Construction of divisible design graphs using affine designs

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Divisible design graphs

A k-regular graph Γ on v vertices is a *divisible design graph* with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if the vertex set of Γ can be partitioned into m classes of size n such that

- any two different vertices from the same class have λ_1 common neighbours,
- any two vertices from different classes have λ_2 common neighbours.

If $m = 1$, $n = 1$, or $\lambda_1 = \lambda_2$, then a divisible design graph is a strongly regular with parameters (v, k, λ_1).

Divisible design graphs

The concept of divisible design graph was introduced by H. Kharaghani.

Divisible design graphs were first studied by W.H. Haemers, H. Kharaghani and M. Meulenberg in 2011.

In particular, the authors have proposed nineteen constructions of divisible design graphs using various combinatorial structures and two sporadic examples.

[HKM] W.H. Haemers, H. Kharaghani, M. Meulenberg, Divisible design graphs, *J. Combinatorial Theory, Series A*, 118 (2011) 978–992.

Divisible design graphs

This class of graphs is particularly interesting for its spectral properties. Just like strongly regular graphs, their eigenvalues can be determined using parameters [Lemma 2.1 HKM].

The spectrum of any divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ can be calculated as follows:

$$\{k^1, \sqrt{k-\lambda_1}^{f_1}, -\sqrt{k-\lambda_1}^{f_2}, \sqrt{k^2-\lambda_2 v}^{g_1}, -\sqrt{k^2-\lambda_2 v}^{g_2}\},$$

where exponents are eigenvalue multiplicities.

For the multiplicities of the eigenvalues, we generally have only the constraints

$$f_1 + f_2 = m(n-1), \quad g_1 + g_2 = m-1.$$

Affine designs

An affine design $\mathcal{D}$ with parameters p and q is a design, where the set of points and the set of blocks have the following two properties:

- (i) every two blocks are either disjoint or intersect in p points;
- (ii) each block together with all blocks disjoint from it forms a parallel class: a set of q mutually disjoint blocks partitioning all points of the design.

Affine designs

If $\mathcal{D}$ is an affine design with parameters p and q, then all parameters of D are expressed in terms of p and q:

q^2p	is the number of points,
qp	is the block size,
$(q^2p-1)/(q-1)$	is the number of parallel classes.

Any d-dimensional affine space over a finite field of order q is point-hyperplane design with $p = q^{d-2}$. This design has q^d points and $(q^d - 1)/(q - 1)$ parallel classes.

Other known examples of affine designs are finite Hadamard 3-designs, Desarguesian and non-Desarguesian finite planes.

Constructions of strongly regular graphs using affine designs

In 1971, W.D. Wallis proposed a new construction of strongly regular graphs based on affine designs and a Steiner 2-design.

30 years later D.G. Fon-Der-Flaass found how to modify a partial case of Wallis construction in order to obtain hyperexponentially many strongly regular graphs with the same parameters.

Few years later M. Muzychuk showed how to modify Fon-Der-Flaass ideas in order to cover all the cases of Wallis construction. He replaced the Steiner 2-design in Wallis's original construction with a partial linear space with a strongly regular block graph, and in addition discovered other new strongly regular graphs.

Constructions of strongly regular graphs using affine designs

[WW] W.D. Wallis, Construction of strongly regular graphs using affine designs, *Bull. Austral. Math. Soc.*, 4 (1971), 41–49.

[FF] D.G. Fon-Der-Flaass New prolific constructions of strongly regular graphs,*Adv. Geom.*, 2 (2002), 301–306.

[MM] M. Muzychuk, A generalization of Wallis-Fon-Der-Flaass construction of strongly regular graphs, *J. Algebr. Comb.*, 25 (2007) 169–187.

[KMR] M. Klin, M. Muzychuk and S. Reichard, Jordan Schemes, *Isr. J. Math.* 249 (2022) 309–342.

Main construction

We use affine designs and a symmetric 2-design to construct divisible design graphs.

Let exist a symmetric 2-design $\mathcal{S}$ with parameters (m, κ, λ) and let A be the symmetric incident matrix of this 2-design. This matrix has exactly $[\kappa]$ nonzero entries in each row. Note that the incidence matrix of a symmetric 2-design is usually not necessarily symmetric, but we just need a symmetric matrix.

Let $L = (e(i,j))$ be a symmetric $(m \times m)$-matrix that is obtained from A by replacing a nonzero element in any row of A with an integer from $[\kappa] = \{1, 2, \ldots, \kappa\}$ such that the mapping of the set of nonzero elements for each row of A to $[\kappa]$ is a bijection.

If $m = \kappa$, then $L = (e(i,j))$ can be considered as the Cayley table of a left quasigroup.

Main construction

Recall that m is the number of points in $\mathcal{S}$.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be a set of affine designs with parameters (p, q) which are not necessarily isomorphic.

Moreover, let κ be equal to the number of parallel classes of blocks for each $\mathcal{D}_i$.

Let $\mathcal{D}_i = (\mathcal{P}_i, \mathcal{B}_i)$ and the parallel classes in each $\mathcal{D}_i$ are enumerated by integers from $[\kappa]$.

For any $x \in \mathcal{P}_i$, denote the block in the parallel class $\mathcal{B}_i^j$ containing x by $B_i^j(x)$.

For each pair i, j for which $e(i, j) \neq 0$, select an arbitrary bijection

$$\sigma_{i,j} : \mathcal{B}_i^{e(i,j)} \to \mathcal{B}_j^{e(j,i)}.$$

We require that $\sigma_{i,j} = \sigma_{j,i}^{-1}$ for $i \neq j$.
If $e(i, i) \neq 0$ then $\sigma_{i,i}$ is the identical map on $\mathcal{B}_i^{e(i,i)}$.

This definition is correct because A is a symmetric matrix so $e(i, j) = 0$ if and only if $e(j, i) = 0$.

Main construction

Let Γ be a graph defined as follows:

- The vertex set of Γ is $V(\Gamma) = \bigcup_{i=1}^{m} \mathcal{P}_i$.
- Two different vertices $x \in \mathcal{P}_i$ and $y \in \mathcal{P}_j$ are adjacent in Γ if and only if $e(i,j) \neq 0$, and

$$y \notin \sigma_{i,j}(B_i^{e(i,j)}(x)) \quad \mathit{for} \quad i,j \in [m].$$

Main construction

Theorem (V.K. 2025)

Let a symmetric 2-design $\mathcal{S}$ have parameters (m, κ, λ).

If Γ is a divisible design graph from Main construction, then Γ has parameters:

- $v = q^2pm,$
- $k = \kappa(q^2p - qp),$
- $\lambda_1 = \lambda(q^2p - qp) + (\kappa - \lambda)(q^2p - 2qp),$
- $\lambda_2 = \lambda p(q-1)^2,$
- $n = q^2p.$

The question of mutual isomorphisms of the graphs constructed here is a difficult one and requires further study. However, the number of non-isomorphic graphs is hyperexponential in the number of vertices.

Main construction

Let a symmetric 2-design $\mathcal{S}$ have parameters
$((q^{d+1}-1)/(q-1), (q^d-1)/(q-1), (q^{d-1}-1)/(q-1))$,
$\mathcal{D}_i$ be the point-hyperplane designs of d-dimensional affine space over a finite field of order q, where $p = q^{d-2}$.
Then Γ has parameters:

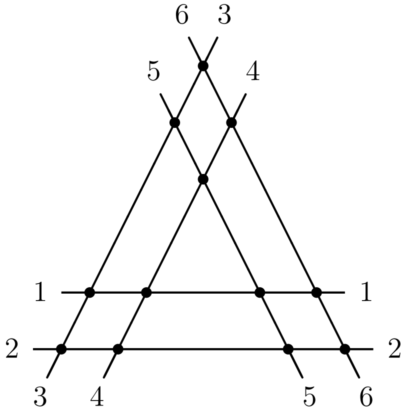
- $v = q^d(q^{d+1}-1)/(q-1)$,
- $k = q^{d-1}(q^d-1)$,
- $\lambda_1 = q^{d-1}(q^d-q^{d-1}-1)$,
- $\lambda_2 = q^{d-2}(q-1)(q^{d-1}-1)$,
- $m = (q^{d+1}-1)/(q-1)$,
- $n = q^d$.

If d is odd and greater than 1, then these parameters match one of the constructions in [BG].

[BG] A. Bhowmik, S. Goryainov, Divisible design graphs from symplectic graphs over rings with precisely three ideals, arXiv:2412.04962v1 [math.CO]

Small example 1

The smallest example of a divisible design graph from Main construction has parameters $(12, 6, 2, 3, 3, 4)$. In this case $m = 3$ so we use 3 copies of finite plane over F_2. This divisible design graph was known in Construction 4.20 [HKM] as sporadic example. This is the line graph of the octahedron.



Small example 2

The smallest example of a divisible design graph from Main construction with nontrivial a symmetric 2-design has parameters $(28, 6, 2, 1, 7, 4)$.

This example based on the Fano plane which can be considered as a symmetric $2-(7, 3, 1)$-design. Since in this case $m = 7$ and $\kappa = 3$, then we can use 7 copies of finite plane over F_2.

This graph was previously found by D.I. Panasenko and L.V. Shalaginov in Construction 23 [PSh] using computer calculations.

[PSh] D.I. Panasenko, L.V. Shalaginov, Classification of divisible design graphs with at most 39 vertices, *J. Comb. Des.*, 30(4) (2022) 205–219.

First partial case of the Main construction

If a symmetric 2-design $\mathcal{S}$ contains only one block, then $m = \kappa$.
Such situation is described in Theorem 1 [VK].
In this case Γ has parameters

$$v = q^d(q^d-1)/(q-1), \quad k = q^{d-1}(q^d-1),$$

$$\lambda_1 = q^{d-1}(q^d - q^{d-1} - 1), \quad \lambda_2 = q^{d-2}(q-1)(q^d-1),$$
$$m = (q^d-1)/(q-1), \quad n = q^d.$$

and four distinct eigenvalues $\{q^{d-1}(q^d-1), q^{d-1}, 0, -q^{d-1}\}$.

[VK] V.V. Kabanov, New versions of the Wallis-Fon-Der-Flaass construction to create divisible design graphs, *Discrete Mathematics*, 345(11) (2022) Article ID 113054.

Second partial case of the Main construction

If $m = \kappa + 1$ and $\lambda = \kappa - 1$, then we have the situation from Theorem 3 [VK] and Γ has parameters

$$v = q^d(q^d + q - 2)/(q - 1), \quad k = q^{d-1}(q^d - 1),$$

$$\lambda_1 = q^{d-1}(q^d - q^{d-1} - 1), \quad \lambda_2 = q^{d-1}(q - 1)(q^{d-1} - 1),$$

$$m = (q^d + q - 2)/(q - 1), \quad n = q^d$$

and five distinct eigenvalues

$$\{q^{d-1}(q^d - 1), \pm(q - 1)q^{d-1}, \pm q^{d-1}\}.$$

If $q = 2$, then Γ is a strongly regular graph for $d > 1$.

[VK] V.V. Kabanov, New versions of the Wallis-Fon-Der-Flaass construction to create divisible design graphs, *Discrete Mathematics*, 345(11) (2022) Article ID 113054.

Cocliques in strongly regular graphs

Let Γ be a strongly regular graph with parameters (v, k, λ, μ), and eigenvalues k, r, s, where $r > 0$ and $s < 0$.

The coclique number in strongly regular graph satisfies the well-known Delsarte-Hoffman bound.

(i) If C is a coclique in Γ, then $|C| \leq v/(1 + \frac{k}{-s})$.

(ii) If the equality holds, then C is called a *Hoffman coclique*.

Strongly regular decomposition

If a strongly regular graph has a Hoffman coclique C, then there is a special case of regular decomposition and the induced subgraph on $V(\Gamma) \setminus C$ has at most four eigenvalues.

W.H. Haemers and D.G. Higman studied the case where the induced subgraph on $V(\Gamma) \setminus C$ has three distinct eigenvalues, which means that a strongly regular graph admits a decomposition into a strongly regular graph and a Hoffman coclique.

[HH] W.H. Haemers and D.G. Higman, Strongly Regular Graphs with Strongly Regular Decomposition, Linear Algebra Appl., 114–115, (1989) 379–398.

Construction of strongly regular graphs with Hoffman coclique

Strongly regular graphs that can be decomposed into a divisible design graph and a Hoffman coclique were found in [VK1].

Namely, a prolific construction of strongly regular graphs with parameters of the complement of the symplectic graph were provided by using the first partial case of our Main construction of divisible design graphs.

[VK1] V.V. Kabanov, A new construction of strongly regular graphs with parameters of the complement symplectic graph, Electron. J. Comb., 30(1) (2023) # P1.25

Are there divisible design graphs other than the first partial case of our Main construction that can be used for this purpose?

Theorem [GK]

Let Γ be a primitive strongly regular graph with parameters (v, k, λ, μ) and the least eigenvalue s. Suppose that Γ contains a Hoffman coclique C and the subgraph Δ induced on $\Gamma \setminus C$ is a divisible design graph. Then there exists n such that the parameters of Γ satisfy: $v = \frac{(-s)(n^2-1)}{n+s}$, $k = (-s)n$, $\lambda = \mu = (-s)(n+s)$ and the parameters of Δ are follows: $(n(-s)(n-1)/(n+s), (-s)(n-1), (-s)(n+s-1), (-s)(n-1)(n+s)/n, (-s)(n-1)/(n+s), n)$. Moreover, if Δ exists, then Γ exists.

[GK] A.L. Gavrilyuk, V.V. Kabanov, Strongly regular graphs decomposable into a divisible design graph and a Hoffman coclique. Des. Codes Cryptogr. 92(5), (2024) 1379–1391.

Strongly regular graphs with Hoffman coclique

There exist a strongly regular graph and a divisible design graph with the parameters as in the above theorem when $-s$ is any prime power by [VK1].

If s is equal to -6, then the parameters of Δ and Γ are $(252, 210, 174, 175; 7, 36)$ and $(259, 216, 180, 180)$, respectively. The existence of both graphs is unknown. This brings us to the problem of 36 officers.

If there were an affine plane with 36 points, we could use our construction. We already know that an affine plane of order 6 does not exist. However, if a divisible design graph with parameters $(252, 210, 174, 175, 7, 36)$ exist, then a strongly regular graph with parameters $(259, 216, 180, 180)$ exist as well.

Thank you for your attantion