Signed Groups I: Basic theory, application to orthogonal matrices

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Abstract

We provide a complete, self-contained, introductory level exposition of the theory of signed groups as pertains to their use in developing, generalizing and analyzing orthogonal matrices (Hadamard matrices, weighing matrices and orthogonal designs) in combinatorics.

At this point I have no thought of trying to publish–it is a working document intended to help people out, and is largely a stream-of-thought explanation of how I have been using signed groups over the years. Many people have "caught on" or worked from the sketchy developments that have appeared in the literature, but this is intended to provide a "canonical" explanation, and it remains under development (note missing internal links, citations, etc).

Please give feedback. What was helpful/unhelpful? Where have I overwritten this thing and could make it less pedantic, more leaner, without detracting from its purpose? There are sure to be numerous goofs, given that I typed this more-or-less in one go, so please point these out.

Parts II and III are intended so outline specific instances of how SG development or SG matrices have/can be used (sort of a "history" of signed groups). Another installment (which almost certainly will be for publication) is intended to map out the relationship to cocyclic development and Ito's Hadamard groups, redoing some of the basic theory using this framework instead of the others. Perhaps sooner, perhaps later, I intend to lay out the "canonical" generalization of this framework to the analogous situation in which $\langle -1 \rangle$ is replaced by a subgroup that is central or, even more generally, normal.

My warm thanks to Dane Flannery for his wisdom, friendship, generosity and feedback, which has significantly improved this presentation, though of course I retain responsibility for any and all errors.

1 Introduction

This largely expository paper is written in propitiation for its author having committed the great sin of introducing, in fairly inaccessible places, certain specialized terms and conventions, and subsequently showing blithe disregard for readers who may be unfamiliar with these sources and may not have access thereto. This is a common and grievous sin in mathematical research writing, concerning which I have on occasion thrown a stone or two. It is therefore necessary that at some point I pack up and move out of the very glass house whose walls I wish to assault.

"Signed groups" are not so much novel objects as a different way of articulating familiar objects in order to focus on information we shall find useful. They first appeared in this way in my thesis [?] and later, with varying degrees of "proper development", in some other places, where they are used as a valuable tool in (for example) establishing asymptotic existence results for Hadamard matrices [**?**], [**?**].

We aim herein to make available a more-or-less complete elementary development of this subject, suitable for novices to gain full access to signed groups and their uses, and for those interested in the practical value of this simple device to become familiar with the breadth of possibilities.

Signed groups are a dandy tool for constructing formerly difficult classes of Hadamard matrices, weighing matrices and orthogonal designs. Further, they themselves are inhabitants of scalar rings, over which matrices may be defined that generalize these well-known objects in very helpful (as I shall demonstrate) ways. They are also an analytic tool by which numerous analytic results may be derived.

In addition to being a source of useful kinds of scalar objects, signed groups provide a helpful and versatile way of developing matrices that naturally generalizes group development. Remarkably, at the very time this system was being developed, halfway around the world Warwick de Launey and Kathy Horadam were laying the groundwork for cocyclic development of designs, which (as it turns out) amounts to much the same thing, but seen from a very different perspective. Eventually (but not in this offering) we shall lay out the connections, and the distinctions, between these two approaches to development of matrices. In the next article, however, I plan to demonstrate that, as useful as signed groups may be for developing matrices, that is by no means the only way in which this theory helps in the study of orthogonal matrices.

2 Signed group basics

A **signed group**, S, of order n, is a group of order $2n$, called the **underlying group**, having a distinguished central element of order 2, which we denote by -1. We write

$$-x := (-1)x = x(-1),$$

observing the consequential rules

$$(-x)y = x(-y) = -xy, \quad -(-x) = x$$

and

$$(-x)(-y) = xy.$$

For study, we generally fix a complete ordered set of representatives, $\mathcal{P} = \{x_1, \ldots, x_s\}$, of cosets of the subgroup $\langle -1 \rangle$, of order 2 (i.e., a transversal)[1].

A **homomorphism** of signed groups S_1 and S_2 is a homomorphism $\pi : S_1 \to S_2$ of the underlying groups *which also preserves* -1. π is a **isomorphism** if it is also one-to-one and onto.

Since $\langle -1 \rangle \lhd S$, it is the kernel of a uniquely determined isomorphism $\pi : S \to P \cong S/\langle -1 \rangle$. We shall call P the **projection group** of S. Noting the isomorphism $C_2 \overset{\imath}{\cong} \langle -1 \rangle$, the cyclic group of order 2, we thus have a short exact sequence[2] (of groups)

$$1 \to C_2 \xrightarrow{\imath} G \xrightarrow{\pi} P \to 1,$$

[1]Mnemonically one might conceive $\mathcal{P}$ as the set of "positive" elements of S, the remainder being the corresponding "negative" elements, observing $x \in \mathcal{P} \Leftrightarrow -x \notin \mathcal{P}$, and usually we take $x_1 = 1$, but in general, the choices of $\mathcal{P}$ and its order are arbitrary. It is important to understand that $\mathcal{P}$ is not an intrinsic part of the structure of S, only an imposed aid to thought much like a partition of a matrix, which is not part of the matrix but expresses a manner in which we are viewing the matrix)

[2]Flannery points out that the converse is also true if we specify that the short exact sequence is central.

i.e., the underlying group G of S is a central extension of $C_2 = \langle x \mid x^2 = 1 \rangle$ by P (with the image of the generator of C_2 under $\imath$ denoted "-1"). There is no constraint on which groups may serve as P. Indeed, if P is any group whatsoever, then $S = C_2 \times P$ is a signed group with projection group P and $-1 = (x, 1_P)$.

Finally, we shall often use standard group presentations

$$\langle \text{ generators } \mid \text{ relations } \rangle$$

to specify signed groups, with all the usual conventions except that it is understood that $-1 \neq 1$ must be an element of the defined object, and of course that it is central and has order 2, whether or not these things are explicitly indicated.

2.1 Examples: a small menagerie of signed groups

For convenience we shall name[3] small and/or standard classes of signed groups after groups or other objects in the general lexicon to which they have some natural relationship. We begin by completely cataloguing the first few by isomorphism class, according to their order.

2.1.1 $n = 1$

The **trivial signed group** is the unique signed group of order 1, which has two elements,

$$S_0 = \langle -1 \rangle = \{1, -1\}.$$

It is isomorphic, as a group, to C_2, and its projection group is the trivial group.

2.1.2 $n = 2$

There are two distinct signed groups of order 2: the **cyclic signed group of order 2**

$$CS_2 = \langle x \mid x^2 = 1 \rangle = \{\pm 1, \pm x\},$$

which is isomorphic (as a group) to the Klein 4-group, $C_2 \times C_2$; and the **negacyclic signed group**

$$NS_2 = \langle x \mid x^2 = -1 \rangle = \{\pm 1, \pm x\},$$

which is isomorphic (as a group) to C_4. We shall also use the notation $S_{\mathbb{C}}$ for NS_2, for reasons which should be obvious shortly.

2.1.3 $n = 3$

There is a unique signed group of order 3: the **cyclic signed group**

$$CS_3 = \langle x \mid x^3 = 1 \rangle = \{\pm 1, \pm x, \pm x^2\},$$

which could also be presented $CS_3 = \langle y \mid y^3 = -1 \rangle$ (here we may take $y = -x$). As a group it is isomorphic to $C_2 \times C_3 \cong C_6$.

[3]NOTE to self: we are diverging here from the original notations by reversing some of the acronyms. Is this a good idea? Also S_Q looks a lot like $S_{\mathbb{Q}}$ here and may confuse with the dicyclic group because of its standard use of Q. Reconsider, perhaps?

2.1.4 $n = 4$

In addition to the cyclic,

$$CS_4 = \langle x \mid x^4 = 1 \rangle = \{\pm 1, \pm x, \pm x^2, \pm x^3\},$$

and negacyclic,

$$NS_4 = \langle x \mid x^4 = -1 \rangle = \{\pm 1, \pm x, \pm x^2, \pm x^3\}$$

signed groups there are four further signed groups of order 4:

The **elementary abelian signed group of order 4**,

$$EAS_4 = \langle x, y \mid x^2 = y^2 = 1, yx = xy \rangle = \{\pm 1, \pm x, \pm y, \pm xy\},$$

which is isomorphic (as a group) to EA_8, but its projection group is EA_4. The **secondary abelian signed group of order 4** is

$$SAS_4 = \langle x, y \mid x^2 = y^2 = -1, yx = xy \rangle = \{\pm 1, \pm x, \pm y, \pm xy\},$$

There is also a **dihedral signed group** (of order 4),

$$DS_4 = \langle x, y \mid x^2 = y^2 = 1, yx = -xy \rangle = \{\pm 1, \pm x, \pm y, \pm xy\},$$

which is isomorphic as groups to D_8, the dihedral group of order 8, and the **quaternion group** (of order 4),

$$S_Q = \langle x, y \mid x^2 = -1, y^2 = -1, yx = -xy \rangle = \{\pm 1, \pm x, \pm y, \pm xy\},$$

which is recognizable as the usual quaternion group of order 8.

2.1.5 Some important classes

Two infinite families are already evident: The **cyclic signed group of order** n,

$$CS_n := \langle x \mid x^n = 1 \rangle = \{\pm 1, \pm x, \dots, \pm x^{n-1}\}$$

and the **negacyclic group of order** n,

$$NS_n := \langle y \mid y^n = -1 \rangle = \{\pm 1, \pm x, \dots, \pm x^{n-1}\}$$

For n odd we have $NS_n \cong CS_n$ (as **signed** groups) via the isomorphism induced by $x \to -y$—but for even $n > 0$, they are never isomorphic.

The multiplicative group of any (associative) ring with unity, of characteristic $\neq 2$, is a signed group.

Let us call any matrix $A \in \{0, \pm 1\}^{n \times n}$ with exactly one nonzero entry per row or column **(real, ternary) monomial**. Equivalently, $A \in \{0, \pm 1\}^{n \times n}$ satisfies $AA^\top = I$. Thus $A^\top = A^{-1}$. Any group of monomial matrices including $-I$ comprises a signed group.

A specific class of monomial matrices of importance to us is the signed group analogue of the symmetric group: the **signed permutation (signed) group** on a set of size n, SP_n, which is a signed group of order $2^{n-1}n!$. The most convenient way to define SP_n is by identifying it with the signed group of $n \times n$ monomial matrices in $\{0, \pm 1\}^{n \times n}$.

Observe that $SP_2 \cong DS_4$.

2.2 General comments on the definitions

Signed groups, as we have defined them, ignoring the business about distinguishing -1, are simply a common class of group. Distinguishing "-1" is the only (mildly) atypical aspect of the objects of interest. Note that it is critical that -1 is a **particular** central element of order 2, and this choice is intrinsic to the object itself[4]. More to the point, if there is more than one such element in an underlying group, a **different** choice for which element to call "-1" can produce a non-isomorphic signed group. Thus, non-isomorphic signed groups may have isomorphic underlying groups. The underlying groups of two of the signed groups of order 4 enumerated above provide a case in point—the smallest such example. Can you see which ones?

It is not for window dressing that the definition of signed group homomorphism requires that -1 be preserved—this is the very thing that distinguishes it from group isomorphism, or rather constricts those isomorphisms to a particular collection, and leads to examples of nonisomorphic signed groups whose underlying groups are isomorphic.

It is important to keep in mind that we have defined the order of a signed group in such a way that it is **half** the cardinality of the underlying set–which is at variance with the way the term is used in groups, semigroups, etc. The motivation for this will become clear in Sections **??**, **??** and **??** as we develop natural rings and matrix algebras whose dimension as modules over $\mathbb{Z}$ or $\mathbb{Q}$ is this number, precisely analogous to the role played by the order of a group when developing group rings or regular representations. Observe that the underlying groups of S_Q and DS_4 are commonly represented exactly as we have done, complete with "-1" and "negation"; one may consider signed groups to be an exact extension of this common convention to a more general case. Consistent with our order convention, DS_4 and EAS_4 are named according to the order of their projection group, not the underlying group. While this is inconsistent with the naming of these objects as groups, it may serve as a reminder of the context.

Subtleties in how we say things are important, so pay attention and when speaking or writing about signed groups, use language carefully. If we say "(signed groups) S_1 and S_2 are not isomorphic" then the fact that we are discussing them as signed groups implies that we mean "isomorphic *in the sense of signed groups*". Whereas if we say "their underlying groups G_1 and G_2 are isomorphic" by implication we mean "...in the sense of groups". In contrast, it is not proper to say (for example) that "S_1 is isomorphic to G_1" without specifying what kind of isomorphism is intended, since the objects are of different types[5].

2.3 Signed group rings

Fix transversal $\mathcal{P} = \{x_1, \ldots, x_n\}$ for $\langle -1 \rangle$ in a signed group S. For any ring R with unity and characteristic $\neq 2$, we associate with S its *signed group ring*,

$$R[S] := \{a_1x_1 + \cdots + a_nx_n \mid a_1, \ldots, a_n \in R\},$$

(a set of formal expressions) with operations addition, defined termwise formally,

$$(a_1x_1 + \cdots + a_nx_n) + (b_1x_1 + \cdots + b_nx_n) = (a_1 + b_1)x_1 + \cdots + (a_n + b_n)x_n.$$

and multiplication defined as the unique formal bilinear extension of multiplication in both R and S, and making the formal identification $-1_S \cdot 1_R = 1_S \cdot -1_R$, abbreviating both expressions as "-1".

[4] In the same way that the choice of root is intrinsic to a rooted tree in graph theory.

[5] Similarly you should not say that "$EA(27)$ is isomorphic to $GF(27)$" without qualification even though it is likely that a well-informed reader will infer the right and natural isomorphism—this is "poor mathematical articulation".

It is a simple exercise to show that $R[S]$ is a ring and an R-module of dimension n with basis $\mathcal{P}$, containing isomorphic copies of S (canonically identified with the multiplicative subgroup $1 \cdot S$ of $R[S]$ via group isomorphism $g \to 1 \cdot g$) and R (canonically identified with $R \cdot 1$ via ring isomorphism $r \to r \cdot 1$).

If $S = \langle -1 \rangle \times G$, G a group, then $\mathbb{Z}[S]$ is congruent, as rings, to the group ring $\mathbb{Z}[G]$. A subtlety: As S is articulated as a signed group, the notation signifies a signed group ring[6] As with group rings, signed group rings may be constructed with respect to *any* ring, but it will suffice for us to focus mainly $\mathbb{Z}$ here. In general, $R[S]$ may alternatively be viewed as the quotient of the group ring of the underlying group with respect to the principal ideal $\langle (1_R \cdot -1_S) - (-1_R \cdot 1_S) \rangle$.

Each signed group homomorphism from S_1 to S_2 extends uniquely to a (ring) homomorphism from $\mathbb{Z}[S_1]$ to $\mathbb{Z}[S_2]$ (i.e., by linear extension).

The field of complex numbers and the skew field of quaternions are signed group rings over $\mathbb{R}$ (using signed groups $S_{\mathbb{C}}$ and S_Q, respectively) even though they are not even embeddable as subrings into group rings over $\mathbb{R}$, suggesting a fundamental significance of signed group rings.

On the signed group ring $\mathbb{Z}[S]$ we define a **(canonical) involution**, $x \to x^*$, by extending the multiplicative inverse of S linearly. This involution in turn induces the canonical map $M \to M^*$, which for a matrix $M = [m_{ij}] \in \mathbb{Z}[S]^{m \times n}$, is defined as $M^* = [m_{ji}{}^*]$. For any n, this is an involution on $\mathbb{Z}[S]^{n \times n}$.

The canonical involution naturally encodes several familiar ideas. For example, if S is trivial, then $*$ is the identity map and the generated matrix map is the transpose. If $S = S_{\mathbb{C}}$ then $\mathbb{Z}[S]$ is the ring of Gaussian integers (identifying the generator of S with $i = \sqrt{-1}$), $*$ amounts to complex conjugation, and the corresponding map on matrices to the Hermitian adjoint (for $S = S_0$ it corresponds to transpose). Similarly when $S = S_Q$ then $\mathbb{Z}[S]$ is the ring of integer quaternions and this involution corresponds (with obvious identifications) to the quaternion conjugate.

Naturally, we refer to both involutions denoted $*$—on the signed group ring $R[S]$ and also on $R[S]^{m \times n}$—as the (natural/induced) **involution** Although we have developed this concept over an arbitrary ring R, in almost every setting $R = \mathbb{Z}$ suffices, although occasionally we need $R = \mathbb{Q}$ and it may be helpful to keep in mind that $R = \mathbb{R}$ or $R = \mathbb{C}$ are available as handy extensions of these. If ring R is equipped with a natural, nontrivial involution we may wish to incorporate that into $*$. For example, typically our convention is that, in $\mathbb{C}[S]$, $(\sum_k \alpha_k x_k)^* = \sum_k \overline{\alpha_k} x_k^*$.

2.4 Matrix representations and remreps

As with groups, signed groups may be represented in various ways as multiplicative signed groups of matrices, with -1 represented by $-I$. Each such representation extends canonically[7] to the signed group ring. There is a natural analogue to the **regular representation** of a group, which represents a signed group of order s as a set of signed permutation matrices of size s [**?**]. SP_n and its signed subgroups also have **natural** representations arising from the identification of each element

[6]This isn't as mysterious as it appears. One reason we use S_Q for the quaternions as a signed group when we already have a common symbol, Q, for the quaternion group and S_Q and Q are indistinguishable as groups, is that it signals. that $\mathbb{Z}[S_Q]$ will be a 4-dimensional module over $\mathbb{Z}$ whereas $\mathbb{Z}[Q]$ is 8-dimensional because Q articulates this object as a group. The ring of quaternions is not the group ring of the quaternion group. However it is the signed group ring of the quaternion signed group. An explanation of the notation might proceed as follows: If R is a ring and X is any set, $R[X]$ denotes the smallest ring obtained by completing (formally, where necessary) operations on the set $R \cup X$ that preserves the ring operations within R, any native addition and multiplication operations in X, elements of R commute with those of X, and any "understood" relations between elements of R and X are respected. So for example we "understand" that -1_R and -1_G will be identified in the constructed object. Thus, for example, if γ is a complex cube root of unity, $\mathbb{Q}[\gamma]$ is a 3-dimensional $\mathbb{Q}$-module, so is $\mathbb{Q}[-\gamma]$ despite that multiplicatively $-\gamma$ generates a group of order 6: it is understood that $(-\gamma)^3$ and $-1_{\mathbb{Q}}$ are to be identified.

[7]i.e., linearly—there is a unique such representation of the ring.

with its defining matrix. Representations of signed groups as sets of $m \times m$ signed permutation matrices, in which case the ring involution corresponds to matrix transpose, are of particular interest to us; we shall call any such representation a **remrep**[8] of **degree** m.

If π is a degree m remrep of a signed group S, it induces a map (also denoted by π) from matrices with entries in $\mathbb{Z}[S]$ to matrices with entries in $\{0, \pm 1\}$ which preserves multiplication and addition, maps conjugation to transpose and maps $\mathbb{Z}[S]^{h\times k}$ into $\{0, \pm 1\}^{hm\times km}$ by replacing each entry with its representation, an $m \times m$ block in the corresponding position of an $h \times k$ partition—that is, if $M = [m_{ij}]$, each $m_{ij} \in \mathbb{Z}[S]$, then we define the partitioned matrix $\pi(M) = [\pi(m_{ij})]$. As a consequence of the elementary fact that the inverse of a signed permutation matrix is its transpose, $\pi(M^*) = \pi(M)^\top$.

Just as every group of order m may be seen as a collection of regular actions on the set consisting of its elements, by right-multiplication, and this produces a linear representation mapping the group to $m \times m$ permutation matrices, analogously a signed group of order m can be thought of as acting (doubly) regularly on $\mathcal{P}$ in a "projective" sense: simply ignore the sign. Perhaps more appropriately $\mathcal{P}$ acts regularly upon S by "projective" right multiplication.

A nice way to directly construct this representation (and simultaneously prove that it exists) is to simply write out a division table D for signed group $S = \{\pm x_1, \dots, \pm x_n\}$ that is in "compact form"—i.e., $n \times n$, with rows and columns indexed by $\{1, 2, \dots, n\}$, and displaying quotients in the form $\pm x$ where $x \in \mathcal{P}$ (from which division of any elements of S may be easily read), manipulate the array as a matrix over $\mathbb{Z}[S]$:

$$D := [x_i x_j^{-1}]_{i,j=1}^n = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} x_1^{-1} & \cdots & x_n^{-1} \end{pmatrix}.$$

For each $k = 1, \dots, n$ it is straightforward to see that, in each row and column, either x_k or $-x_k$ appears, exactly once. Accordingly, D may be regarded as a linear combination of signed permutation matrices (taken over the scalar ring $\mathbb{Z}[S]$).

That is, there exist monomial matrices $P_1, \dots, P_n \in \{0, \pm 1\}^{n\times n}$ such that

$$D = \sum_{k=1}^{n} x_k P_k.$$

Observe that

$$D^2 = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} x_1^{-1} \cdots x_n^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} x_1^{-1} \cdots x_n^{-1} \end{pmatrix} = nD = \sum_k n x_k P_k = \sum_{u,v} x_u x_v P_u P_v.$$

As the latter sum consists of n^2 monomial terms, by elementary counting it follows that for each $u, v = 1, \dots, n$ there exists $k \in \{1, \dots n\}$ such that $x_u x_v P_u P_v = x_k P_k$. That is, $x_u x_v = \pm x_k \Leftrightarrow P_u P_v = \pm P_k$ (with agreement in sign). That is, $\pi : \pm x_k \to \pm P_k$ is an injective signed group homomorphism (and, more to the point, a remrep) $S \to \mathbb{Z}[S]^{n\times n}$.

Theorem 1 *For any signed group S of order n there exists an injective real monomial representation $\pi : S \to \{0, \pm 1\}^{n\times n}$ whose image projects to a transitive subset of S_n.*

[8] Which is a contraction of "real monomial representation", with a nod to the fact that linear representations are naturally regarded as maps to complex-valued matrices, and that nonzero entries of monomial group (or signed group) matrix representations are necessarily complex numbers of modulus 1.

π may be called the **Cayley representation** of S (by analogy to Cayley's representation of groups, which can be derived in the same fashion from a group's division table[9]) or, also commonly, its **(right-)regular representation**[10].

Here is a catalogue of the division tables for the groups listed in section **??** (given only for a single choice of $\mathcal{P}$ in each case). The corresponding remrep π for each element $x \in P$ by writing ± 1 for $\pm x$ and 0 for all other elements of P. To determine $\pi(x_i)$, "set" $x_i = 1$ and $x_j = 0$ for $j \neq i$. We treat the listing of a signed group as $S = \{\pm x_1, \ldots, \pm x_n\}$ as an implicit determination of $\mathcal{P} = \{x_1, \ldots, x_n\}$, which by default indexes rows and columns of the constructed matrices.

2.4.1 $n = 1$

$S_0 = \langle -1 \rangle = \{\pm 1\}$:

$$D = (1)$$

2.4.2 $n = 2$

$CS_2 = \langle x \mid x^2 = 1 \rangle = \{\pm 1, \pm x\}$:

$$D = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$NS_2 = \langle x \mid x^2 = -1 \rangle = \{\pm 1, \pm x\}$:

$$D = \begin{pmatrix} 1 & -x \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

2.4.3 $n = 3$

$CS_3 = \langle x \mid x^3 = 1 \rangle = \{\pm 1, \pm x, \pm x^2\}$:

$$D = \begin{pmatrix} 1 & x^2 & x \\ x & 1 & x^2 \\ x^2 & x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + x^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

2.4.4 $n = 4$

$CS_4 = \langle x \mid x^4 = 1 \rangle = \{\pm 1, \pm x, \pm x^2, \pm x^3\}$:

$$D = \begin{pmatrix} 1 & x^3 & x^2 & x \\ x & 1 & x^3 & x^2 \\ x^2 & x & 1 & x^3 \\ x^3 & x^2 & x & 1 \end{pmatrix}.$$

$NS_4 = \langle x \mid x^4 = -1 \rangle = \{\pm 1, \pm x, \pm x^2, \pm x^3\}$:

$$D = \begin{pmatrix} 1 & -x^3 & -x^2 & -x \\ x & 1 & -x^3 & -x^2 \\ x^2 & x & 1 & -x^3 \\ x^3 & x^2 & x & 1 \end{pmatrix}.$$

[9] Dane Flannery points that the division table construction predates Cayley, at least to Dedekind and Frobenius.

[10] Although the naming conventions of signed group terminology are not used in algebra it is interesting to note that the corresponding 4-dimensional representation of the Quaternion group of order 8 is commonly referred to in exactly this manner even though, strictly speaking, the right-regular representation of a group of order n has dimension n. Could it be that algebraists display a subconscious affinity toward the naming conventions of signed groups?

$EAS_4 = \langle x, y \mid x^2 = y^2 = 1, yx = xy\rangle = \{\pm 1, \pm x, \pm y, \pm xy\}$:

$$D = \begin{pmatrix} 1 & x & y & xy \\ x & 1 & xy & y \\ y & xy & 1 & x \\ xy & y & x & 1 \end{pmatrix}.$$

$SAS_4 = \langle x, y \mid x^2 = y^2 = -1, yx = xy\rangle = \{\pm 1, \pm x, \pm y, \pm xy\}$:

$$D = \begin{pmatrix} 1 & -x & -y & xy \\ x & 1 & -xy & y \\ y & xy & 1 & -x \\ xy & -y & x & 1 \end{pmatrix}.$$

$DS_4 = \langle x, y \mid x^2 = y^2 = 1, yx = -xy\rangle = \{\pm 1, \pm x, \pm y, \pm xy\}$:

$$D = \begin{pmatrix} 1 & x & y & -xy \\ x & 1 & xy & -y \\ y & -xy & 1 & x \\ xy & -y & x & 1 \end{pmatrix}.$$

$S_Q = \langle x, y \mid x^2 = -1, y^2 = -1, yx = -xy\rangle = \{\pm 1, \pm x, \pm y, \pm xy\}$:

$$D = \begin{pmatrix} 1 & -x & -y & -xy \\ x & 1 & -xy & y \\ y & xy & 1 & -x \\ xy & -y & x & 1 \end{pmatrix}.$$

2.5 Signed group development of matrices

There are two principal ways in which signed groups are used. First, to generalize group development of matrices, and second as a source of scalar entries for matrices that generalize classes of interest. There are numerous other ways they arise in the theory of orthogonal matrices, but their uses almost always boil down to one, or both, of these approaches.

First, let us consider what it means to develop a matrix over a group. Some say that $A = [a_{ij}]$ is "G-developed", where $G = \{g_1 \dots g_n\}$ is a group, if there is a function $f : G \to R$ (any ring of scalars) such that for all i, j we have $a_{ij} = f(g_i g_j)$. That is, the matrix displays the "shape" of the multiplication table of G.

However, in light of the Cayley representation of a group we argue that it is more elegant and useful to insist that **G-development** should properly correspond to the *division table*—that is, for some f, we have $a_{ij} = f(g_i g_j^{-1})$. Since the multiplication and division tables of a group are equivalent modulo a permutation of their columns, the two approaches are equivalent but in the former conception G-developed matrices do not generally form a subring of the matrix ring, whereas in the latter they do; we shall adhere to the latter. In particular, circulant $n \times n$ matrices shall be C_n-developed whereas back-circulant matrices are not, rather than the other way around.

Equivalently, if $\pi : G \to \{0, 1\}^{n \times n}$ is the Cayley representation of G, this means that $A = \sum_{g \in G} f(g)\pi(g)$ for some map f from G to a scalar ring R. The set of all G-developed matrices, then, is seen to form a ring, which is an n-dimensional module over R, and which is isomorphic to the group ring $R[G]$, where the map is obvious (extend π linearly to $R[G]$).

Similarly, if $S = \{\pm x_1, \ldots, \pm x_n\}$ is a signed group we say that $A = [a_{ij}]$ is **S-developed** if there is a function $f : S \to R$ (any ring of scalars) such that $A = \sum_{k=1}^{n} f(x_k)\pi(x_k)$, where π is the Cayley representation of S. As with the case of groups, the set of all S-developed matrices forms a ring isomorphic to $R[S]$ via the linear extension of π to the signed group ring.

Using "π" for the Cayley representation on a signed group S and the natural extension to $\mathbb{R}[S]$, since π is a remrep, it follows that conjugation (i.e., the involution *) in $R[S]$ corresponds, for S-developed matrices, to matrix transpose.

Theorem 2 *Let R be a ring with unity, having characteristic $\neq 2$, and let S be a signed group of order n. Then $\pi : R[S] \to R^{n\times n}$ is a ring homomorphism such that $\pi(\alpha^*) = \pi(\alpha)^\top$ for all $\alpha \in R[S]$. Further, π maps $\mathcal{P}$ to a set of n disjoint monomial $\{0, \pm 1\}$-matrices and is a faithful linear representation of the underlying group of S.*

Replacing the elements of the specified transversal with indeterminates A, B, C, D below gives arrays showing the **general case**[11] of S-developed matrices where $S = CS_4,\ SAS_4,\ DS_4$ and S_Q, respectively:

$$\begin{pmatrix} A & B & C & D \\ D & A & B & C \\ C & D & A & B \\ B & C & D & A \end{pmatrix}, \begin{pmatrix} A & -B & -C & D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix}, \begin{pmatrix} A & B & C & -D \\ B & A & D & -C \\ C & -D & A & B \\ D & -C & B & A \end{pmatrix}, \begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & B \\ D & -C & B & A \end{pmatrix}. \quad (1)$$

The following result is straightforward to show and indicates the role of the choice of transversal—and shows that, in some sense[12], this is arbitrary.

Theorem 3 *If $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ are transversals of $\langle -1 \rangle$ in signed group S resulting in Cayley representations π_X and π_Y respectively, and if $A = \sum_k a_k \pi_X(x_k)$, then there exist scalars $b_1, \ldots, b_n$ and a signed permutation matrix P such that for each i, $b_i = \pm a_i$, and $B = \sum_k b_k \pi_Y(x_k) = PAP^{-1}$.*

That is, a matrix being S-developed relative to some ordered transversal of $\langle -1 \rangle$ is equivalent to it being S-developed relative to any given ordered transversal of S via conjugation by a permutation.

When we choose $x_1 = 1$, the first row and column of the division table of a signed group consists of the transversal $\mathcal{P}$, in natural order, and the inverse of transversal elements, again in natural order. This makes it easy to manually build and verify the division table D by starting with this row and column then filling in the rest as quotients.

There are other semi-canonical ways to "develop" matrices from a signed group (which will be explored in a later instalment of this series, along with the network of relationships between the various developments and how they can be used together).

2.6 Signed group matrices and signed group orthogonality

The other major way in which signed groups are used is by constructing matrices whose elements come from that signed group or signed group ring. A **signed group matrix** is a matrix whose entries come from a signed group S, with scalar arithmetic performed in $R[S]$ for some ring R.

[11] That is, relative to the given (ordered) transversal.

[12] Specifically, for most classes of orthogonal matrices of interest, permutation and negation of rows and columns is an equivalence operation, and for most purposes one matrix is as good as any other in its equivalence class.

When wishing to strictly distinguish constraints on entries of a matrix we may specify this with such articulation as S**-matrices**, $(0, S)$**-matrices** or $R[S]$**-matrices**.

A **signed group weighing matrix** of **order** n and **weight** w, over the signed group S, and denoted by $SW(n, w; S)$, is a matrix $W \in (\{0\} \cup S)^{n\times n} \subseteq \mathbb{Z}[S]^{n\times n}$ satisfying $WW^* = wI$.

When $w = n$, we say that W is a **signed group Hadamard matrix**, and write $W = SH(n, S)$. Obviously, $SW(n, w; \langle -1\rangle)$ is another way to denote the weighing matrices $W(n, w)$. Similarly, a $SH(n, \langle -1\rangle)$ is a Hadamard matrix of order n, $H(n)$.

The following key result shows that signed group weighing matrices (and signed group Hadamard matrices) imply **real** ones.

Theorem 4 *Let π be a degree m remrep of the signed group S, and let $V = W(m, v)$. If $W = SW(n, w; S)$, then $\pi(W)(I_n \otimes V) = W(mn, vw)$. Conversely*[13]*, if π is faithful and $U = W(mn, vw)$ is of the form $U = [W_{ij}](I_n \otimes V)$, where each W_{ij} is the image of some element of S π, then $[W_{ij}] = \pi(W)$, where $W = SW(n, w; S)$.*

Proof: By construction, $\pi(W)(I_n \otimes V) \in \{0, \pm 1\}^{mn\times mn}$. Further,

$$\begin{aligned} \pi(W)(I_n \otimes V)[\pi(W)(I_n \otimes V)]^\top &= \pi(W)(I_n \otimes V)(I_n \otimes V^\top)\pi(W)^\top \\ &= v\pi(W)\pi(W^*) = v\pi(WW^*) = v\pi(wI_n) = wvI_{mn}, \end{aligned}$$

which establishes the first part. The second part is obtained by similar algebra and applying the inverse of π to each block W_{ij} of the first array. $\square$

Consider the signed group $SP_2 = \langle x, y \mid x^2 = y^2 = 1, xy = -yx\rangle$ (by definition, SP_2 is isomorphic to the set of signed permutation matrices of order 2, which may be verified using the representation π, induced by[14] $\pi(x) = \begin{pmatrix} 1 & 0 \\ 0 & - \end{pmatrix}$ and $\pi(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; this is the natural representation). $\mathbb{Z}[SP_2]$ contains a copy of the Gaussian integers; this is seen by making the identification $\sqrt{-1} = i \leftrightarrow xy$.

Note that the above representation is of half the dimension of the Cayley representation of SP_2. Since there exists a $H(2)$ we can thus infer from Theorem 4 that every $SW(n, w; SP_2)$ gives a $W(2n, 2w)$. Conversely, since every $H(2)$ can be obtained by premultiplication of $\begin{pmatrix} 1 & - \\ 1 & 1 \end{pmatrix}$ by some 2×2 monomial matrix, any weighing (or Hadamard) matrix $W(2n, 2w)$ that can be decomposed so that nonzero 2×2 blocks are (± 1)-matrices with rank 2 the existence of a $SW(n, w; SP_2)$. As we shall see, this "reverse construction" can be be extremely useful, but for now let us note that it has an unusual and powerful property: While converting orthogonality of matrices in one setting to orthogonality of matrices in a different setting, the size of those matrices is **cut in half**. This form is more common than you might think: It is not currently known whether there exists a Hadamard matrix that cannot be decomposed into 2×2 submatrices in this way[15], and it is known that *every* $W(4n, 4n - 2)$ can be so decomposed[16].

[13]not exactly.

[14]It is conventional in the context of studying ternary matrices in combinatorics to use "$-$" as an abbreviation for "-1" when displaying matrices or sequences used to obtain them.

[15]I will cite a paper covering my Hadamard 2022 (Krakow) talk on this topic when such paper is citeable.

[16]This is shown in R. Craigen, *The Structure of Weighing Matrices having Large Weights*, Designs, Codes and Cryptography 5 (1995), 199–216.

3 Applications: signed groups everywhere!

We study signed groups as a tool for studying orthogonal matrices in combinatorics because they appear naturally at many different points in the standard theory, they provide a helpful tool for analyzing the underlying mechanisms for why things work, they are useful in constructions for objects difficult to obtain in other ways, and they show avenues for broadening the theory in directions that are certain to bear new and helpful fruit.

Have not worked on this section yet – it will be a litany of brief references to places where signed groups have been or could/should be invoked in the literature either to do something new or to provide insight into something old. I probably have a dozen instances of this, but there ought to be more, if I was not so bleedin' slow at publishing my ideas and results...the second paper in this series is intended to go into much greater detail

4 Exercises: to be filled out, but here's a few

1. Identify which two signed groups in our menagerie are isomorphic as groups but not as signed groups. And show a specific group automorphism.

2. Explicitly give an isomorphism showing that $SP_2 \cong DS_4$.

3. Complete the proofs of each result given here.

4. Complete the missing information in our menagerie: the underlying groups of CS_4, NS_4, DS_4, S_Q and the projection groups for all except the one for which this is already given, EAS_4.

5. Show that $CS_n \cong NS_n$ if and only if n is odd.

6. Explicitly derive the representation π for each of the signed groups in §2.4.4, by expressing D as a linear combination of monomial matrices with coefficients in $\mathcal{P}$, taking $\mathcal{P}$ either as $\{1, x, x^2, x^3\}$ or $\{1, x, y, xy\}$, as appropriate.

7. Provide a presentation for SP_n in the general case. What is the order of SP_n?

8. The symmetric group S_n is not the underlying group of a signed group, for any $n > 2$. Why not?

9. Why do we specify that the ring over which a signed group ring is developed may not have characteristic 2? Why do we ask that it be a ring with unity?

10. If p is an odd prime, how many non-isomorphic signed groups are there of order p? Of order $2p$? Write out representative presentations and division tables for each isomorphism class of order 6 signed groups.

11. Count the (signed group) automorphisms of each of the signed groups in our menagerie.

12. Prove that the property of being a signed group weighing matrix, and the weight of such a matrix, is preserved under the entrywise application of any signed group homomorphism to the matrix.