Divisible design graphs from symplectic graphs over rings with precisely three ideals

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8th Workshop on Design Theory, Hadamard Matrices and Applications
(Hadamard 2025)

May 27, 2025

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Group Divisible designs

Definition 1

A group divisible design (GDD) with parameters $(v, b; r, k; \lambda_1, \lambda_2; m, n)$ is an incidence structure that consists of v points and b blocks with constant block size k; each point appears in r blocks; and the set of points can be partitioned into m groups each of size $n \geq 2$, such that two points from a group occur together precisely in λ_1 blocks and two points from different groups occur together precisely in λ_2 blocks.

- Bose and Connor introduced GDDs in [BC52].
- A GDD is called symmetric if $r = k$ (whence, $b = v$). So, a symmetric GDD has parameters $(v, k; \lambda_1, \lambda_2; m, n)$.
- A symmetric GDD is said to have the dual property if its dual is also a symmetric GDD with the same parameters. In [B77], Bose considered symmetric GDDs with the dual property.

[BC52] R. C. Bose and W. S. Connor, *Combinatorial properties of group divisible incomplete block designs*, The Annals of Mathematical Statistics (1952), p. 367–383.

[B77] R.C. Bose, *Symmetric group divisible designs with the dual property*, Journal of Statistical Planning and Inference 1 (1977), 87–101.

Divisible design graphs (I)

- We consider simple graphs, that is graphs, without loops and multiple edges.
- A graph can be interpreted as a design (or incidence structure), by taking the vertices of the graph as points, and the neighbourhoods of the vertices as blocks.
- Divisible design graphs were first introduced in [M08], and then studied in more detail in [HKM11]. These graphs act as a bridge between graph theory and the theory of (group) divisible designs.

Definition 2

A k-regular graph on v vertices is called a divisible design graph (DDG) with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if its vertex set can be partitioned into m classes of size n such that any two vertices from the same class have λ_1 common neighbours and any two vertices from different classes have λ_2 common neighbours.

[HKM11] W. H. Haemers, H. Kharaghani, and M. A. Meulenberg, *Divisible design graphs*, Journal of Combinatorial Theory, Series A 118, no. 3 (2011), 978–992.

[M08] M. A. Meulenberg, *Divisible design graphs*, Master's thesis, Tilburg University (2008).

Divisible design graphs (II)

- In other words, a divisible design graph is a graph whose adjacency matrix is the incidence matrix of a symmetric GDD with dual property.
- A DDG with $m = 1$ or $n = 1$ or $\lambda_1 = \lambda_2$ is called an improper DDG, otherwise it is called a proper DDG.
- The partition of the vertex set of a DDG into classes is called the canonical partition.

Some existing results on DDGs

- A list of known DDGs can be found in [P23]. In particular, a number of characterization results and explicit constructions are known.
- For instance, most DDGs with at most 39 vertices were enumerated in [PS22].
- Let V be a $2e$-dimensional vector space over the finite field $\mathbb{F}_q$, where $e \geq 1$ and q is a prime power. For any nonzero $v \in V$, denote by $[v]$ the 1-dimensional subspace generated by v. Let

$$K = \begin{bmatrix} 0 & I_e \\ -I_e & 0 \end{bmatrix},$$

where I_e is the $e \times e$ identity matrix. The symplectic graph $Sp^{(2e)}(\mathbb{F}_q)$ relative to K over $\mathbb{F}_q$ is the graph with the set of 1-dimensional subspaces of V as its vertex set and the adjacency defined by $[v] \sim [u]$ if $vKu^t \neq 0$ for 1-dimensional subspaces $[u]$ and $[v]$. Very recently, [DGHS24] gives new DDGs arising from the symplectic graph $Sp^{(2e)}(\mathbb{F}_q)$.

[DGHS24] B. De Bruyn, S. Goryainov, W. Haemers, and L. Shalaginov, *Divisible design graphs from the symplectic graph*, Designs, Codes and Cryptography (2024), pp. 1-24.

[P23] D. Panasenko, *Bibliography on Deza graphs*,
http://alg.imm.uran.ru/dezagraphs/biblio.html

[PS22] D. Panasenko and L. Shalaginov, *Classification of divisible design graphs with at most 39 vertices*, Journal of Combinatorial Designs 30, no. 4 (2022), 205–219.

Graphs closely related to DDGs (I)

Definition 3

A k-regular graph on v vertices is called a strongly regular graph with parameters (v, k, λ, μ) if any two adjacent vertices have exactly λ common neighbours, and any two distinct non-adjacent vertices have exactly μ common neighbours.

- A proper DDG is strongly regular if and only if the graph or the complement is mK_n, the disjoint union of m complete graphs of size n.

Graphs closely related to DDGs (II)

Definition 4

A Deza graph with parameters (v, k, b, a) is a k-regular graph with v vertices, such that the number of common neighbours of two distinct vertices takes precisely two values, b or a, where $b \geq a$.

- A Deza graph of diameter 2 which is not a strongly regular graph is called a strictly Deza graph.
- Such Deza graphs are regular graphs which are not strongly regular, so proper DDGs, which are not isomorphic to mK_n or the complement, are strictly Deza graphs.
- [GWL10] and [LW08] give constructions of Deza graphs based on symplectic spaces. The construction from [GWL10] generalises the one from [LW08].

[GWL10] J. Guo, K. Wang, and F. Li, *Deza graphs based on symplectic spaces*, European Journal of Combinatorics 31, no. 8 (2010), 1969–1980.

[LW08] F. Li and Y. Wang, *Subconstituents of symplectic graphs*, European Journal of Combinatorics 29, no. 5 (2008), 1092–1103.

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Symplectic graphs over $R = \mathbb{F}_{p^n}$ or $\mathbb{Z}_{p^n}$ (I)

- Let p be a prime number and let R be the ring of integers modulo p^n, $\mathbb{Z}_{p^n}$, or the field of p^n elements, $\mathbb{F}_{p^n}$, where n is a positive integer. Let $R^\times$ denote the set of units of R.
- For $e \geq 1$, let V' denote the set of $2e$-tuples $(a_1, a_2, \ldots, a_{2e})$ of elements in R such that $a_i \in R^\times$ for some $i \in \{1, 2, \ldots, 2e\}$.
- We consider an equivalence relation $\sim_{p^n}$ on V' by $(a_1, a_2, \ldots, a_{2e}) \sim_{p^n} (b_1, b_2, \ldots, b_{2e})$ if and only if $(a_1, a_2, \ldots, a_{2e}) = \lambda(b_1, b_2, \ldots, b_{2e})$ for some $\lambda \in R^\times$.
- Let $[a_1, a_2, \ldots, a_{2e}]$ denote the equivalence class of $(a_1, a_2, \ldots, a_{2e})$ modulo $\sim_{p^n}$, and let V be the set of all such equivalence classes.
- Let K be the $2e \times 2e$ matrix with entries in R, given by

$$K = \begin{bmatrix} 0 & I_e \\ -I_e & 0 \end{bmatrix},$$

  I_e being the $e \times e$ identity matrix.
- The symplectic graph on V relative to K, denoted by $Sp^{(2e)}(R)$, is the graph whose vertex set is V and with adjacency defined by $[a_1, \ldots, a_{2e}]$ is adjacent to $[b_1, \ldots, b_{2e}]$ if and only if $(a_1, \ldots, a_{2e})K(b_1, \ldots, b_{2e})^t \in R^\times$.

Symplectic graphs over $R = \mathbb{F}_{p^n}$ or $\mathbb{Z}_{p^n}$ (II)

- $Sp^{(2e)}(\mathbb{F}_{p^n})$ is a strongly regular graph with $\lambda = \mu$ ([TW06]).
- $Sp^{(2)}(\mathbb{Z}_{p^n})$ is a strongly regular graph with $k = \mu$ ([MP11]).
- For $e \geq 2$ and $n \geq 2$, $Sp^{(2e)}(\mathbb{Z}_{p^n})$ is a strictly Deza graph with $k = b$ (LWG12).
- As a more general case, symplectic graphs over local rings have also been defined using symplectic spaces over such rings. Let the dimension of the symplectic space be $2e$ ($e \geq 1$).
  If $e = 1$, these graphs are strongly regular with $k = \mu$.
  If $e \geq 2$, these graphs are strictly Deza with $k = b$. Such a Deza graph is necessarily the composition of a strongly regular graph with $\lambda = \mu$ and a coclique.

[TW06] Z. Tang, Z. Wan, *Symplectic graphs and their automorphisms*, European Journal of Combinatorics 27 (2006), 38–50.

[MP11] Y. Meemark and T. Prinyasart, *On symplectic graphs modulo p^n*, Discrete Mathematics 311 (2011), 1874–1878.

[LWG12] F. Li, K. Wang, and J. Guo, *More on symplectic graphs modulo p^n*, Linear Algebra and its Applications 438 (2012), 2651–2660.

[MP13] Y. Meemark and T. Puirod, *Symplectic graphs over finite local rings*, European Journal of Combinatorics 34, no. 7 (2013), 1114–1124.

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Structure and properties of rings with precisely three ideals

- Our results involve finite commutative rings with identity having precisely three ideals. Hereon, let R denote such a ring, with $R^\times$ denoting its subset of units. Let $J = \langle r \rangle$ be its unique ideal which is not equal to $\langle 0 \rangle$ or R.
- Then, R is a local ring, so R equals the disjoint union of the two subsets: the subset of units, and J.
  We also have $R/J \cong \mathbb{F}_q$, the finite field of order q, where q is a prime power.
- $R \cong P/\langle p^2 \rangle$ for some principal ideal domain P and prime $p \in P$.
- Let $R/J = \{z_1 + J, z_2 + J, \ldots, z_q + J\}$ where $\{z_1, z_2, \ldots, z_q\}$ is a set of coset representatives with $z_1 = 0$. Then $J = \{z_1 r, z_2 r, \ldots, z_q r\}$ and $R = \{z_{j_1} + z_{j_2} r : j_1, j_2 \in \{1, 2, \ldots, q\}\}$, where $|J| = q$ and $|R| = q^2$. Hereon, we shall use these notations to define our results.
- Let $j_1, j_2 \in \{1, 2, \ldots, q\}$.
  $z_{j_1} + z_{j_2} r \in J$ if and only if $z_{j_1} = 0$.
  $z_{j_1} + z_{j_2} r = 0$ if and only if $z_{j_1} = z_{j_2} = 0$.

- Let $e \geq 1$ be an integer.
- Let $V' = \{(a_1, a_2, \ldots, a_{2e}) : a_1, a_2, \ldots, a_{2e} \in R$ and $a_j \in R^\times$ for some $j \in \{1, 2, \ldots, 2e\}\}$.
  We define an equivalence relation $\sim$ on V' as: $(a_1, a_2, \ldots, a_{2e}) \sim (b_1, b_2, \ldots, b_{2e})$ if and only if there exists $\lambda \in R^\times$ such that $a_j = \lambda b_j, 1 \leq j \leq 2e$.
  Let V denote the set of equivalence classes on V' corresponding to the equivalence relation $\sim$. We denote by $[a_1, a_2, \ldots, a_{2e}]$ the equivalence class of $(a_1, a_2, \ldots, a_{2e})$.
- Let K be the $2e \times 2e$ non-singular matrix with entries in R, given by

$$K = \begin{bmatrix} 0 & I_e \\ -I_e & 0 \end{bmatrix}$$

where I_e is the $e \times e$ identity matrix.

Defining two new families of graphs over rings with precisely three ideals (II)

- Let $\mathcal{P}((a_1, a_2, \ldots, a_{2e}), (b_1, b_2, \ldots, b_{2e}))$ denote the matrix product
$$(a_1, a_2, \ldots, a_{2e})K(b_1, b_2, \ldots, b_{2e})^t,$$
where $(a_1, a_2, \ldots, a_{2e}),\ (b_1, b_2, \ldots, b_{2e}) \in V'$. Then,
$$\mathcal{P} = (a_1 b_{e+1} - a_{e+1} b_1) + (a_2 b_{e+2} - a_{e+2} b_2) + \cdots + (a_e b_{2e} - a_{2e} b_e).$$

- We define two graphs $X(2e, R)$ and $Y(2e, R)$, both with vertex set V.
- In the graph $X(2e, R)$, $[a_1, a_2, \ldots, a_{2e}]$ is adjacent to $[b_1, b_2, \ldots, b_{2e}]$ if and only if
$$\mathcal{P}((a_1, a_2, \ldots, a_{2e}), (b_1, b_2, \ldots, b_{2e})) \in S_{X(2e,R)},$$
where $S_{X(2e,R)} = R \setminus \{0\}$.
- In the graph $Y(2e, R)$, $[a_1, a_2, \ldots, a_{2e}]$ is adjacent to $[b_1, b_2, \ldots, b_{2e}]$ if and only if
$$\mathcal{P}((a_1, a_2, \ldots, a_{2e}), (b_1, b_2, \ldots, b_{2e})) \in S_{Y(2e,R)},$$
where $S_{Y(2e,R)}$ is the set of nonzero elements in J that is, $S_{Y(2e,R)} = \{z_j r : j \in \{2, \ldots, q\}\}$.
- Clearly, $Y(2e, R)$ is a spanning subgraph of $X(2e, R)$.

The DDG $X(2e, R)$

Theorem 5 ([BG24, Theorem 1])

Let R be a finite commutative ring with identity having precisely three ideals, $\langle 0 \rangle$, J and R. Let $R/J \cong \mathbb{F}_q$ for a prime power q. Then the graph $X(2e, R)$ is a divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ where

$$\begin{aligned}
v &= \frac{q^{2e-1}(q^{2e}-1)}{q-1},\\
k &= q^{4e-2} + q^{4e-3} - q^{2e-2},\\
\lambda_1 &= q^{4e-2} + q^{4e-3} - q^{4e-4} - q^{2e-2},\\
\lambda_2 &= q^{4e-2} + q^{4e-3} - q^{4e-4} - q^{4e-5} - q^{2e-2} + q^{2e-3},\\
m &= \frac{q^{2e}-1}{q-1},\\
n &= q^{2e-1}.
\end{aligned}$$

[BG24] A. Bhowmik and S. Goryainov, *Divisible design graphs from symplectic graphs over rings with precisely three ideals*, December 2024, arXiv:2412.04962.

The DDG $Y(2e, R)$

Theorem 6 ([BG24, Theorem 2])

Let R be a finite commutative ring with identity having precisely three ideals, $\langle 0 \rangle$, J and R. Let $R/J \cong \mathbb{F}_q$ for a prime power q. Then the graph $Y(2e, R)$ is a divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ where

$$
\begin{aligned}
v &= \frac{q^{2e-1}(q^{2e}-1)}{q-1}, \\
k &= q^{4e-3} - q^{2e-2}, \\
\lambda_1 &= q^{4e-3} - q^{4e-4} - q^{2e-2}, \\
\lambda_2 &= q^{4e-4} - q^{4e-5} - q^{2e-2} + q^{2e-3}, \\
m &= \frac{q^{2e}-1}{q-1}, \\
n &= q^{2e-1}.
\end{aligned}
$$

[BG24] A. Bhowmik and S. Goryainov, *Divisible design graphs from symplectic graphs over rings with precisely three ideals*, December 2024, arXiv:2412.04962.

Key ideas contributing to proofs (I)

- $T := \underbrace{J \times J \times \ldots \times J}_{2e \text{ times}}$ is a vector space over R/J of dimension $2e$, with the scalar multiplication defined as,

$$(z_j + J).(x_1, x_2, \ldots, x_{2e}) = (z_j x_1, z_j x_2, \ldots, z_j x_{2e}),$$

  where $j \in \{1, 2, \ldots, q\}$ and $x_1, x_2, \ldots, x_{2e} \in J$. We say that a $(2e-1)$-dimensional subspace of T is a hyperplane. We define the canonical partitions of the graphs using such hyperplanes.
- Let H be a hyperplane of T and let $u \in V'$. Let $C(H, u) := \{[u + h] : h \in H\}$. The elements of $C(H, u)$ are distinct if and only if $ru \notin H$.
- Let H_1, H_2 be hyperplanes of T and let $u_1 \in V'$, $u_2 \in V'$ such that $ru_1 \notin H_1$ and $ru_2 \notin H_2$. If the sets $C(H_1, u_1)$ and $C(H_2, u_2)$ have a vertex in common, then they coincide.
- There exists a uniquely determined partition of the vertex sets of $X(2e, R)$ and $Y(2e, R)$ into classes $C(H, u)$. Moreover, the size of each class is q^{2e-1} and the number of such classes in the partition is the Gaussian coefficient $\begin{bmatrix} 2e \\ 2e-1 \end{bmatrix}_q = \frac{q^{2e}-1}{q-1}$. We denote this partition by $\Pi(2e, R)$. This partition is the canonical partition for the graphs $X(2e, R)$ and $Y(2e, R)$.

Choosing hyperplanes and tuples

We show the existence of a partition into classes $C(H, u)$, which is sufficient to prove our main results. However, the disadvantage of our results is that the canonical partition is given implicitly. The following problem naturally arises.

Question 7

Let m be the number of classes of the canonical partition of the graphs $X(2e, R)$ and $Y(2e, R)$, which is equal to the number of $(2e-1)$-dimensional subspaces in a $2e$ dimensional vector space over the finite field $\mathbb{F}_q$. How can we explicitly choose hyperplanes $H_1, H_2, \ldots, H_m$ and tuples $u_1, u_2, \ldots, u_m$ such that $C(H_1, u_1), C(H_2, u_2), \ldots, C(H_m, u_m)$ is the canonical partition of the graphs $X(2e, R)$ and $Y(2e, R)$?

Key ideas contributing to proofs (II)

The graph $X(2e, R)$:

- We find the number of common neighbours of two distinct vertices in the complement graph $\overline{X(2e, R)}$.
- Number of common neighbours of two distinct vertices in the *same class* of the partition $\Pi(2e, R)$:
  We count the number of solutions of some linear equations with coefficients in R, where the equations are reduced modulo J.
- Number of common neighbours of two distinct vertices in *different classes* of the partition $\Pi(2e, R)$:
  We use the fact that $C(H, u)$ can be naturally identified with the set of $2e$-tuples $\{\sigma(u + h) : \sigma \in R^\times, h \in H\}$, and this set can be partitioned into $q - 1$ cosets of T, where a set of representatives of these cosets is $\{z_j u : j \in \{2, \ldots, q\}\}$.

The graph $Y(2e, R)$:

The ideas of proofs are similar- they also involve counting the number of solutions of some linear equations with coefficients in R.

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Particular instances of DDGs we constructed

- Let p be a prime and let q be a prime power. The rings $R = \mathbb{Z}_{p^2}$ and $R = \dfrac{\mathbb{F}_q[x]}{\langle x^2 \rangle}$ are finite commutative rings with identity having precisely three ideals.
- By numerical computations for some small values of primes p, we find that the graphs $X(2e, \mathbb{Z}_{p^2})$ and $Y(2e, \mathbb{Z}_{p^2})$ have the same parameters with, but are not isomorphic to $X\left(2e, \dfrac{\mathbb{F}_p[x]}{\langle x^2 \rangle}\right)$ and $Y\left(2e, \dfrac{\mathbb{F}_p[x]}{\langle x^2 \rangle}\right)$ (note that the rings $\mathbb{Z}_{p^2}$ and $\dfrac{\mathbb{F}_p[x]}{\langle x^2 \rangle}$ are not isomorphic).
- By numerical computations, we did not find small local rings having precisely three ideals, other than $\mathbb{Z}_{p^2}$ and $\dfrac{\mathbb{F}_q[x]}{\langle x^2 \rangle}$, leading to divisible design graphs. We referred to [N18] for a classification of small local rings.

[N18] A. Nowicki, *Tables of finite commutative local rings of small orders*, 2018 (online).

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Vertex transitivity of the graphs $X(2e, R)$ and $Y(2e, R)$

- Let S be a commutative ring with identity. Let K be the $2e \times 2e$ matrix with entries in S, given by $K = \begin{bmatrix} 0 & I_e \\ -I_e & 0 \end{bmatrix}$, where I_e is the $e \times e$ identity matrix.
- The symplectic group of degree $2e$ over S with respect to K, denoted by $Sp_{2e}(S)$, consists of all $2e \times 2e$ matrices M over S satisfying $MKM^t = K$.
- Each element M in $Sp_{2e}(S)$ induces a graph automorphism $\sigma_M : [x_1, x_2, \ldots, x_{2e}] \longmapsto [(x_1, x_2, \ldots, x_{2e})M]$.
- A graph G is called vertex-transitive if given any two vertices v_1 and v_2 of G, there is a graph automorphism f such that $f(v_1) = v_2$.
- Tang and Wan [TW06], and Li et al. [LWG13] have shown, respectively, that the symplectic graphs over the field $\mathbb{F}_{p^n}$ and the ring $\mathbb{Z}_{p^n}$ are vertex-transitive.
- Recently, we have proved that the graphs $X(2e, R)$ and $Y(2e, R)$ are vertex-transitive. The proof goes along similar lines as in [LWG13].

[TW06] Z. Tang and Z. Wan, *Symplectic graphs and their automorphisms*, European Journal of Combinatorics 27, no. 1 (2006), p. 38–50.

[LWG13] F. Li, K. Wang and J. Guo, *More on symplectic graphs modulo p^n*, Linear Algebra and its Applications 438, no. 6 (2013), p. 2651–2660.

Thank you for your attention!