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Exact SIC-POVMs from Permutation Symmetries

Markus Grassl

International Centre for Theory of Quantum Technologies

University of Gdansk

markus.grassl@ug.edu.pl sicpovm.markus-grassl.de

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in collaboration with

Marcus Appleby, Ingemar Bengtsson, Michael Harrison, Gary McConnell, Andrew Scott

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Overview

- SIC-POVMs
- group orbits
- Zauner's conjecture
- numerics
- number theory
- linking numerics and number theory
- deterministic recipe
- conclusions & outlook

Symmetric Informationally-Complete POVMs

- informationally complete POVM with d^2 rank-one elements $\Pi_i = \alpha|\psi_i\rangle\langle\psi_i|$
- symmetry: overlaps $\mathrm{tr}(\Pi_i\Pi_j) = \text{const} = \beta$ for $i \neq j$
- using $\sum \Pi_i = \mathbb{1}$ it follows that $\alpha = \frac{1}{d}$ and $\beta = \frac{1}{d^2(d+1)}$

$$\Longrightarrow \quad |\langle\psi_i|\psi_j\rangle|^2 = \frac{1+\delta_{ij}d}{1+d}$$

- reconstruction of quantum states from measurement statistics

$$\varrho = \sum_{j=1}^{d^2} \underbrace{\Bigl(d(d+1)\,\mathrm{Tr}(\varrho\,\Pi_j) - 1\Bigr)}_{=c_j} \Pi_j$$

$\Longrightarrow$ coefficient c_j is an affine function of the probability $p_j = \mathrm{tr}(\varrho\,\Pi_j)$

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A Simple to State Problem

Are there d^2 vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d^2} \in \mathbb{C}^d$ in the complex vector space of dimension d such that:

(i)	$\langle \boldsymbol{v}_j \vert \boldsymbol{v}_j \rangle = 1$	for $j = 1, \ldots, d^2$
(ii)	$\vert\langle \boldsymbol{v}_j \vert \boldsymbol{v}_k \rangle\vert^2 = \dfrac{1}{d+1}$	for $1 \leq j < k \leq d^2$

The vectors $\boldsymbol{v}_j$ form an equiangular tight frame and a 2-design.

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All solutions form a real algebraic variety, using $2d$ real variables per vector

$$\boldsymbol{v}_j = (a_{j,1} + ib_{j,1}, a_{j,2} + ib_{j,2}, \ldots, a_{j,d} + ib_{j,d})^T \qquad (i = \sqrt{-1})$$

$2d^3$ variables, d^2 equations (i) of degree 2 and $\binom{d^2}{2}$ equations (ii) of degree 4.

$\Longrightarrow$ exact solutions via Gröbner bases (but too complicated)

Group Orbits

instead of d^2 vectors, consider the (projective) orbit of a single *fiducial* vector with respect to a finite group $G \leq U(d)$

- irreducible projective representation ρ of a group $H \cong G/\mathcal{Z}(G)$ of order d^2
- matrices $\rho(H)$ form a trace orthogonal basis of the complex $d \times d$ matrices [Zauner 1999] (*nice unitary error basis*)
- almost all *known* SIC-POVMs are orbits of the Weyl-Heisenberg group
- exception: Hoggar lines in dimension $d = 8$ (H_2^3)
- for $d = 2, 3$, all SIC-POVMs are orbits of the Weyl-Heisenberg group [Lane P. Hughston & Simon M. Salamon, Surveying points in the complex projective plane, Advances in Mathematics **286**:1017–1052 (2016)] [Ferenc Szöllősi, All complex equiangular tight frames in dimension 3, arXiv:1402.6429]
- numerics for $d = 4, 5, (6, 7)$: [Solomon B. Samuel & Zafer Gedik, Group Theoretical Classification of SIC-POVMs, arXiv:2401.11026]

Weyl-Heisenberg Group

- generators: $$H_d := \langle X, Z \rangle$$

  where $X := \sum_{j=0}^{d-1} |j+1\rangle\langle j|$ and $Z := \sum_{j=0}^{d-1} \omega_d^j |j\rangle\langle j|$

  $$(\omega_d := \exp(2\pi i/d))$$

- relations:

$$\left(\omega_d^c X^a Z^b\right)\left(\omega_d^{c'} X^{a'} Z^{b'}\right) = \omega_d^{a'b-b'a}\left(\omega_d^{c'} X^{a'} Z^{b'}\right)\left(\omega_d^c X^a Z^b\right)$$

- basis:

$$H_d \big/ \mathcal{Z}(H_d) = \left\{X^a Z^b \colon a, b \in \{0, \ldots, d-1\}\right\} \cong \mathbb{Z}_d \times \mathbb{Z}_d$$

  trace-orthogonal basis of all $d \times d$ matrices

- exists in all dimensions

Constructing SIC-POVMs

Ansatz:

SIC-POVM that is the orbit under the Weyl-Heisenberg group H_d, i. e.,

$$
\begin{aligned}
|\boldsymbol{v}^{(a,b)}\rangle &:= X^a Z^b |\boldsymbol{v}^{(0,0)}\rangle \\
|\langle \boldsymbol{v}^{(a,b)} | \boldsymbol{v}^{(a',b')}\rangle|^2 &= \begin{cases} 1 & \text{for } (a,b) = (a',b'), \\ 1/(d+1) & \text{for } (a,b) \neq (a',b') \end{cases}
\end{aligned}
$$

$$
|\boldsymbol{v}^{(0,0)}\rangle = \sum_{j=0}^{d-1} (x_{2j} + i x_{2j+1}) |j\rangle,
$$

($x_0, \ldots, x_{2d-1}$ are real variables, $x_1 = 0$)

$\Longrightarrow$ we have to find only one *fiducial* vector $|\boldsymbol{v}^{(0,0)}\rangle$ instead of d^2 vectors

$\Longrightarrow$ polynomial equations with $2d-1$ variables, but already quite complicated for $d = 6$

Clifford Group (or Jacobi Group)

- automorphism group of the Weyl-Heisenberg group H_d, i. e.

$$\forall T \in J_d : T^\dagger H_d T = H_d$$

- the action of J_d on H_d modulo phases corresponds to the symplectic group $\mathrm{Sp}(2,\mathbb{Z}_d) \cong \mathrm{SL}(2,\mathbb{Z}_d)$, i. e.

$$T^\dagger X^a Z^b T = \omega_d^c X^{a'} Z^{b'} \qquad \text{where } \begin{pmatrix} a' \\ b' \end{pmatrix} = \tilde{T} \begin{pmatrix} a \\ b \end{pmatrix}, \ \tilde{T} \in \mathrm{SL}(2,\mathbb{Z}_d)$$

$\Longrightarrow$ homomorphism $J_d \to \mathrm{SL}(2,\mathbb{Z}_d)$

- additionally: complex conjugation (anti-unitary)

$$X^a Z^b \mapsto X^a Z^{-b} \qquad \text{corresponding to } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Zauner's Conjecture

[G. Zauner, Dissertation, Universität Wien, 1999]

Conjecture:
For every dimension $d \geq 2$ there exists a SIC-POVM whose elements are the orbit of a rank-one operator E_0 under the Weyl-Heisenberg group H_d.
What is more, E_0 commutes with an element S of the Clifford group J_d.
The action of S on H_d modulo the center has order three. support for this

conjecture (to date):

- numerical solutions for all dimensions $d \leq 196$, plus a few more
- exact algebraic solutions for quite a few ~~some~~ dimensions (see below)

one of the prize problems in
[Paweł Horodecki, Łukasz Rudnicki, Karol Życzkowski, Five open problems in quantum information, arXiv:2002.03233]

Numerical Search for SIC-POVMs

- “second frame potential” f for 2-designs

$$\sum_{i,j=1}^{d^2} |\langle v^{(i)} \otimes v^{(i)} | v^{(j)} \otimes v^{(j)} \rangle|^2 = \sum_{i,j=1}^{d^2} |\langle v^{(i)} | v^{(j)} \rangle|^4 = d^2 \sum_{a,b=1}^{d} |\langle \psi | X^a Z^b | \psi \rangle|^4$$

- for any state $|\psi\rangle \in \mathbb{C}^d$

$$f(|\psi\rangle) = \sum_{j,k=1}^{d} \Big| \sum_{\ell=1}^{d} \langle \psi | j+\ell \rangle \langle \ell | \psi \rangle \langle \psi | k+\ell \rangle \langle j+k+\ell | \psi \rangle \Big|^2$$

$$= \sum_{j,k=1}^{d} \Big| \underbrace{\sum_{\ell=1}^{d} \overline{\psi}_{j+\ell}\, \psi_\ell\, \overline{\psi}_{k+\ell}\, \psi_{j+k+\ell}}_{=:G(j,k)} \Big|^2 \qquad \geq \frac{2}{d+1}$$

with equality iff $|\psi\rangle$ is a fiducial vector for a Weyl-Heisenberg SIC-POVM

- gradient descent to minimize $f(|\psi\rangle)$, subject to unit norm

Numerical Search for SIC-POVMs

- efficient implementation of $F(\vec{x})$ and its gradient in `C++` by Andrew Scott
- parallel computation of the function/gradient using OpenMP/CUDA
- minimization using limited-memory Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm
- search runs into local minima, we need many random initial points
- running many instances on HPC clusters by MPG and GWDG
-
  - for $d = 189$: approx. 23.3×10^6 trials, 3.48 CPU years
  - for $d = 190$: approx. 66.8×10^6 trials, 10.51 CPU years
  - for $d = 193$: approx. 78.3×10^6 trials, 13.00 CPU years
  - for $d = 5779$: 55065 trials, 17.69 GPU years, no success

Increasing Numerical Precision

various methods to increase precision

(A) re-run BFGS with multi-precision arithmetic (multi-threaded version)
$\Longrightarrow$ use "$3d$ conjecture" by Chris Fuchs

(B) additive correction (linear system of d^2 equations)

(C) "alternating projections" for SIC-POVMs

for (B) and (C), recall that $X^a Z^b$ forms on ONB for $d \times d$ matrices

$$|\psi\rangle\langle\psi| = \sum_{a,b} c_{a,b} X^a Z^b$$

where

$$c_{a,b} = \frac{1}{d} \operatorname{Tr}\left(Z^{-b} X^{-a} |\psi\rangle\langle\psi|\right) = \frac{1}{d} \langle\psi| Z^{-b} X^{-a} |\psi\rangle = \frac{\exp(i\theta_{a,b})}{d\sqrt{d+1}} \quad (a,b) \neq (0,0)$$

Ray Class Field Conjecture

[Appleby, Flammia, McConnell & Yard, arXiv:1604.06098 & arXiv:1701.052000]

Ray class field conjecture

let $\mathbb{E}$ be the field containing all rank-one projection operators of a SIC-POVM

$$\mathbb{Q} \lhd \mathbb{K} = \mathbb{Q}(\sqrt{D}) \lhd \mathbb{E}_0 \lhd \mathbb{E}_1 \lhd \mathbb{E}$$

for the minimal field:

- $\mathbb{E}$ is the ray class field over $\mathbb{Q}(\sqrt{D})$ with conductor[a] d' with ramification at both infinite places, D is the squarefree part of $(d+1)(d-3)$
- $\mathbb{E}_1$ contains the overlap phases and equals the ray class field with ramification only allowed at the infinite place taking $\sqrt{D}$ to a positive real number
- $\mathbb{E}_0$ is the Hilbert class field $H_{\mathbb{K}}$, in particular $h = [\mathbb{E}_0 : \mathbb{K}]$ equals the class number of $\mathbb{K}$

[a] $d' = d$, or $d' = 2d$ for d even

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$$\mathbb{Q} \lhd \mathbb{K} = \mathbb{Q}(\sqrt{D}) \lhd \mathbb{E}_0 \lhd \mathbb{E}_1 \lhd \mathbb{E}$$

- **"Fact 8"**: $\mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0)$ permutes the overlaps.
  For each $\sigma \in \mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0)$ there is a matrix $G_\sigma \in \mathrm{GL}(2, \mathbb{Z}/d'\mathbb{Z})$ such that[a]

$$\sigma\big(\langle\psi|D_{\boldsymbol{p}}|\psi\rangle\big) = \langle\psi|D_{G_\sigma \boldsymbol{p}}|\psi\rangle.$$

  G_σ commutes with matrices F related to symmetries U_F of the fiducial vector $|\psi\rangle$.

[a] $D_{\boldsymbol{p}} = D_{a,b} = (e^{\frac{i\pi}{d}})^{ab} X^a Z^b$

Exact Solutions from Numerical Solutions

[Appleby, Chien, Flammia & Waldron, J. Phys. A. 51, 2018, arXiv:1703.05981]

- matrix group $\mathcal{M} = \langle G_\sigma : \sigma \in \mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0)\rangle$, commutes with the symmetry
- projection operator $\Pi = |\psi\rangle\langle\psi|$

"Fact 8:" $\quad \sigma\big(\mathrm{Tr}(\Pi D_{\boldsymbol{p}})\big) = \mathrm{Tr}\big(\Pi D_{G_\sigma \boldsymbol{p}}\big)$

- expansion coefficients $c_{\boldsymbol{p}} = \mathrm{Tr}(\Pi D_{\boldsymbol{p}})$ in the same orbit under $\mathcal{M}$ are related by Galois conjugation
- the coefficients of the polynomial $f_{\boldsymbol{p}_0}(z) = \prod_{\boldsymbol{p}\in\boldsymbol{p}_0^{\mathcal{M}}} (z - c_{\boldsymbol{p}})$

  lie in a number field of "small" degree
- find the exact minimal polynomials of those coefficients (requires high-precision numerical solution)
- find the roots of the exact polynomials $f_{\boldsymbol{p}_0}(z)$ in the ray class field
- compute Π from the d^2 expansion coefficients $c_{\boldsymbol{p}}$
- exact solutions for *some* $d \leq 48$

More Exact Solutions from Numerical Solutions

- when G_σ has determinant 1, there exists a unitary $U_{G_\sigma} := T_\sigma$ with

$$\sigma\left(\mathrm{Tr}(\Pi D_{\boldsymbol{p}})\right) = \mathrm{Tr}(\Pi D_{G_\sigma \boldsymbol{p}}) = \mathrm{Tr}(\Pi T_\sigma D_{\boldsymbol{p}} T_\sigma^\dagger) = \mathrm{Tr}(T_\sigma^\dagger \Pi T_\sigma D_{\boldsymbol{p}})$$

  $\Longrightarrow$ action of $T_\sigma^\dagger$ on the projection Π and on the state $|\psi\rangle$

- when $G_\sigma = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$ is additionally diagonal, then T_σ is a permutation matrix
  (depends on primes in the factorisation of the dimension d)

- moreover, *assume* that $\sigma(D_{\boldsymbol{p}}) = D_{\boldsymbol{p}}$; then

$$\sigma(\Pi) = T_\sigma^\dagger \Pi T_\sigma$$

  and hence

$$\sigma(\Pi_{j,k}) = \Pi_{\alpha j, \alpha k}$$

  where the indices are computed modulo d

More Exact Solutions from Numerical Solutions

- for the first column of Π we have

$$\sigma(\Pi_{j,0}) = \Pi_{\alpha j,0} \qquad \text{for } j = 0, \ldots, d-1$$

- we can take the first column as (unnormalised) fiducial vector $\boldsymbol{v}$, unless it is zero (which was observed for $d = 26, 28, 62, 98, 228$)
  $\Longrightarrow \sigma$ permutes the components of the fiducial vector, stabilising the first coordinate

- when the first column is zero, consider a non-zero column k:

$$\sigma(\Pi_{j,k}) = \Pi_{\alpha j,\alpha k} \stackrel{(*)}{=} \gamma\Pi_{\alpha j,k} \qquad \text{for } j = 0, \ldots, d-1$$

  $\Longrightarrow \sigma$ gives rise to a projective permutation action
  $\Longrightarrow$ consider the action on ratios $v_j/v_{j'}$

$^{(*)}$ Π has rank one, so column αk is proportional to column k, i.e., $\Pi_{j,\alpha k} = \gamma\Pi_{j,k}$

More Exact Solutions from Numerical Solutions

outline of the procedure:

- compute a numerical fiducial vector with prescribed symmetry S
- determine the diagonal matrices $G_\sigma \in \mathrm{SL}(2, \mathbb{Z}/d'\mathbb{Z})$ in the centraliser of S
- the diagonal matrices correspond to a subgroup $H \leq (Z/d'\mathbb{Z})^\times$
- consider the rescaled fiducial vector[a] $\boldsymbol{v}$ with $v_0 = 1$
- the coefficients of the polynomial $f_j(z) = \prod_{\alpha\in H} (z - v_{\alpha j})$ lie in a number field of "small" degree, fixed by (a subgroup of) the Galois group
- similar as before, find the exact coefficients of $f_j(z)$ from a high-precision numerical solution, and then compute its exact roots
$\Longrightarrow$ only $\mathcal{O}(d)$ numbers in a field of smaller degree

[a] assuming $v_0 \neq 0$ for simplicity here

More Exact Solutions from Numerical Solutions

- the *assumption* that $\sigma(D_{\boldsymbol{p}}) = D_{\boldsymbol{p}}$ appears to be true
- new exact solutions for 58 additional dimensions (so far)

$$\begin{aligned} d = {} & 26, 38, 42, 49, 52, 56, 57, 61, 62, 63, 65, 67, 73, 74, 78, 79, 84, 86, 91, 93, \\ & 95, 97, 98, 103, 109, 111, 122, 127, 129, 133, 134, 139, 143, 146, 147, \\ & 151, 155, 157, 163, 168, 169, 172, 181, 182, 183, 193, 199, \\ & 201, 228, 259, 292, 327, 364, 399, 403, 489, 844, 1299 \end{aligned}$$

- fiducial vectors lie in a proper ("small") subfield of the ray class field from before, that intersects with the cyclotomic field $\mathbb{Q}(\zeta_{d'})$ trivially or in a smaller cyclotomic field
- "small ray class field conjecture":
  The minimal field containing a (suitably rescaled) fiducial vector is a ray class field whose conductor is a particular factor of the ideal $d\mathcal{O}_{\mathbb{K}}$ with ramification at both infinite places.

Solutions to the SIC-POVM Problem

- **numerical solutions:**
  - $d \leq 45$: Joesph Renes et al. quant-ph/0310075
  - $d \leq 67$: Andrew Scott & Markus Grassl arXiv:0910.5784
  - $d \leq 121$ plus a few more: Andrew Scott arXiv:1703.03993
  - $d \leq 151$: Christopher Fuchs et al. arXiv:1703.07901
  - $d \leq 193$, $d = 204, 224, 255, 288, 528, 1155, 2208$ MG, unpublished
- **exact algebraic solutions:**
  - early 2017: $d = 2$–$24, 28, 30, 31, 35, 37, 39, 43, 48, 124$ (32 dimensions)
  - early 2021: all dimensions $d \leq 53, 57, 61, 62, 63, 65, 67, 73, 74, 76, 78, 79, 80, 84, 86, 91, 93, 95, 97, 98, 99, 103, 109, 111, 120, 122, 124, 127, 129, 133, 134, 139, 143, 146, 147, 151, 155, 157, 163, 168, 169, 172, 181, 182, 183, 193, 195, 199, 201, 228, 259, 292, 323, 327, 364, 399, 403, 487, 489, 725, 787, 844, 1299, 5779$ (115 dimensions)

Prime Dimensions $p \equiv 1 \bmod 3$

- for prime dimensions $d = p \equiv 1 \bmod 3$, the Zauner symmetry F_z is conjugate to a diagonal matrix $\widetilde{F}_z$
- the centraliser of $\widetilde{F}_z$ contains all diagonal matrices in $\mathrm{SL}(2, \mathbb{Z}/d\mathbb{Z})$
- the components v_j, $j = 1, \ldots, d-1$, of the fiducial vector (with $v_0 = 1$) are on a single orbit with respect to the Galois group, i.e.,

$$v_{\theta^k} = \sigma^k(v_1)$$

  for generators θ and σ of $(\mathbb{Z}/d\mathbb{Z})^\times$ and the Galois group, resp.
- for a permutation symmetry of order 3ℓ, we need only $m = \frac{d-1}{3\ell}$ numbers

dream:

find a *direct* way to determine the algebraic number v_1, as well as σ and θ

Prime Dimensions $p = n^2 + 3$

[Appleby, Bengtsson, Grassl, Harrison, McConnell, "SIC-POVMs from Stark Units", J. Math. Physics 63 (2022), 112205]

Conjecture:

- for prime dimensions $p = n^2 + 3$ $(n > 0)$, there is an *almost flat* fiducial vector $\boldsymbol{v}$ with

$$v_j = \begin{cases} -2 - \sqrt{d+1} & j = 0 \\ \sqrt{v_0 e^{i\vartheta_j}} & j > 0 \end{cases}$$

- the components of $\boldsymbol{v}$ generate a "small" ray class field $\mathbb{K}^{\mathfrak{m}}$ with finite modulus $1 + \sqrt{d+1}$ and ramification at one infinite place
- the phases $e^{i\vartheta_j}$ are Galois conjugates of (real) *Stark units* for the ray class field $\mathbb{K}^{\mathfrak{m}}$

Stark units can be computed via L-functions

Application of Stark's Conjectures

- for certain ray class fields $\mathbb{K}^{\mathfrak{m}}$ over the real quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, $D > 0$, one can compute numerical approximations of *Stark units* ϵ_σ via special values of derivatives of L-functions
- the Stark units are labelled by elements σ of the Galois group $\mathrm{Gal}(\mathbb{K}^{\mathfrak{m}}/\mathbb{K})$ such that $\epsilon_\sigma = \sigma(\epsilon_0)$
- from numerical Stark units with sufficiently high precision, we can deduce their exact minimal polynomial over $\mathbb{K}$
- we have a heuristic that allows us to deduce the required precision from numerical Stark units with low precision
- the complexity of the calculation appears to be roughly $\mathcal{O}\left(\deg(\mathbb{K}^{\mathfrak{m}}/\mathbb{K}) \times (\#\text{digits})^{3.3}\right)$

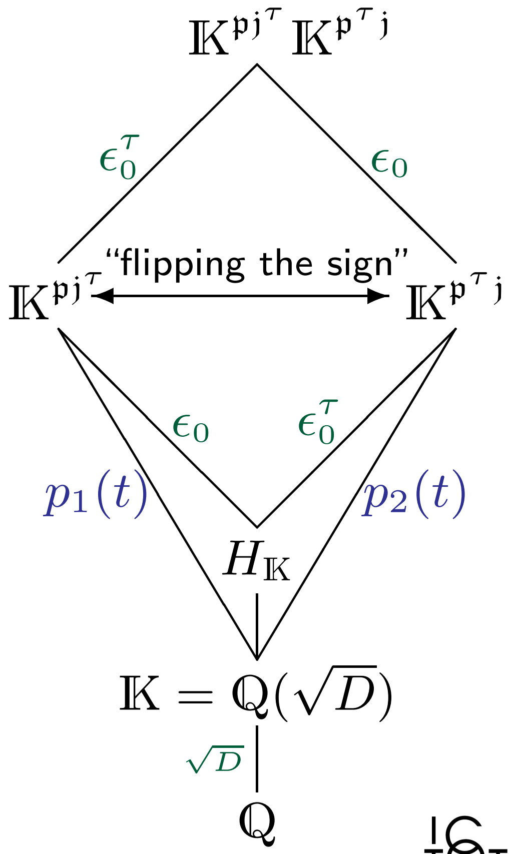
Runtime L-Functions

total CPU time to compute the numerical derivative of L-functions using Magma and PARI/GP (last four cases)

d	$\deg(\mathbb{K}^{\mathfrak{m}}/\mathbb{K})$	$\log_{10}$ height	precision	CPU time
487	324	424	1000 digits	251 hours
787	262	299	1000 digits	118 hours
2707	902	1861	3800 digits	900 days
4099	1366	974	2000 digits	170 days
5779	214	127	300 digits	18 min
1447	964	2158	4600 digits	111 days
2503	3336	6464	13000 digits	60.5 years
19324	4831	10815	22000 digits	328 years
19603	2178	1754	4000 digits	82 days

Flipping the Sign

- real quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ with non-trivial automorphism
  $\tau\colon \sqrt{D} \mapsto -\sqrt{D}$
- embedding $\mathfrak{j}\colon \mathbb{K} \hookrightarrow \mathbb{R}, \mathfrak{j}(\sqrt{D}) > 0$, $\mathfrak{j}^\tau(\sqrt{D}) = \mathfrak{j}\big(\big(\sqrt{D}\big)^\tau\big) < 0$
  - "real" Stark units ϵ_σ: $\mathfrak{j}(\epsilon_\sigma) > 0$
  - "complex" Stark units ϵ_σ^τ: $\mathfrak{j}(\epsilon_\sigma^\tau) = \mathfrak{j}^\tau(\epsilon_\sigma) \in \mathbb{C} \setminus \mathbb{R}$
- minimal polynomial of ϵ_σ: $\qquad p_1(t) \in \mathbb{K}[t]$
  $\Longrightarrow$ minimal polynomial of ϵ_σ^τ: $p_2(t) = p_1^\tau(t)$
- **obstacle:**
  operation of σ on ϵ_0^τ would require factoring $p_2(t)$



New Solutions

Marcus Appleby, Steven T. Flammia, Gene. S Kopp, "A Constructive Approach to Zauner's Conjecture via the Stark Conjectures", arXiv:2501.03970

- method to compute overlap phases of the SIC-POVM using double-sine function
- implemented in julia package 'Zauner.jl'
- intermediate step is a numerical "ghost" fiducial
- computing exact minimal polynomial(s) for Galois orbit(s) of overlap phases
- apply the same techniques based on permutation action on the "ghost" fiducial to obtain exact expressions
- apply the Galois automorphism τ to the exact solution

$\Longrightarrow$ new numerical solution for $d = 194$ and exact solutions for $d = 196$, 211, 247, 271, 307, 337, 362, 703, 721, and 728

Solutions for $d = n^2 + 3$

- the method using Stark units can be generalised to composite dimensions $d = n^2 + 3$
- even dimensions $d = n^2 + 3$ are divisible by 4, but not by 8; almost flat fiducial vector after change of basis
- for composite dimensions, one has to compute Stark units for certain subfields as well

so far, our method has been successfully applied in 72 dimensions (including the first 53 of the sequence):

$d = 4$, 7, 12, 19, 28, 39, 52, 67, 84, 103, 124, 147, 172, 199, 228, 259, 292, 327, 364, 403, 444, 487, 532, 579, 628, 679, 732, 787, 844, 903, 964, 1027, 1092, 1159, 1228, 1299, 1372, 1447, 1524, 1603, 1684, 1767, 1852, 1939, 2028, 2119, 2212, 2307, 2404, 2503, 2604, 2707, 2812, 3028, 3252, 3484, 3603, 3724, 3972, 4099, 4492, 4627, 5332, 5779, 6727, 7399, 7924, 12324, 19324, 19603, 39604, and 45372 ($d = 54759$ in progress)

Conclusions & Outlook

- numerical search allowed to derive conjectures about possible symmetries
- imposing symmetries allowed to find first exact solutions
- exact solutions lead to ray class field conjecture
- conjectures about the action of the Galois group
- combination of Galois theory and numerics yields exact solutions

- deterministic procedure to compute SIC-POVMs from Stark units
- successfully applied in 72 dimensions $d = n^2 + 3$; did not fail in any
- exact solutions for many other dimensions with permutation symmetries

- assuming Stark's conjectures to be true, can be prove that our construction always works?
- can we extend the method to other dimensions?

Thank you!
Dziekuje! Danke!
¡Gracias!

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