on a plane graph G is said to be *Pfaffian* if the boundary of all the bounded faces has an odd number of clockwise oriented edges. In other words, a Pfaffian orientation is said to possess the *clockwise-odd property*.

Result by Kasteleyn



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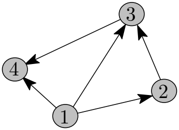

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Theorem (Kasteleyn, 1963 [3])

Let G be a simple plane weighted graph with Pfaffian orientation $\mathcal{O}$ then

$$Z_G = |Pf(K_G)|,$$

where K_G is a matrix defined as

$$(K_G)_{u,v} = \begin{cases} w_{(u,v)} & \text{if } u \to v \text{ in } \mathcal{O}, \\ -w_{(u,v)} & \text{if } v \to u \text{ in } \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

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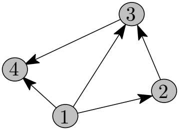

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Loop-vertex configuration

Let G be a simple weighted labelled graph on n vertices with an orientation[1] $\mathcal{O}$, vertex-weight $x(v)$ for $v \in V(G)$ and edge-weight $a(v, v') \equiv a(v', v)$ for $(v, v') \in E(G)$.

[1] Orientation is assignment of arrows to the edges.

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Loop-vertex configuration C of G:-

is a subgraph of G consisting of

- Directed loops of even lengths(with length > 2),
- Doubled edges(can be thought of as loop of length 2),
- Isolated vertices

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The set of all loop vertex configurations of G will be denoted as $\mathcal{L}(G)$.

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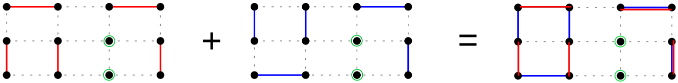
A loop-vertex configuration can be thought of as superposition of two matchings having non-matched vertices at same locations.



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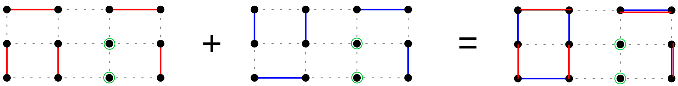
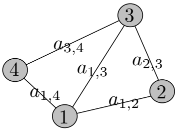
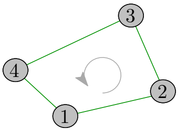


Figure: superposition of two matchings

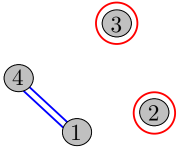
The following figure shows a graph and two of its loop-vertex configurations.



(a) Graph G



(b) The directed cycle (1234)



(c) The doubled edge (14) and isolated vertices 2 and 3

Figure: A graph with edge weights and two loop-vertex configurations on it.

Loop-vertex model

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Loop-vertex model:-

The loop-vertex model on the (edge- and vertex-weighted) graph G with an orientation $\mathcal{O}$ is the collection $\mathcal{L}$ of configurations with the weight of each configuration defined as:

$$w(C) = \prod_{\ell = \text{loop in } C} w(\ell) \prod_{\substack{v \text{ an} \\ \text{isolated vertex} \\ \text{in } C}} x(v)$$

$$\text{where } w(\ell) = -\prod_{i=1}^{2m} sgn(v_i, v_{i+1})\, a(v_i, v_{i+1}) \quad \text{for } \ell = (v_1, v_2, .., v_{2m}, v_1).$$

Partition function of the Loop-vertex Model

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Theorem(A. Ayyer)[2]

The partition function of loop-vertex model on $(G, \mathcal{O})$ is

$$\mathcal{Z}_{G,\mathcal{O}} = \det K,$$

where K is the signed adjacency matrix of $(G, \mathcal{O})$ defined as:

$$K(v, v') = \begin{cases} x(v) & \text{if v=v'}, \\ a(v, v') & \text{if } v \rightarrow v' \, in \, \mathcal{O}, \\ -a(v, v') & \text{if } v' \rightarrow v \, in \, \mathcal{O}, \\ 0 & \text{if } v \nsim v'. \end{cases}$$

Monopole-dimer model

If G is a plane graph with a Pfaffian orientation (having clockwise-odd property), then loop-vertex model is called the *monopole-dimer model* and

$$w(\ell) = (-1)^{\substack{\#\text{ of vertices} \\ \text{enclosed by } \ell}} \prod_{j=1}^{2m} a(v_j, v_{j+1}) \quad \text{for } \ell = (v_1, v_2, .., v_{2m}, v_1).$$

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Partition function

The Partition function of monopole-dimer model on G is

$$\mathcal{Z}_G := \sum_{C \in \mathcal{L}(G)} w(C).$$

Let G be a plane bipartite graph.

Definition

The *sign* of a cycle decomposition $\mathcal{D} = \{d_1, \dots, d_k\}$ of G is given by

$$sgn(\mathcal{D}) := \prod_{i=1}^{k} (-1)^{\#\text{ of vertices enclosed by } d_i + 1}. \tag{1}$$

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Lemma (**Arora, Ayyer**)

Let G be a connected bipartite even plane graph. Then all cycle decompositions $\mathcal{D}$ of G will have same sign [1].

Let $G_1, \ldots, G_k$ be k plane simple naturally labelled bipartite graphs with Pfaffian orientations $\mathcal{O}_1, \ldots, \mathcal{O}_k$ respectively and P be their oriented Cartesian product. For $i \in [k]$, Let $G_1, \dots, G_k$ be k plane simple naturally labelled bipartite graphs with Pfaffian orientations $\mathcal{O}_1, \dots, \mathcal{O}_k$ respectively and P be their oriented Cartesian product. For $i \in [k]$,

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The *i-projection* of a subgraph S of the Cartesian product P is the graph obtained by contracting all but G_i-edges of S and is denoted $\tilde{S}_i$.

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Then the *sign* of ℓ is defined by

Extended monopole-dimer model

$$sgn(\ell) := -\prod_{i=1}^{k-1}(-1)^{e_i} \prod_{j=1}^{k} sgn(\mathcal{D}_j), \qquad (2)$$

where

$$e_i = \sum_{\substack{\hat{v} \in V(\hat{G}^{(i)}) \\ v_{i+1}+\cdots+v_k+(k-i)\equiv 1 \pmod 2}} e_i(\hat{v}).$$

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Theorem

The partition function of the extended monopole-dimer model [1] for the weighted oriented Cartesian product $(P, \mathcal{O})$ of $G_1, \dots, G_k$ is given by

$$\mathcal{Z}_P = \det K_P. \tag{3}$$

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For the non-bipartite case, (2) holds for a smaller collection of cycle decompositions.

Monopole dimer model on three dimensional grids

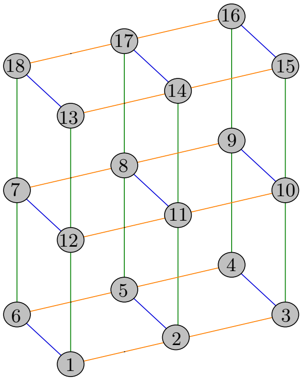

Figure: The boustrophedon labelling on $P_3 \square P_2 \square P_3$.

Theorem

Let $(G, \mathcal{O})$ be the three dimensional grid graph with even lengths (say $2\ell, 2m$ and $2n$) and $\mathcal{O}$ be the canonical orientation induced from the boustrophedon labelling. Let vertex weights be x for all vertices of G, and edge weights be a, b, c for the edges along the three coordinate axes. Then the partition function of the monopole-dimer model on G is given by

$$\prod_{j=1}^{n} \prod_{s=1}^{m} \prod_{k=1}^{\ell} \left(x^2 + 4a^2 \cos^2 \frac{\pi k}{2\ell+1} + 4b^2 \cos^2 \frac{\pi s}{2m+1} + 4c^2 \cos^2 \frac{\pi j}{2n+1} \right)^4$$

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Similarly partition function of the monopole-dimer model on the d-dimensional grid graph with P_{m_i}-edges having edge-weights a_i can be expressed as a product.

d-dimensional grid graph

Theorem

Let $(G, \mathcal{O})$ be the d-dimensional grid graph with even lengths (say $2m_1, \ldots, 2m_d$) and $\mathcal{O}$ be the canonical orientation induced from the boustrophedon labelling. Let vertex weights be x for all vertices of G and edge weights be a_i for the P_{2m_i}-edges. Then the partition function of the monopole-dimer model on G is given by

$$\mathcal{Z}_G = \prod_{i_1=1}^{m_1} \cdots \prod_{i_d=1}^{m_d} \left(x^2 + \sum_{q=1}^{d} 4a_q^2 \cos^2 \frac{i_q \pi}{2m_q + 1} \right)^{2^{d-1}}. \tag{4}$$

References I

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Thank You!