How to design a graph with three eigenvalues

Gary Greaves

Nanyang Technological University

Singapore

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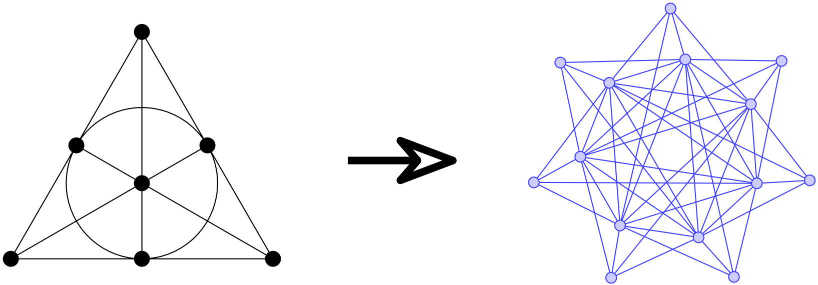

Nonregular graphs with three eigenvalues

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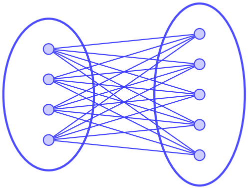
Strongly regular graph: a regular graph (V, E) for which $\exists \lambda, \mu$ such that, $\forall x, y \in V$ with $x \neq y$, the number of common neighbours of x and y is

$$\begin{cases} \lambda, & \text{if } x \sim y \\ \mu, & \text{if } x \not\sim y. \end{cases}$$

Complete bipartite graphs $K_{a,b}$ have spectrum

$$\left\{ [\sqrt{ab}]^1, [0]^{a+b-2}, [-\sqrt{ab}]^1 \right\}.$$



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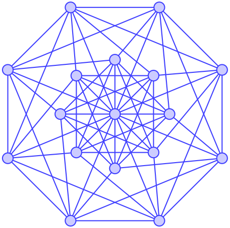
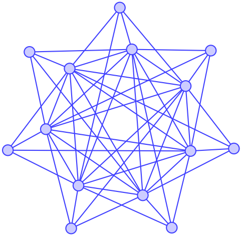
Muzychuk-Klin (1998): infinite families of examples with
Van Dam (1998): **two valencies** and a positive number of examples with **three valencies**.



Shrikhande cone



Fano graph

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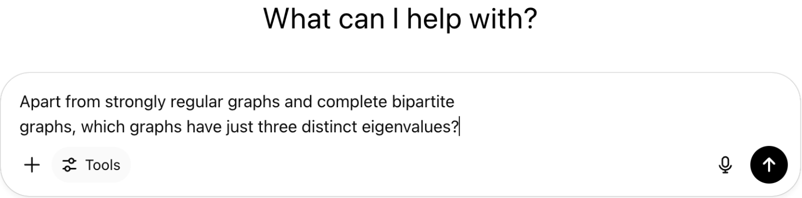

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Apart from strongly regular graphs and complete bipartite graphs, which graphs have just three distinct eigenvalues?

Graphs with exactly three distinct eigenvalues are quite special, and this property often implies strong regularity or high symmetry. Besides **strongly regular graphs** and **complete bipartite graphs** (like $K_{m,n}$), here are some **other families or examples** of graphs with exactly three distinct eigenvalues:

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Besides strongly regular graphs and complete bipartite graphs, graphs with exactly three distinct eigenvalues include:

- Friendship graphs
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these are strongly regular! (pointing to: Certain circulant and Cayley graphs; Complements of SRGs; Conference graphs; Disjoint unions of complete graphs)

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[Klin and Gyürki 2015]

Weisfeiler-Leman stabilisation

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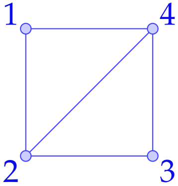

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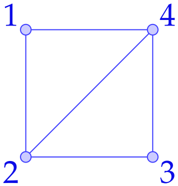

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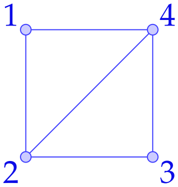

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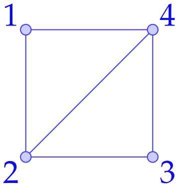

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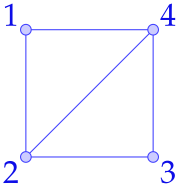

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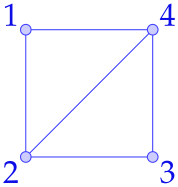

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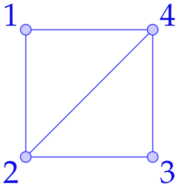

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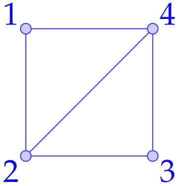

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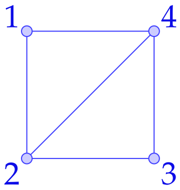

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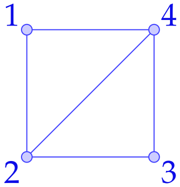

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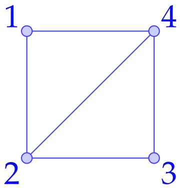

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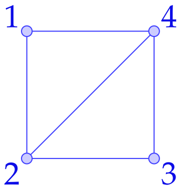

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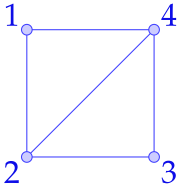

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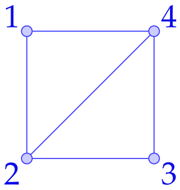

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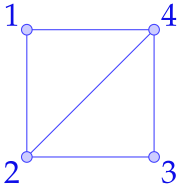

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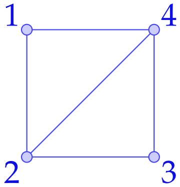

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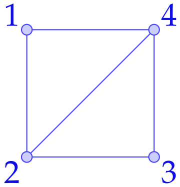

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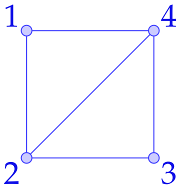
$$M_3 = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix} = M_2$$

Coherent rank

$$\mathcal{W}(\Gamma) = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$

Γ



$$\overset{a}{\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}}, \overset{b}{\begin{bmatrix} 0&1&0&1\\0&0&0&0\\0&1&0&1\\0&0&0&0 \end{bmatrix}}, \overset{c}{\begin{bmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}}, \overset{d}{\begin{bmatrix} 0&0&0&0\\1&0&1&0\\0&0&0&0\\1&0&1&0 \end{bmatrix}}, \overset{e}{\begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}}, \overset{f}{\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix}}$$

We say "Γ has **coherent rank** 6".

Coherent rank of graphs with three eigenvalues

Theorem (Muzychuk-Klin 1998)
Let Γ be a connected graph w/ three distinct eigenvalues.
Then the coherent rank of Γ is

- $= 3$ iff Γ is strongly regular;
- $\neq 4$;
- $= 5$ iff $\Gamma \cong K_{1,b}$ with $b > 1$;
- $= 6$ iff $\Gamma \cong K_{a,b}$ with $2 \leqslant a < b$ or
  Γ is a cone over a strongly regular graph;
- $\neq 7$.

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Proposition 6.2. *For a non-standard graph Γ the cases* $\dim(W(\Gamma)) = r$, $r \in \{7, 8\}$ *are impossible.*

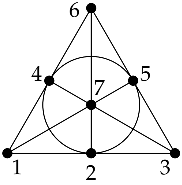
non-standard: connected w/ 3 evs, not srg, not $K_{a,b}$.

Total graph of a symmetric design

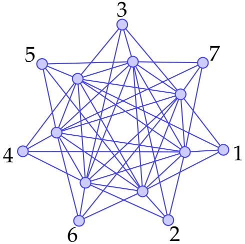
B: incidence matrix of a symmetric 2-design $\mathcal{D}$.

Total graph of $\mathcal{D}$: $\begin{bmatrix} O & B \\ B^\intercal & J-I \end{bmatrix}$.

Theorem (Van Dam 1998)
Total graph of a symmetric 2-(q^3-q+1, q^2, q) design
spectrum: $\left\{ [q^3]^1, [q-1]^{(q-1)q(q+1)}, [-q]^{(q-1)q(q+1)+1} \right\}$.



Fano graph:
$(q=2)$



Graphs with coherent rank 8

B: incidence matrix of a symmetric 2-design $\mathcal{D}$.

Total graph of $\mathcal{D}$: $\begin{bmatrix} O & B \\ B^\intercal & J-I \end{bmatrix}$.

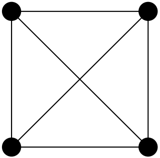
Theorem (GG and Yip 2025+)
Let Γ be a connected graph w/ three distinct eigenvalues.
Then $\mathcal{W}(\Gamma)$ has rank 8 if and only if Γ is the total graph of a symmetric 2-(q^3-q+1, q^2, q) design.

$$\mathcal{W}(\Gamma) = \boxed{a} \begin{bmatrix} I & O \\ O & O \end{bmatrix} + \boxed{b} \begin{bmatrix} J-I & O \\ O & O \end{bmatrix} + \boxed{c} \begin{bmatrix} O & B \\ O & O \end{bmatrix} + \boxed{d} \begin{bmatrix} O & J-B \\ O & O \end{bmatrix} + \boxed{e} \begin{bmatrix} O & O \\ B^\intercal & O \end{bmatrix} + \boxed{f} \begin{bmatrix} O & O \\ J-B^\intercal & O \end{bmatrix} + \boxed{g} \begin{bmatrix} O & O \\ O & I \end{bmatrix} + \boxed{h} \begin{bmatrix} O & O \\ O & J-I \end{bmatrix}$$

Quasi-symmetric designs

Definition (quasi-symmetric design)
A 2-(v, k, λ) design $(X, \mathcal{B})$ is called **quasi-symmetric** if $\forall B_1 \neq B_2$ in $\mathcal{B}$ we have $|B_1 \cap B_2| \in \{x, y\}$ with $x \neq y$.

x and y are called **intersection numbers**.



quasi-symmetric 2-$(4, 2, 1)$ design

intersection numbers: 0 and 1

Definition (block graph)
The x-**block graph** of $(X, \mathcal{B})$ has vertex set $\mathcal{B}$, and two blocks are adjacent iff they intersect in x points.

Total graph of a quasi-symmetric design

B: incidence matrix of a quasi-symmetric 2-design $\mathcal{Q}$.
C: adjacency matrix of the x-block graph of $\mathcal{Q}$.

x**-total graph** of $\mathcal{Q}$: $\begin{bmatrix} O & B \\ B^\intercal & C \end{bmatrix}$.

Theorem (Van Dam 1998)
The q-total graph of a quasi-symmetric 2-$(q^3, q^2, q+1)$ design with intersection numbers 0 and q has spectrum:

$$\left\{[q^3+q^2+q]^1, [q]^{q^3-1}, [-q]^{q^3+q^2+q}\right\}.$$

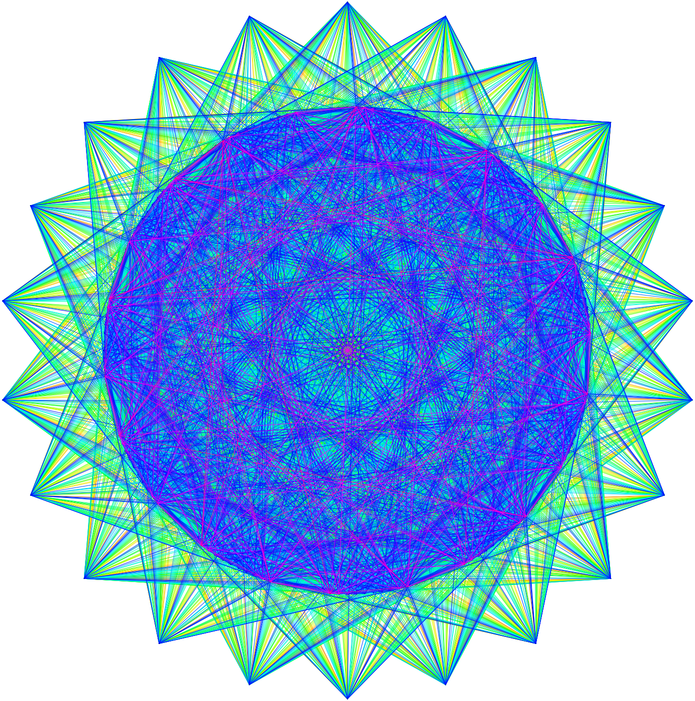
- Case $q = 2$ discovered earlier by Bridges and Mena (1981)

Graphs with coherent rank 9

Theorem (GG and Yip 2025+)
Let Γ be a connected graph w/ three distinct eigenvalues. Then $\mathcal{W}(\Gamma)$ has rank 9 if and only if Γ or $\overline{\Gamma}$ is *a* total graph of certain* quasi-symmetric 2-designs.

- Muzychuk-Klin 1998: quasi-sym 2-$(8, 6, 15)$ design with intersection numbers 4 and 5.
- Van Dam 1998: quasi-sym 2-$(q^3, q^2, q+1)$ designs with intersection numbers 0 and q.

- GG and Yip 2025+: quasi-sym 2-$(22, 15, 80)$ design with intersection numbers 9 and 11.



Graphs with coherent rank 9

Theorem (GG and Yip 2025+)
Let Γ be a connected graph w/ three distinct eigenvalues. Then $\mathcal{W}(\Gamma)$ has rank 9 if and only if Γ or $\overline{\Gamma}$ is *a* total graph of certain* quasi-symmetric 2-designs.

$(v,k,\lambda;x,y)$	Spectrum	Exists
$(76,40,52;24,20)$	$\{[125]^1,[11]^{75},[-5]^{190}\}$	?
$(120,50,35;25,20)$	$\{[153]^1,[9]^{119},[-6]^{204}\}$	?
$(141,45,33;9,15)$	$\{[175]^1,[5]^{329},[-13]^{140}\}$	?
$(121,46,69;16,21)$	$\{[368]^1,[5]^{483},[-23]^{121}\}$	?
$(85,40,130;15,20)$	$\{[224]^1,[4]^{595},[-31]^{84}\}$	?
$(225,36,10;0,6)$	$\{[384]^1,[9]^{224},[-6]^{400}\}$	?
$(120,75,370;50,45)$	$\{[476]^1,[44]^{119},[-6]^{952}\}$	?
$(232,112,296;48,56)$	$\{[539]^1,[7]^{1276},[-41]^{231}\}$	?
$\vdots$	$\vdots$	$\vdots$

Graphs with three valencies

Theorem (GG and Yip 2025+)
Let Γ be connected w/ three distinct eigenvalues and three distinct valencies. Then $\operatorname{rank}(\mathcal{W}(\Gamma)) \geqslant 14$.

Valencies	Spectrum	Coherent rank
$\{[45]^1, [25]^{18}, [13]^{27}\}$	$\{[21]^1, [3]^{19}, [-3]^{26}\}$	16
$\{[15]^4, [10]^{16}, [7]^4\}$	$\{[11]^1, [3]^7, [-2]^{16}\}$	18
$\{[96]^1, [61]^{64}, [21]^{32}\}$	$\{[56]^1, [4]^{41}, [-4]^{55}\}$	20
$\{[24]^{18}, [14]^9, [8]^9\}$	$\{[20]^1, [2]^{17}, [-3]^{18}\}$	29
$\{[24]^{18}, [14]^9, [8]^9\}$	$\{[20]^1, [2]^{17}, [-3]^{18}\}$	240
$\{[35]^1, [26]^7, [19]^{35}\}$	$\left\{[21]^1, [\frac{-1\pm\sqrt{41}}{2}]^{21}\right\}$	949
$\{[35]^1, [26]^7, [19]^{35}\}$	$\left\{[21]^1, [\frac{-1\pm\sqrt{41}}{2}]^{21}\right\}$	1849

Bridges-Mena, Aequationes Math. (1981); Van Dam, JCTB (1998); De Caen-Van Dam-Spence, JCTA (1999)
Cheng-Gavrilyuk-GG-Koolen, European J. Combin. (2016)

Graphs with three valencies

Theorem (GG and Yip 2025+)
Let Γ be connected w/ three distinct eigenvalues and three distinct valencies. Then $\operatorname{rank}(\mathcal{W}(\Gamma)) \geqslant 14$.

$\mathcal{Q}$: quasi-symmetric 2-$(85, 35, 34)$ design with intersection numbers 10 and 15.

Γ: cone over the total graph of $\mathcal{Q}$.

???

Does $\mathcal{Q}$ exist?

Properties of Γ:

- valencies $\{[289]^1, [169]^{85}, [64]^{204}\}$;
- spectrum $\{[119]^1, [4]^{204}, [-11]^{85}\}$;
- coherent rank 14.

Switching strongly regular graphs

Switching: $$\begin{bmatrix} A & B \\ B^\intercal & C \end{bmatrix} \mapsto \begin{bmatrix} A & J-B \\ J^\intercal - B^\intercal & C \end{bmatrix}$$

	$\mathrm{srg}(v,k,\lambda,\mu)$	switched spectrum	rank
Muzychuk-Klin	$(36,14,7,4)$	$\{[21]^1,[5]^7,[-2]^{28}\}$	9
Van Dam	$(176,49,12,14)$	$\{[61]^1,[5]^{97},[-7]^{78}\}$	134
Van Dam	$(126,45,12,18)$	$\{[57]^1,[3]^{89},[-9]^{36}\}$	1222
Van Dam	$(256,105,44,42)$	$\{[121]^1,[9]^{104},[-7]^{151}\}$	2048
Martin	$(105,72,51,45)$	$\{[60]^1,[9]^{21},[-3]^{83}\}$	2893
Van Dam	$(625,288,133,132)$	$\{[313]^1,[13]^{287},[-12]^{337}\}$	15625
Van Dam	$(729,390,207,210)$	$\{[363]^1,[12]^{391},[-15]^{337}\}$	19683

Question: Is arbitrarily large rank possible?

Van Dam, JCTB (1998)
Muzychuk and Klin, Discrete Math (1998)

Switching Latin square graphs

Theorem (GG and Yip 2025+)

For $N = \frac{q^2}{2} - \frac{q\sqrt{3(q^2+2)}}{6}$, switching $\mathcal{L}_{\frac{q^2-1}{2}}(q^2)$ w.r.t. NK_{q^2} results in a graph w/ 3 distinct eigenvalues.

- q an odd prime power $\implies \mathcal{L}_{\frac{q^2-1}{2}}(q^2)$ exists.
- $q = a_k \implies N \in \mathbb{N}$, where: $a_k = 4a_{k-1} - a_{k-2}$ and $a_0 = 1$, $a_1 = 5$.

- Examples: $q = 5, 19, 71, 3691, 1911861, 138907099, \ldots$
- Hone et al. (2018) **conjecture** a_k is prime infinitely often.
- Shorey and Stewart (1983): a_k is a *proper* power for only finitely many k.

Thanks!