Isolated (complex) Hadamard Matrices

Wojciech Bruzda

Center for Theoretical Physics of the Polish Academy of Sciences, Warszawa

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outline

- motivation
- isolation
- special family L
- special family V
- conclusions

based on:

1. WB; J. Math. Phys. **64**, 052201 (2023)
2. WB, et al.; Open Syst. Inf. Dyn. **31**, 2450008 (2024)

(complex) Hadamard matrix – notation

Df. (real) **Hadamard** matrix H or size N is a ∓ 1-valued square matrix with row (columns) mutually orthogonal;

$$H H^{\mathrm{T}} = N\,\mathbb{I}_N$$

Df. H is called **complex** HM, (written $H \in \mathbb{H}(N)$), iff

$$H \in \sqrt{N}\mathbb{U}(N) \cap \mathbb{T}^N$$

Df. CHM $H \in \mathbb{H}(N)$ is of **Butson type**, ($H \in \mathrm{B}\mathbb{H}(N, q)$), iff its entries are q^{th}-roots of unity

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additional structure(s) are distinguished: R-dual ($\mathbb{H}^{\mathrm{R}}(N^2)$), Γ-dual ($\mathbb{H}^{\Gamma}(N^2)$), two-unitary ($\mathbb{H}^2(N^2)$) Hadamard matrices

can be mixed with Butson classes, e.g., $\mathrm{B}\mathbb{H}^2(9, 6)$

motivation

$\mathbb{H}(6)$ classification
= { affine and non-affine families } $\cup$ { isolated representative }

$$\mathrm{LOG}\left(S_6\right) = \frac{2}{3}\pi \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & 1 & 1 & 2 & 2 \\ \bullet & 1 & \bullet & 2 & 2 & 1 \\ \bullet & 1 & 2 & \bullet & 1 & 2 \\ \bullet & 2 & 2 & 1 & \bullet & 1 \\ \bullet & 2 & 1 & 2 & 1 & \bullet \end{bmatrix}$$

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proper credits

1. **A.T. Butson**; Proc. AMS **13**, 894–898 (1962)
2. **G.E. Moorhouse**; unpublished notes (2001)
3. **T. Tao**; Math. Res. Lett. **11**, 251–258 (2004)

isolation; $\{\forall_{H' \in \mathrm{neigh}(H)} : H' \simeq H\} \implies H$ is isolated

tools

- **defect** $\mathrm{d} : \mathbb{H} \to \{0\} \cup \mathbb{N}$

  one-way criterion:

  if defect vanishes then matrix is isolated[1]
- commuting squares[2]

constructions

- numerical search
- theorem[1]: $N \in \mathbb{P} \implies F_N = F_N^{(0)}$
- specific construction from MUPB[3]
- this contribution

[1] **W. Tadej et al.**; Open Syst. Inf. Dyn. **13**, 133–177 (2006)

[2] **R. Nicoară**; Indiana Univ. Math. J., **60**, 847–858, (2011)

[3] **D. McNulty et al.**; J. Math. Phys. **53**, 122202 (2012)

application – ?

- classification
- **conjecture: only non-isolated CHM provide two-unitary structures**

$$N_9^{(0)} \simeq \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & y & y^2 & -1 & -y & y & y^3 & y^3 & y \\ 1 & y^2 & y^4 & -y & y^2 & -y^3 & y^4 & y^2 & 1 \\ 1 & -y^4 & -y^3 & y^3 & -y^3 & -y^2 & -y^4 & -y^2 & -1 \\ 1 & -1 & y^2 & -1/y & 1 & 1 & y & y^2 & 1/y \\ 1 & y^3 & -y & -1 & y^2 & y^3 & y & y & y \\ 1 & y & 1 & -1/y & y^2 & y^2 & -1 & 1 & 1/y \\ 1 & y^4 & y^2 & -y & y^4 & y^2 & y^2 & -y^3 & 1 \\ 1 & y^3 & y^4 & -y^3 & y^4 & y^2 & y^3 & y^2 & -1 \end{bmatrix}, \quad y = -\frac{1}{4} + i\frac{\sqrt{15}}{4}$$

$$D_L = \text{diag}\left\{1, 1, 1, 1, -y^4, -y^3, 1, y, 1\right\}$$

$$D_R = \text{diag}\left\{1, 1, 1, 1, -1, -y, -y^3, \xi, \xi y\right\} \quad : \quad \xi = \frac{7}{2^7} + i\frac{33\sqrt{15}}{2^7} \quad \Longrightarrow \quad D_L N_9^{(0)} D_R \in \mathbb{H}^{\Gamma}(9)$$

- self-testing and quantum non-locality – is there any quantum advantage that can be obtained when using isolated structures?

special construction (L)

$N = 9$

$$
Y_{\texttt{numeric}} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & a & d & \underline{a} & \underline{c} & \underline{b} & c & b & \underline{d} \\
1 & b & c & \underline{b} & a & \underline{d} & \underline{a} & d & \underline{c} \\
1 & c & \underline{b} & \underline{c} & d & a & \underline{d} & \underline{a} & b \\
1 & \underline{b} & \underline{c} & b & \underline{a} & d & a & \underline{d} & c \\
1 & d & \underline{a} & \underline{d} & b & \underline{c} & \underline{b} & c & a \\
1 & \underline{a} & \underline{d} & a & c & b & \underline{c} & \underline{b} & d \\
1 & \underline{c} & b & c & \underline{d} & \underline{a} & d & a & \underline{b} \\
1 & \underline{d} & a & d & \underline{b} & c & b & \underline{c} & \underline{a}
\end{bmatrix}
\simeq
\left[\begin{array}{c|cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & a & \underline{a} & b & \underline{b} & c & \underline{c} & d & \underline{d} \\
1 & \underline{a} & a & \underline{b} & b & \underline{c} & c & \underline{d} & d \\ \hline
1 & b & \underline{b} & d & \underline{d} & \underline{a} & a & c & \underline{c} \\
1 & \underline{b} & b & \underline{d} & d & a & \underline{a} & \underline{c} & c \\ \hline
1 & c & \underline{c} & \underline{a} & a & \underline{d} & d & \underline{b} & b \\
1 & \underline{c} & c & a & \underline{a} & d & \underline{d} & b & \underline{b} \\ \hline
1 & d & \underline{d} & c & \underline{c} & \underline{b} & b & \underline{a} & a \\
1 & \underline{d} & d & \underline{c} & c & b & \underline{b} & a & \underline{a}
\end{array}\right]
$$

$$
\left.\begin{aligned}
-1 &= a + b + c + d + \text{c.c.} \\
-1 &= a^2 + b^2 + c^2 + d^2 + \text{c.c.} \\
-1 &= a/b + b/d + c/d + ac + \text{c.c.} \\
-1 &= a/d + b/c + bc + ad + \text{c.c.} \\
-1 &= a/c + ab + cd + bd + \text{c.c.}
\end{aligned}\right\} \implies \text{analytic solution (isolated)}
$$

special construction (L)

$$\mathrm{core}\Big(Y\Big) = \begin{bmatrix} \boxed{\texttt{A}} & \boxed{\texttt{B}} & \boxed{\texttt{C}} & \boxed{\texttt{D}} \\ \boxed{\texttt{B}} & \boxed{\texttt{D}} & \underline{\underline{\boxed{\texttt{A}}}} & \boxed{\texttt{C}} \\ \boxed{\texttt{C}} & \underline{\underline{\boxed{\texttt{A}}}} & \underline{\underline{\boxed{\texttt{D}}}} & \underline{\underline{\boxed{\texttt{B}}}} \\ \boxed{\texttt{D}} & \boxed{\texttt{C}} & \underline{\underline{\boxed{\texttt{B}}}} & \underline{\underline{\boxed{\texttt{A}}}} \end{bmatrix} \qquad : \qquad \boxed{\texttt{X}} = \begin{bmatrix} x & \underline{x} \\ \underline{x} & x \end{bmatrix}$$

special construction (L)

$$L_N = \left[\begin{array}{ll} 1 & 1^{1\times(N-1)} \\ & \\ 1^{(N-1)\times 1} & \mathrm{core}(L_N) \end{array} \right] \in \mathbb{C}^{N\times N}$$

$$\mathrm{core}(L_N) = \mathrm{circ}\left(\left[\begin{array}{cc} c_0 & c_0^* \\ c_0^* & c_0 \end{array} \right], \left[\begin{array}{cc} c_1 & c_1^* \\ c_1^* & c_1 \end{array} \right], \ldots, \left[\begin{array}{cc} c_{n-1} & c_{n-1}^* \\ c_{n-1}^* & c_{n-1} \end{array} \right] \right) \;:\; c_j \in \mathbb{C}$$

special construction (L)

$$
\begin{cases}
 & \sum_{j=0}^{2k} \cos \alpha_j = -\frac{1}{2} \\
 & \sum_{j=0}^{2k} \cos (2\alpha_j) = -\frac{1}{2} \\
k \text{ equations with } 0 \leqslant i < k & : \quad \sum_{j=0}^{2k} \cos \left(\alpha_j + \alpha_{(j+1+i) \bmod (2k+1)} \right) = -\frac{1}{2} \\
k \text{ equations with } 0 \leqslant i < k & : \quad \sum_{j=0}^{2k} \cos \left(\alpha_j - \alpha_{(j+1+i) \bmod (2k+1)} \right) = -\frac{1}{2}
\end{cases}
$$

$\alpha_j \in [0, 2\pi)$

$2(k+1) = (N+1)/2$ trigonometric equations with $2k+1 = (N-1)/2$ real variables

special construction (L)

for $N = 3 + 4k$

$k = 0 \Longrightarrow L \simeq F_3^{(0)}$

$k = 1 \Longrightarrow L \simeq F_7^{(0)}$

$k = 2 \Longrightarrow L_{11}$ analytic result (unpublishable)

$((1/153964389743738930357557015269047805443 8 \times (92179560806032202054774413769812973 3 \times \ldots$

$k = 3 \Longrightarrow L_{15}$ analytic result (Markus Grassl and Magma)

$$\left.\begin{array}{l} k = 4 \\ \vdots \\ k = 31 \end{array}\right\} \Longrightarrow \text{numerical confirmation that isolated solution exists}$$

all isolated

> conjecture:
> $\forall_{k \in \{0,1,2,\ldots\}} \exists_{L \in \mathbb{H}(3+4k)} : L = L^{(0)} \notin \mathrm{B}\mathbb{H}(3 + 4k, q)$ for any $q \geqslant 2$

special construction (V)

$$V_N = \text{circ}\left[c_0, c_1, c_2, \ldots, c_{N-1}\right] \in \mathbb{C}^{N\times N} \quad : \quad c_j \in \mathbb{C}$$

this undephased form provides unitary constraints:

$$\sum_{j=0}^{N-1} \frac{c_j}{c_{(j+k) \bmod N}} \overset{\star}{=} 0 \qquad \text{for} \qquad k \in \{1, 2, \ldots, N-1\}$$

cyclic-N-roots...

> conjecture:
> $\forall_{N\geqslant 6}$ the solution of $\star$ contains at least one isolated CHM matrix $V^{(0)}$, which is not of Butson type

confirmed numerically for $6 \leqslant N \leqslant 64$

algebra of monic palindromic polynomials

$$\begin{cases} \mathbf{p}_Y = (1, -3, 9, -16, -12, 6, \underline{3}, 6, -12, -16, 9, -3, 1) \\ \mathbf{p}_Y = (1, 6, 15, 26, 3, -24, \underline{-27}, -24, 3, 26, 15, 6, 1) \end{cases}$$

$$\mathbf{p}_{V_8} = (1, 16, 64, 16, -332, -1040, -1984, -2832, \underline{-3194}, \ldots)$$

conclusion

- (probably) infinite constructive family $L^{(0)}_{3+4k}$ of isolated CHM
- the same for $V^{(0)}_N$
- the same for many other configurations...

future directions

? real and practical application(s)

? analytic methods

? more tools

? classification of isolated cases

? are there any isolated $\mathrm{B}\mathbb{H}(8, q)$?

? mystery of $N = 11$

are there other “reclusive” dimensions like this?

thank you!