Signed Groups & their applications to Hadamardish stuff

HADAMARD 2025
Sevilla Spain

R. Craigen

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Each in whole or in part arises via a development function.

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But ... what <u>is</u> this "other way" of developing over groups?

Signed groups and cocycles

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- If all entries of a $UH(n)$ are kth roots of unity, we say that it is a **Butson Hadamard matrix** over the kth roots of unity

Some comments/opinions on Hadamardish notation

- Retain historical "$H(n)$" $\leftrightarrow$ "**Hadamard matrix of order** n" ($H \in \{\pm 1\}^n$ s.t. $HH^\top = nI$).
- Krakow group is not wrong use "Hadamard matrix" $\leftrightarrow H \in U^{n\times n}$ s.t. $HH^* = nI$, where $U = \{\mu \in \mathbb{C} : |\mu| = 1\}$—exactly Hadamard's subject in 1893—writing $\mathcal{H}(n)$ for the set of these.
- Personal (minority) preference "$UH(n)$" $\leftrightarrow$ "**complex Hadamard matrix of order** n"—that is, a member of $\mathcal{H}(n)$.
- Once upon a time $CH(n)$, "complex Hadamard matrix of order n" referred to $H \in \{\pm 1, \pm i\}^{n\times n}$ satisfying $HH^* = nI$ but this is conflicting/anachronistic and should be consigned to history.
- For finite group G, $GH(n, G) \leftrightarrow$ "$H \in G^{n\times n}$ satisfying $HH^* = nI$ with scalar arithmetic over the group ring $\mathbb{Z}[G]$, reduced $(\text{mod } \Sigma G)$(ΣG is the sum of the elements of G)".
- If all entries of a $UH(n)$ are kth roots of unity, we say that it is a **Butson Hadamard matrix** over the kth roots of unity, denoted $BH(n, k)$ (de Launey argued for $GH(n, k)$). k is the **phase**. So retired "$CH(n)$" is now **four-phase** $UH(n)$, $GH(n, 4)$ or $BH(n, 4)$.

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Consider what it means for two signed groups to be "the same"

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Signed group S embodies this machinery in a simple algebraic vehicle which may be driven without "looking under the hood".

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A question from Dane Flannery

(How) can signed groups be generalized to take care of distinguished central subgroups bigger than $\langle -1 \rangle$?

- It is evident from the start that a higher order central element may be used in place of -1, generalizing C from $\langle -1 \rangle$ to an arbitrary cyclic group.
- This yields development functions for matrices whose entries are roots of unity, such as $BH(n,k)$s.
- Similarly one may do the same when a central abelian group N takes the place of C. Development functions produce $(0, N)$-matrices.
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- Similarly one may do the same when a central abelian group N takes the place of C. Development functions produce $(0, N)$-matrices.
- A nonabelian normal subgroup could stand in for C but I think at this point the relation to cocycles might need to be abandoned. Perhaps this will be useful for addressing GH matrices over nonabelian groups?

Menagerie of signed groups of order $n < 4$

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$\underline{n = 1}$ $\quad$ unique

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($\cong$ **negacyclic** signed group of order 3, $NC_3 = \langle -x \rangle$)

The six signed groups of order 4

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to generalize (ordinary) group development of matrices.

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(Both representations should be familiar!)

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EG: For $S_\mathbb{C}$-or-S_Q-matrices conjugation corresponds to, respectively, the Hermitian adjoint and the corresponding matrix operation for quaternion matrices.

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EG: Williamson's 50-year old construction gives Hadamard matrices of all orders $4q$, $q \leq 33$, which are seen to be developed over $S_Q \times C_q$.

Forms of 4×4 matrices developed over signed groups

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So signed group = cocyclic development (+ permuting columns)

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To obtain a Hadamard matrix from H: (a) Apply a remrep of some order m to every entry; (b) multiply every resulting block by a Hadamard matrix of order m.
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$$H = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} \to \left(\begin{array}{cc|cc} 1 & 0 & 0 & - \\ 0 & 1 & 1 & 0 \\ \hline 0 & - & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \cdot \left(\begin{array}{cc|cc} 1 & - & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & - \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & - & - & - \\ 1 & 1 & 1 & - \\ \hline - & - & 1 & - \\ 1 & - & 1 & 1 \end{array}\right) \text{—a } H(4).$$

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At least 3 subsequent authors have produced extensions of the method giving even better asymptotics or other asymptotic results (EG for orthogonal designs).

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- Etc.

Some references

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