Signed Groups & their applications to Hadamardish stuff

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Hadamard matrices

Some objects of interest

(in arrays "$-$" represents "-1")

$$A = \begin{pmatrix} 1 & - \\ 1 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & - & - & - \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \\ 1 & - & 1 & 1 \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & - & - & - \\ \hline 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \\ 1 & - & 1 & 1 \end{array}\right);$$

$$C = \left(\begin{array}{cc|cc} 1 & - & - & 1 \\ 1 & 1 & - & - \\ \hline 1 & - & 1 & - \\ 1 & 1 & 1 & 1 \end{array}\right); \qquad D = \begin{pmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{pmatrix};$$

$$\begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix}; \qquad \begin{pmatrix} 1 & \lambda & -1 & \lambda \\ \lambda & 1 & \lambda & -1 \\ -1 & \lambda & 1 & \lambda \\ \lambda & -1 & \lambda & 1 \end{pmatrix}$$

Each in whole or in part arises via a development function.

Structure: circulant, group development, and ...?

Circulant matrices are "developed over" the cyclic group C_n.

We also find analogous development over other groups such as $EA(4)$: $\begin{pmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{pmatrix}$. There is also mysterious "other" group structure like the quaternion (Williamson) array shown, not strictly "group developed", yet there is a group—the group of quaternions Q of order 8—whose structure it reflects.

In the original Williamson Construction we insert symmetric circulant matrices A, B, C, D, yielding matrices "developed" over groups $C_n \times Q$—but of half the size of the group ($8n$).

Group development won't solve the general case

While circulants appear in many constructions for Hadamard matrices the only known examples of circulant $H(n)$ are for $n = 1, 4$. Ryser (circa 1960) conjectured there are no others.

Circulant and group-developed matrices suffer one fundamental weakness: They have constant row/column sum.

Suppose $HH^\top = nI$ where $H \in \mathbb{R}^{n \times n}$ has constant row/column sum r. That is, $H\mathbf{e} = H^\top\mathbf{e} = r\mathbf{e}$ where $\mathbf{e}$ is a column of ones.

So $HH^\top\mathbf{e} = H(r\mathbf{e}) = rH\mathbf{e} = r^2\mathbf{e} = (nI)\mathbf{e} = n\mathbf{e}$.

Thus, $n = r^2$. Hadamard matrices may be group developed <u>only</u> if the order is a perfect square.

But ... what <u>is</u> this "other way" of developing over groups?

Signed groups and cocycles

About the same time, somewhat prior to 1990

Warwick de Launey: “useful” assumptions about indexing functions for symbolic arrays used for orthogonal matrices.

Finds algebraic conditions he later learns [via K. Horadam] correspond to the cocycle equation known in cohomology

R. Craigen: “backwards compiles” existing types of arrays to infer generalizations of group development.

Finds a simple group variant with an additional assumption about “sign” that appears to capture what is needed.

Both concepts have “mathematical legs”, not exhausted today.

Same essential mathematical structure—though initially developed independently and subsequent work not harmonized

Very different results perhaps because two completely different formulations lead to different insights.

Some comments/opinions on Hadamardish notation

- Retain historical "$H(n)$" $\leftrightarrow$ "**Hadamard matrix of order** n" ($H \in \{\pm 1\}^n$ s.t. $HH^\top = nI$).
- Krakow group is not wrong use "Hadamard matrix" $\leftrightarrow H \in U^{n\times n}$ s.t. $HH^* = nI$, where $U = \{\mu \in \mathbb{C} : |\mu| = 1\}$—exactly Hadamard's subject in 1893—writing $\mathcal{H}(n)$ for the set of these.
- Personal (minority) preference "$UH(n)$" $\leftrightarrow$ "**complex Hadamard matrix of order** n"—that is, a member of $\mathcal{H}(n)$.
- Once upon a time $CH(n)$, "complex Hadamard matrix of order n" referred to $H \in \{\pm 1, \pm i\}^{n\times n}$ satisfying $HH^* = nI$ but this is conflicting/anachronistic and should be consigned to history.
- For finite group G, $GH(n, G) \leftrightarrow$ "$H \in G^{n\times n}$ satisfying $HH^* = nI$ with scalar arithmetic over the group ring $\mathbb{Z}[G]$, reduced $(\text{mod } \Sigma G)$(ΣG is the sum of the elements of G)".
- If all entries of a $UH(n)$ are kth roots of unity, we say that it is a **Butson Hadamard matrix** over the kth roots of unity, denoted $BH(n, k)$ (de Launey argued for $GH(n, k)$). k is the **phase**. So retired "$CH(n)$" is now **four-phase** $UH(n)$, $GH(n, 4)$ or $BH(n, 4)$.

Objectives

- What is a signed group?
- Signed group rings
- Signed group development of matrices
- Signed groups and cocyles
- Signed group matrices
- How are signed groups used?
- What have they accomplished?

Defining signed groups

A **signed group** S is a multiplicative group G having a distinguished central involution, denoted "-1"$= -1_S$.

For $a \in S$ write $-a = (-1)a$. Therefore $(-a)(-b) = ab$ (& etc)

EG: The quaternion group Q of order 8 is often written this way

$$S_Q = \langle x, y \mid x^2 = -1, y^2 = -1, yx = -xy \rangle = \{\pm 1, \pm x, \pm y, \pm xy\}$$

or, more popularly, $\{\pm 1, \pm i, \pm j, \pm k\}$

We say that G is the **underlying group** of signed group S.

Why it is useful to distinguish between S and G?

Consider what it means for two signed groups to be "the same"

When are two signed groups essentially the same?

Let S, T be signed groups.

Map $\varphi : S \to T$ is a **(signed group) homomorphism** if

1. φ is a group homomorphism; and
2. $\varphi(-1) = -1$. (i.e., $\phi(-1_S) = -1_T$)

Homomorphism φ is an **isomorphism** if it is a bijection, and S, T are **isomorphic** signed groups.

It is possible for two non-isomorphic signed groups to have the same (isomorphic) underlying group(s)!

EG: Let $G = \langle a, b \mid a^2 = b^4 = 1 \rangle = \langle a \rangle \times \langle b \rangle$ $(= C_2 \times C_4)$

Taking $a = -1$ gives signed group
$S_1 = \langle b \mid b^4 = 1 \rangle = \{\pm 1, \pm b, \pm b^2, \pm b^3\}$

Taking $b^2 = -1$ gives
$S_2 = \langle a, b \mid a^2 = 1, b^2 = -1, ab = ba \rangle = \{\pm 1, \pm a, \pm b, \pm ab\}$

Order and the projection group

The **projection group** of signed group S with underlying group G is $P = G/\langle -1 \rangle$.

If $|G| = 2n$ then $|P| = n$.

The **order** of a signed group S is the order of its projection group P (and therefore half its own cardinality)

EG: Underlying group $G = \langle a, b \mid a^2 = b^4 = 1 \rangle = C_2 \times C_4$ of order 8 supports signed groups
$S_1 = \langle b \mid b^4 = 1 \rangle$ and $S_2 = \langle a, b \mid a^2 = 1, b^2 = -1, ab = ba \rangle$
of order 4 with respective projection groups
$P_1 = C_4$ and $P_2 = C_2 \times C_2$

The group structure of G

G is a central extension of $C_2 = \langle -1 \rangle$ by P.

S "embodies" the (central) short exact sequence

$$1 \to C_2 \xrightarrow{\imath} G \xrightarrow{\pi} P \to 1$$

Two apparently different perspectives on the above structure:

Cocycles The object of interest is the projection group P and the action (via a cocycle) of C_2 (or another group in its place) on P; G is an artifact of this interaction.

SGs The fundemental object of interest is S, which is isomorphic to G as a group; the injection map $C_2 \to G$ distinguishes the central involution -1; P is an artifact.

The relationship to cocycles

If $f : P \times P \to \langle C \rangle$ is a (normalized) cocycle (where $C = \langle -1 \rangle$) then we can form a group on the set $G = P \times C$ with operation

$$(x, a)(y, b) := (xy, f(x, y)ab)$$

It may be verified that G is indeed a group and $(1_P, -1)$ is a central involution. Casting G as a signed group S, P is the projection group and the resulting sequence

$$1 \to C \to G \to P \to 1$$

is short, exact and central.

Signed group S embodies this machinery in a simple algebraic vehicle which may be driven without “looking under the hood”.

A question from Dane Flannery

(How) can signed groups be generalized to take care of distinguished central subgroups bigger than $\langle -1 \rangle$?

- It is evident from the start that a higher order central element may be used in place of -1, generalizing C from $\langle -1 \rangle$ to an arbitrary cyclic group.
- This yields development functions for matrices whose entries are roots of unity, such as $BH(n, k)$s.
- Similarly one may do the same when a central abelian group N takes the place of C. Development functions produce $(0, N)$-matrices.
- A nonabelian normal subgroup could stand in for C but I think at this point the relation to cocycles might need to be abandoned. Perhaps this will be useful for addressing GH matrices over nonabelian groups?

Menagerie of signed groups of order $n < 4$

$\underline{n = 1}$ unique

Trivial signed group

$$S_0 = \langle -1 \rangle = \{1, -1\}.$$

Underlying and projection groups: $G \cong C_2$; $P \cong \{1\}$

$\underline{n = 2}$ two cases

Cyclic signed group:

$$CS_2 = \langle x \mid x^2 = 1 \rangle = \{\pm 1, \pm x\}; G \cong C_2 \times C_2; P \cong C_2$$

Negacyclic signed group:

$$NS_2 = \langle x \mid x^2 = -1 \rangle = \{\pm 1, \pm x\}; G \cong C_4; P \cong C_2$$

$\underline{n = 3}$ unique

Cyclic signed group:

$$CS_3 = \langle x \mid x^3 = 1 \rangle = \{\pm 1, \pm x, \pm x^2\}; G \equiv C_2 \times C_3; P \cong C_3$$

($\cong$ **negacyclic** signed group of order 3, $NC_3 = \langle -x \rangle$)

The six signed groups of order 4

Cyclic and negacyclic:

$$CS_4 = \langle x \mid x^4 = 1 \rangle = \{\pm 1, \pm x, \pm x^2, \pm x^3\}$$

$$NS_4 = \langle x \mid x^4 = -1 \rangle = \{\pm 1, \pm x, \pm x^2, \pm x^3\}$$

Elementary abelian signed group of order 4:

$$EAS_4 = \langle x, y : x^2 = y^2 = 1, yx = xy \rangle = \{\pm 1, \pm x, \pm y, \pm xy\}$$

$(G \cong EA_8;\ P \cong EA_4)$

Secondary abelian signed group of order 4:

$$SAS_4 = \langle x, y \mid x^2 = y^2 = -1, yx = xy \rangle = \{\pm 1, \pm x, \pm y, \pm xy\}$$

Dihedral signed group of order 4:

$$DS_4 = \langle x, y \mid x^2 = y^2 = 1, yx = -xy \rangle = \{\pm 1, \pm x, \pm y, \pm xy\}$$

Quaternion signed group of order 4:

$$S_Q = \langle x, y \mid x^2 = -1, y^2 = -1, yx = -xy \rangle = \{\pm 1, \pm x, \pm y, \pm xy\}$$

The signed group ring $R[S]$

$\mathcal{P} = \{x_1, \ldots, x_n\}$: a transversal for $\langle -1 \rangle$ in S.

R: any ring with unity and characteristic $\neq 2$.

The **signed group ring** of S relative to R:

$$R[S] := \{a_1 x_1 + \cdots + a_n x_n \mid a_1, \ldots, a_n \in R\}$$

- addition formally extends addition in R to these expressions
- multiplication formally extends products of both R and S
- -1_R is identified with -1_S (so, also, are 1_R and 1_S) and written $-1 = -1_R = -1_S = (-1_S) \cdot 1_R = 1_S \cdot (-1_R)$
- If n is the order of S then $R[S]$ has dimension n (with basis $\mathcal{P}$) whereas the group ring $R[G]$ has dimension $2n$ (and basis $\{\pm x_1, \ldots, \pm x_n\}$

EGs: $\mathbb{Z}[NS_2] = \{a + bx \mid a, b \in \mathbb{Z}; x^2 = -1\}$ = Gaussian integers

$\mathbb{R}[S_Q] = \{a + bx + cy + dxy \mid x^2 = y^2 = -1\}$ = Quaternions

Monomial $\{0, \pm 1\}$ representations of signed groups

A **degree** m **representation** of signed group S is a homomorphism $\pi : S \to \mathbb{C}^{m \times m}$ such that $\phi(-1_S) = -I_m$.

We especially desire real monomial representations (**remreps**)

$$\pi : S \to \text{signed permutation matrices},$$

to generalize (ordinary) group development of matrices.

EGs: For $(S_{\mathbb{C}} =) NS_2 = \langle x \mid x^2 = -1 \rangle$, take $\pi(x) = \begin{pmatrix} 0 & - \\ 1 & 0 \end{pmatrix}$

For $S_Q = \langle x, y \mid x^2 = -1, y^2 = -1, yx = -xy \rangle$ take

$$\pi(x) = \begin{pmatrix} 0 & - & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \pi(y) = \begin{pmatrix} 0 & 0 & - & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{pmatrix}$$

(Both representations should be familiar!)

The (right-)regular representation

Analogous to the regular representation of groups. For signed group S with $\mathcal{P} = \{x_1, \ldots, x_n\}$ (with $x_1 = 1$) proceed as follows:

Let D be the $\mathcal{P} \times \mathcal{P}$ division table

$$\left(\begin{array}{c|ccc} \div & x_1 & \cdots & x_n \\ \hline x_1 & x_1 x_1^{-1} & \cdots & x_1 x_n^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n x_1^{-1} & \cdots & x_n x_n^{-1} \end{array}\right). \text{ i.e., } D = \begin{pmatrix} x_1 x_1^{-1} & \cdots & x_1 x_n^{-1} \\ \vdots & \ddots & \vdots \\ x_n x_1^{-1} & \cdots & x_n x_n^{-1} \end{pmatrix}$$

In each row/column $\pm x_i$ appears (once!) for each i.
So there exist signed permutation matrices $P_1, \ldots, P_n$ such that

$$D = x_1 P_1 + \cdots + x_n P_n$$

(With matrix arithmetic over the ring of scalars $\mathbb{Z}[S]$).

Claim: $\pi : S \to \{0, \pm 1\}^{n \times n}$ where $\pi(x_i) = P_i$ is a (signed group) homomorphism (injection).

Proof of claim

We have $D = x_1 P_1 + \cdots + x_n P_n$, $\pi(x_i) = P_i$ and $D = VV^*$

where $V = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $V^* = \begin{pmatrix} {x_1}^{-1} & {x_2}^{-1} & \cdots & {x_n}^{-1} \end{pmatrix}$. Now,

$$D^2 = (x_1 P_1 + \cdots + x_n P_n)^2 = \sum_{i,j=1}^{n} x_i x_j P_i P_j = \sum_k x_k \sum_{x_i x_j = e_{ij} x_k} e_{ij} P_i P_j$$

$$= (VV^*)^2 = V(V^* V)V^* = V(x_1^{-1} x_1 + \cdots + x_n^{-1} x_n)V^* = nVV^*$$

$$= n \sum_k x_k P_k = \sum_k (n x_k) P_k$$

By direct counting, $x_i x_j = e_{ij} x_k \Rightarrow P_i P_j = e_{ij} P_k \qquad (e_{ij} \in \{\pm 1\})$.

That is, $\pi(x_i)\pi(x_j) = \pi(x_i x_j)$, and the result follows. □

The standard involution

and signed group matrix conjugates

On $\mathbb{Z}[S]$ define the **(standard) involution** $(\sum a_i x_i)^* := \sum a_i x_i^{-1}$

EG: the involution on $\mathbb{Z}[S_\mathbb{C}]$ corresponds to complex conjugation of Gaussian integers.

EG: the involution on $\mathbb{R}[S_Q]$ corresponds to the usual conjugation of quaternions

For matrices $M = [m_{ij}] \in \mathbb{Z}[S]^{h \times k}$ define the **conjugate**

$$M^* := [m_{ji}^*] \in \mathbb{Z}[S]^{k \times h}$$

EG: For $S_\mathbb{C}$-or-S_Q-matrices conjugation corresponds to, respectively, the Hermitian adjoint and the corresponding matrix operation for quaternion matrices.

Signed group development of matrices

Linearly extending the regular representation
$\pi(R[S]) \subset R^{n\times n}$—a matrix ring isomorphic to $\pi(R[S])$.

Matrices in $\pi(R[S])$ are **developed over** signed group S.

For real monomial matrix P, $P^\top = P^{-1}$ (easy).

It follows that, for $\alpha \in R[S]$, $\pi(\alpha^*) = \pi(\alpha)^\top$.

If $A = \pi(\alpha)$ then $AA^\top = nI \Leftrightarrow \alpha\alpha^* = n$. (**Gram products**)

(±1)-matrices developed over a signed group are <u>not</u> subject to the order constraints of the group-developed case.

EG: Williamson's 50-year old construction gives Hadamard matrices of all orders $4q$, $q \leq 33$, which are seen to be developed over $S_Q \times C_q$.

Forms of 4×4 matrices developed over signed groups

In general, the set of matrices $M \in R^{n \times n}$ developed over a signed group of order n is an n-dimensional subspace. Using A, B, C, D for the coefficients of $x_1, \ldots, x_4$ in our construction:

$S{=}CS_4$:
$$\begin{pmatrix} A & B & C & D \\ D & A & B & C \\ C & D & A & B \\ B & C & D & A \end{pmatrix}$$

$S{=}NS_4$:
$$\begin{pmatrix} A & -B & -C & -D \\ D & A & -B & -C \\ C & D & A & -B \\ B & C & D & A \end{pmatrix}$$

$S{=}EAS_4$:
$$\begin{pmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{pmatrix}$$

$S{=}SAS_4$:
$$\begin{pmatrix} A & -B & -C & D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix}$$

$S{=}SD_4$
$$\begin{pmatrix} A & B & C & -D \\ B & A & D & -C \\ C & -D & A & B \\ D & -C & B & A \end{pmatrix}$$

$S{=}S_Q$
$$\begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & B \\ D & -C & B & A \end{pmatrix}$$

Cocyclic development of matrices

A map $f : G \times G \to C$ is a **cocycle** (to C) if for all $x, y, z \in G$,

$$f(x, y)f(xy, z) = f(y, z)f(x, yz)$$

(The **cocycle equation**)

de Launey/Flannery/Horadam (etc):
Index the rows of $M = [m_{x,y}]_{x,y \in G} \in R^{|G| \times |G|}$ by the elements of G. We say that M is **cocyclic** if there exists a function $g : G \to R$ and a cocycle f such that, for all $x, y \in G$,

$$m_{x,y} = f(x, y)g(xy)$$

(The value of $m_{x,y}$ is $g(xy)$ (depends only on xy) times the value of a cocycle.) For the trivial cocycle $f : G \times G \to 1$ this reduces to **group development** over group G.

Cocycles and signed group development

Suppose S is a signed group with projection group $P = \{x_1, \ldots, x_n\}$. Write the elements of S as $\pm x_1, \ldots, \pm x_n$ in the obvious way.

If, in P, $x_i x_j = x_k$ then, in S, $x_i x_j = e_{ij} x_k$ where $e_{ij} \in \{\pm 1\}$.

Define $f : P \times P \to C_2 = \langle -1 \rangle$ by $f(x_i, x_j) = e_{ij}$.

The cocycle equation gives the condition necessary for the above construction to give associative operation on S. Therefore, f is a cocycle. Conversely, any cocycle in the above construction makes S a signed group!

If $M \in R^{n \times n}$ is developed over S, $M = [m_{x_i, x_j}] = \sum a_k P_k$.

Thus, $m_{x_i, x_j} = f(x_i, x_j^{-1}) g(x_i x_j^{-1})$ where $g(x_i x_j^{-1}) = g(x_k) = a_k$.

So signed group = cocyclic development (+ permuting columns)

Signed group matrices

Besides developing matrices over signed groups, we consider matrices whose entries come from a signed group S—and perform algebra over $\mathbb{Z}[S]$.

Suppose $H \in S^{n \times n}$ and that $HH^* = nI$. We say that H is a **signed group Hadamard matrix** over S, and write $SH(n, S)$.

EG: Let $S = NS_2 =< x \mid x^2 = -1 >$. Take $H = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$. Then

$$HH^* = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & -x \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I$$

So $H = SH(2, NS_2)$.

$SH(n, S)$ can give larger Hadamard matrices

To obtain a Hadamard matrix from H: (a) Apply a remrep of some order m to every entry; (b) multiply every resulting block by a Hadamard matrix of order m.
Continuing the previous example:

$$H = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} \to \left(\begin{array}{cc|cc} 1 & 0 & 0 & - \\ 0 & 1 & 1 & 0 \\ \hline 0 & - & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \cdot \left(\begin{array}{cc|cc} 1 & - & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & - \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & - & - & - \\ 1 & 1 & 1 & - \\ \hline - & - & 1 & - \\ 1 & - & 1 & 1 \end{array}\right) \text{—a } H(4).$$

Circulant signed group Hadamard matrices always exist!

Theorem (Craigen, 1994): For any odd positive integer q, there exists a circulant $SH(2q, S)$ (for some signed group $S = S(q)$).
In EVERY even order?? CIRCULANT? But ...Ryser ... ??

Devil in details: S depends on q; so, then, does the order of a resulting Hadamard matrix. And the final structure will definitely not be circulant.

S arises from a witch's brew of ingredients needed to form the matrix, "assembled" by picking a signed group to multiply the parts by so they "behave right".

S can be chose to have a remrep of degree a power of 2. Consequently we obtain an asymptotic result...

Asymptotic existence of Hadamard matrices

Theorem (Craigen, 1994): For any odd $q \in \mathbb{Z}^+$, there exists a Hadamard matrix of order $2^t q$, where $t \leq \frac{2}{5} \log_2 q$.

... improving the first result of this type by Jennie Seberry, $\approx$ 1995 where $t \approx 2 \log_2 q$.

The bound has been further improved (Livinskyi, 2011) using the signed group approach.

At least 3 subsequent authors have produced extensions of the method giving even better asymptotics or other asymptotic results (EG for orthogonal designs).

Further applications of signed group matrices

- Making Hadamard matrices from large weight Weighing matrices
- Maps between classes of generalized Hadamard matrices
- Making possible sequence constructions for Hadamard matrices that involve complex sequences.
- Expanding the possibilities for various tensor constructions for orthogonal matrices.
- 2×2 block conjecture for Hadamard matrices:
  Every $H(2n)$ can be partitioned
  into rank two 2×2 submatrices.
  Such a partition produces a $SH(n, DS_4)$ (**Dihedral Hadamard matrix**). It's not true that every $H(2n)$ can be obtained by "inflating" a $BH(n, 4)$. But it may be true that every $H(2n)$ can be obtained by "inflating" a $SH(n, DS_4)$.
- Etc.

Some references

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