Bent partitions, vectorial dual-bent functions, and LP-packings

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- Bent functions and bent partitions

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- Secondary constructions of bent partitions, vectorial dual-bent functions and LP-packings

Definition: Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_m^{(p)}$ be a function. The Walsh transform of F is the complex valued function

$$\mathcal{W}_F(a,b) = \sum_{x \in \mathbb{V}_n^{(p)}} \epsilon_p^{\langle a, F(x) \rangle_m - \langle b, x \rangle_n}, \quad \epsilon_p = e^{2\pi i/p},$$

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where $\langle , \rangle_k$ denotes a non-degenerate inner product in $\mathbb{V}_k^{(p)}$.
If $\mathbb{V}_k^{(p)} = \mathbb{F}_p^k$, one may take the conventional dot product.
If $\mathbb{V}_k^{(p)} = \mathbb{F}_{p^k}$, the standard inner product is $\langle b, x\rangle_k = \mathrm{Tr}_1^k(bx)$, where Tr_r^k denotes the trace function from $\mathbb{F}_{p^k}$ to $\mathbb{F}_{p^r}$.

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A function $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_m^{(p)}$ is called a bent function if $|\mathcal{W}_F(a,b)| = p^{n/2}$ for all nonzero $a \in \mathbb{V}_m^{(p)}$ and $b \in \mathbb{V}_n^{(p)}$.

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If $m = 1$, then F is also called a p-ary bent function. The Walsh transform of a p-ary function $F : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ is of the form

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If $m > 1$, then F is also called a vectorial bent function. The p-ary functions $F_a(x) =< a, F(x) >_m$ for nonzero $a \in \mathbb{V}_m^{(p)}$ are called the component functions of F.

Construction of p-ary bent functions with a complete spread:

Let $n = 2m$. Consider the partition $\Omega = \{U_0, U_1^*, \ldots, U_{p^m}^*\}$ of $\mathbb{V}_n^{(p)}$, where

- $U_i \leq \mathbb{V}_n^{(p)}$ and $\dim(U_i) = m$ for all $0 \leq i \leq p^m$,
- $U_i \cap U_j = \{0\}$ for all $0 \leq i < j \leq p^m$,
- $U_i^* = U_i \setminus \{0\}$, for all $1 \leq i \leq p^m$, (i.e., $\{U_0, U_1, \ldots, U_{p^m}\}$ is a complete spread of $\mathbb{V}_n^{(p)}$).

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One can obtain a bent function from $\mathbb{V}_n^{(p)}$ to $\mathbb{F}_p$ as follows.

I) For every $c \in \mathbb{F}_p$, the elements of exactly p^{m-1} of U_j^*, $1 \leq j \leq p^m$ are mapped to c.

II) The elements of U_0 are mapped to a fixed $c_0 \in \mathbb{F}_p$.

Definition: Let $\circ$ be a binary operation on an m-dimensional vector space $\mathbb{V}_m^{(p)}$, without loss of generality $\mathbb{F}_{p^m}$, satisfying

I) $x \circ y = 0 \Rightarrow x = 0$ or $y = 0$,

II) $(x + y) \circ s = (x \circ s) + (y \circ s)$ and $s \circ (x + y) = (s \circ x) + (s \circ y)$,

for all $x, y, s \in \mathbb{F}_{p^m}$. Then $P = (\mathbb{F}_{p^m}, +, \circ)$ is called a presemifield.

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Let $n = 2m$ and $\mathbb{V}_n^{(p)} = \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$. Consider $\{U, U_s : s \in \mathbb{F}_{p^m}\}$, where $U_s = \{(x, s \circ x) : x \in \mathbb{F}_{p^m}\}$ and $U = \{(0, y) : y \in \mathbb{F}_{p^m}\}$.
Then $\{U, U_s : s \in \mathbb{F}_{p^m}\}$ is the semifield spread.

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If the semifield is the finite field, i.e., $s \circ x = sx$, then $\{U, U_s : s \in \mathbb{F}_{p^m}\}$ is the Desarguesian spread.

Recall: Construction of p-ary bent functions with a complete spread

Let $n = 2m$. Consider the partition $\Omega = \{U, U_1^*, \dots, U_{p^m}^*\}$ of $\mathbb{V}_n^{(p)}$. One can obtain a bent function from $\mathbb{V}_n^{(p)}$ to $\mathbb{F}_p$ as follows.

I) For every $c \in \mathbb{F}_p$, the elements of exactly p^{m-1} of U_j^*, $1 \leq j \leq p^m$ are mapped to c.

II) The elements of U are mapped to a fixed $c_0 \in \mathbb{F}_p$.

Definition (Anbar, Meidl, 2022): Let $n = 2m$, $U \leq \mathbb{V}_n^{(p)}$ and $\dim(U) = m$. A partition $\Omega = \{U, A_1, \dots, A_K\}$ of $\mathbb{V}_n^{(p)}$ is called a normal bent partition of depth K if every function from $\mathbb{V}_n^{(p)}$ to $\mathbb{F}_p$ with the following properties, is a bent function.

I) Every $c \in \mathbb{F}_p$ has exactly K/p of the sets $A_1, \dots, A_K$ in its preimage set $f^{-1}(c) = \{x \in \mathbb{V}_n^{(p)} : f(x) = c\}$,

II) $f(x) = c_0$ for all $x \in U$ and some fixed $c_0 \in \mathbb{F}_p$.

Generalized semifield spreads

Given a (pre)semifield $P = (\mathbb{F}_{p^m}, +, \circ)$, consider the (pre)semifield $P^d = (\mathbb{F}_{p^m}, +, \star)$ obtained by defining $x \star y$ with the equation

$$\mathrm{Tr}_1^m(x(b \star y)) = \mathrm{Tr}_1^m(b(x \circ y)) \text{ for all } b, x, y \in \mathbb{F}_{p^m}.$$

Then P^d is called the dual of P.

Let $P = (\mathbb{F}_{p^m}, +, \circ)$ be a (pre)semifield, m, $k, l \in \mathbb{Z}^+$ such that $k \mid m$, $e \equiv p^l \bmod (p^k - 1)$, $\gcd(p^m - 1, e) = 1$. Consider the following partition of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.

$$\Omega_1 = \{U, \mathcal{A}(\gamma) : \gamma \in \mathbb{F}_{p^k}\}$$

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$$\mathcal{A}(\gamma) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \gamma} U_s^*, \qquad U_s = \{(x, s \circ x^e) : x \in \mathbb{F}_{p^m}\},$$

$$U = \{(0, y) : y \in \mathbb{F}_{p^m}\}, \qquad U_s^* = U_s \setminus \{(0,0)\},$$

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Theorem (Anbar, K., Meidl, 2023): Suppose that $P = (\mathbb{F}_{p^m}, +, \circ)$ is a (pre)semifield such that the dual $P^d = (\mathbb{F}_{p^m}, +, \star)$ satisfies

$$x \star (cy) = c(x \star y) \quad \text{for all } x, y \in \mathbb{F}_{p^m}, c \in \mathbb{F}_{p^k},$$

(i.e., P^d is right $\mathbb{F}_{p^k}$-linear).
Then Ω_1 is a bent partition of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.

Let $P = (\mathbb{F}_{p^m}, +, \circ)$ be a (pre)semifield, m, $k, l \in \mathbb{Z}^+$ such that $k \mid m$, $e \equiv p^l \mod (p^k - 1)$, $\gcd(p^m - 1, e) = 1$. and d satisfies $de \equiv 1 \mod (p^m - 1)$. Consider the following partition of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.

$$\Omega_2 = \{V, \mathcal{B}(\gamma) : \gamma \in \mathbb{F}_{p^k}\}$$

$$\mathcal{B}(\gamma) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \gamma} V_s^*, \qquad V_s = \{(s \circ x^d, x) : x \in \mathbb{F}_{p^m}\},$$

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(i.e., P^d is right $\mathbb{F}_{p^k}$-linear).
Then Ω_2 is a bent partition of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.

Recall: Let $P = (\mathbb{F}_{p^m}, +, \circ)$ be a (pre)semifield, m, $k, l \in \mathbb{Z}^+$ such that $k \mid m$, $e \equiv p^l \mod (p^k - 1)$, $\gcd(p^m - 1, e) = 1$, and $de \equiv 1 \mod (p^m - 1)$.

- $\Omega_1 = \{U, \mathcal{A}(\gamma) : \gamma \in \mathbb{F}_{p^k}\}$, where $\mathcal{A}(\gamma) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \gamma} U_s^*$, $U_s = \{(x, s \circ x^e) : x \in \mathbb{F}_{p^m}\}$, $U_s^* = U_s \setminus \{(0,0)\}$, $U = \{(0, y) : y \in \mathbb{F}_{p^m}\}$,
- $\Omega_2 = \{V, \mathcal{B}(\gamma) : \gamma \in \mathbb{F}_{p^k}\}$, where $\mathcal{B}(\gamma) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \gamma} V_s^*$, $V_s^* = V_s \setminus \{(0,0)\}$, $V_s = \{(s \circ x^d, x) : x \in \mathbb{F}_{p^m}\}$, $V = \{(x, 0) : x \in \mathbb{F}_{p^m}\}$.

Remark: The semifield spread $\{U, U_s : s \in \mathbb{F}_{p^m}\}$, where

$$U = \{(0, y) : y \in \mathbb{F}_{p^m}\} \text{ and } U_s = \{(x, s \circ x) : x \in \mathbb{F}_{p^m}\},$$

can be obtained from the partitions Ω_1 and Ω_2 by taking $k = m$.

Recall: Let $P = (\mathbb{F}_{p^m}, +, \circ)$ be a (pre)semifield, m, $k, l \in \mathbb{Z}^+$ such that $k \mid m$, $e \equiv p^l \mod (p^k - 1)$, $\gcd(p^m - 1, e) = 1$, and $de \equiv 1 \mod (p^m - 1)$.

- $\Omega_1 = \{U, \mathcal{A}(\gamma) : \gamma \in \mathbb{F}_{p^k}\}$, where $\mathcal{A}(\gamma) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \gamma} U_s^*$, $U_s = \{(x, s \circ x^e) : x \in \mathbb{F}_{p^m}\}$, $U_s^* = U_s \setminus \{(0,0)\}$, $U = \{(0, y) : y \in \mathbb{F}_{p^m}\}$,
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Remark: The semifield spread $\{U, U_s : s \in \mathbb{F}_{p^m}\}$, where

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can be obtained from the partitions Ω_1 and Ω_2 by taking $k = m$.

Proposition (Anbar, Meidl, 2022): Let $k < m$ and consider the bent partitions Ω_1 and Ω_2 constructed with the finite field operation. Then the functions obtained from Ω_1 and Ω_2 can not be obtained from any spread.

Recall: If $F : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ is a p-ary bent function, then

$$|\mathcal{W}_F(b)| = |\sum_{x \in \mathbb{V}_n^{(p)}} \epsilon_p^{F(x) - \langle b, x \rangle_n}| = p^{n/2},$$

for all $b \in \mathbb{V}_n^{(p)}$.

The Walsh transform of a p-ary bent function $F : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ satisfies

$$\mathcal{W}_F(b) = \zeta p^{n/2} \epsilon_p^{F^*(b)}$$

for a function $F^*\colon \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ and for some $\zeta \in \{\pm 1, \pm i\}$ for all $b \in \mathbb{V}_n^{(p)}$. The function F^* is called the dual of F.

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If $\mathcal{W}_F(b) = \zeta p^{n/2} \epsilon_p^{F^*(b)}$ for all $b \in \mathbb{V}_n^{(p)}$ for some $\zeta \in \{\pm 1, \pm i\}$, then F is called weakly regular.

Recall: If $F : \mathbb{V}_n^{(p)} \to \mathbb{F}_p$ is a p-ary bent function, then

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If $\zeta = 1$, then F is called regular.

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Proposition: If F is a weakly regular bent function, then F^* is a bent function.

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Question: Is the set of the duals of the bent functions obtained from Ω_1 and Ω_2 obtained from a bent partition?

Recall: Let $P = (\mathbb{F}_{p^m}, +, \circ)$ be a (pre)semifield, $m,\ k, l \in \mathbb{Z}^+$ such that $k \mid m$, $e = p^k + p^l - 1$, $\gcd(p^m - 1, e) = 1$, and $de \equiv 1 \mod (p^m - 1)$.

- $\Omega_1 = \{U, \mathcal{A}(\gamma) : \gamma \in \mathbb{F}_{p^k}\}$, where $\mathcal{A}(\gamma) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \gamma} U_s^*$,
  $U_s = \{(x, s \circ x^e) : x \in \mathbb{F}_{p^m}\}$, $U_s^* = U_s \setminus \{(0,0)\}$,
  $U = \{(0, y) : y \in \mathbb{F}_{p^m}\}$,
- $\Omega_2 = \{V, \mathcal{B}(\gamma) : \gamma \in \mathbb{F}_{p^k}\}$, where
  $\mathcal{B}(\gamma) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \gamma} V_s^*$, $V_s^* = V_s \setminus \{(0,0)\}$,
  $V_s = \{(s \circ x^d, x) : x \in \mathbb{F}_{p^m}\}$, $V = \{(x, 0) : x \in \mathbb{F}_{p^m}\}$.

Theorem (Anbar, Meidl, 2022): If $P = (\mathbb{F}_{p^m}, +, \circ)$ is the finite field, then the set of the duals of the bent functions obtained from Ω_1 is the set of the duals of the bent functions obtained from Ω_2 and vice versa.

Recall: Suppose that $P = (\mathbb{F}_{p^m}, +, \circ)$ is a (pre)semifield such that the dual $P^d = (\mathbb{F}_{p^m}, +, \star)$ is right $\mathbb{F}_{p^k}$-linear, m, $k, l \in \mathbb{Z}^+$ such that $k \mid m$, $e \equiv p^l \mod (p^k - 1)$, $\gcd(p^m - 1, e) = 1$. and d satisfies $de \equiv 1 \mod (p^m - 1)$.

$$\Omega_1 = \{U, \mathcal{A}(\gamma) : \gamma \in \mathbb{F}_{p^k}\}, \text{where } \mathcal{A}(\gamma) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \gamma} U_s^*,$$
$$U_s = \{(x, s \circ x^e) : x \in \mathbb{F}_{p^m}\}, U_s^* = U_s \setminus \{(0,0)\},$$
$$U = \{(0, y) : y \in \mathbb{F}_{p^m}\},$$

Given $x \in \mathbb{F}_{p^m}^*$, let $\eta_x \in \mathbb{F}_{p^m}$ be the element satisfying $x \star \eta_x^{-1} = 1$ (for convention set $\eta_0 = 0$). Consider the following partition of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.

$$\Theta_1 = \{W, \mathcal{C}(\beta) \ : \ \beta \in \mathbb{F}_{p^k}\}$$

$$\mathcal{C}(\beta) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \beta} W_s^* \qquad W_s = \{(s{\eta_x}^d, x) : x \in \mathbb{F}_{p^m}\},$$
$$W = \{(0, y) : y \in \mathbb{F}_{p^m}\}, \qquad W_s^* = W_s \setminus \{(0,0)\},$$

Theorem (Anbar, K., Meidl, 2023): The partition Θ_1 is a bent partition of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ and the set of the duals of the bent functions obtained from Ω_1 is the set of the duals of the bent functions obtained from Θ_1 and vice versa.

Recall: Suppose that $P = (\mathbb{F}_{p^m}, +, \circ)$ is a (pre)semifield such that the dual $P^d = (\mathbb{F}_{p^m}, +, \star)$ is right $\mathbb{F}_{p^k}$-linear, $m,\ k, l \in \mathbb{Z}^+$ such that $k \mid m$, $e = p^k + p^l - 1$, $\gcd(p^m - 1, e) = 1$.

$$\Omega_2 = \{V, \mathcal{B}(\gamma) \mid \gamma \in \mathbb{F}_{p^k}\}, \text{where}$$
$$\mathcal{B}(\gamma) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \gamma} V_s^*, V_s^* = V_s \setminus \{(0,0)\}$$
$$V_s = \{(s \circ x^d, x) : x \in \mathbb{F}_{p^m}\},\ V = \{(x, 0) : x \in \mathbb{F}_{p^m}\}, .$$

Given $x \in \mathbb{F}_{p^m}^*$, let $\eta_x \in \mathbb{F}_{p^m}$ be the element satisfying $x \star \eta_x^{-1} = 1$ (for convention set $\eta_0 = 0$). Consider the following partition of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.

$$\Theta_2 = \{T, \mathcal{D}(\beta)\ :\ \beta \in \mathbb{F}_{p^k}\},$$

$$\mathcal{D}(\beta) = \bigcup_{s \in \mathbb{F}_{p^m} : \mathrm{Tr}_k^m(s) = \beta} T_s^*, \qquad T_s = \{(x, s\eta_x^e) : x \in \mathbb{F}_{p^m}\},$$
$$T = \{(x, 0) : x \in \mathbb{F}_{p^m}\}, \qquad T_s^* = T_s \setminus \{(0,0)\}.$$

Theorem (Anbar, K., Meidl, 2023): The partition Θ_2 is a bent partition of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ and the set of the duals of the bent functions obtained from Ω_2 is the set of the duals of the bent functions obtained from Θ_2 and vice versa.

Partial difference sets and generalized semifield spreads

Definition:

- Let G be a finite abelian group of order v. A κ-subset D of G is called a $(v, \kappa, \lambda, \mu)$ partial difference set if

  I) Every nonzero element of D can be written as a difference of elements in D in exactly λ ways,

  II) Every nonzero element of $G \setminus D$ can be written as a difference of elements in D in exactly μ ways.

- A partial difference set D is called regular if $0 \notin D$ and $-D = D$.

Lemma (Ma, 1994): Let G be an abelian group of order v. Suppose that D is a κ-subset of G which satisfies $-D = D$ and $0 \notin D$. Then D is a $(v, \kappa, \lambda, \mu)$ partial difference set if and only if for each non-principal character χ of G we have

$$\chi(D) = \frac{\beta \pm \sqrt{\Delta}}{2},$$

where $\beta = \lambda - \mu$, $\delta = \kappa - \mu$ and $\Delta = \beta^2 + 4\delta$.

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Theorem (Anbar, K., Meidl, 2024): Let Ω_1 $(\Omega_2, \Theta_1, \Theta_2)$ be a bent partition. Then any union of sets from $\Omega_1(\Omega_2, \Theta_1, \Theta_2)$ is a regular partial difference set.

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Sketch of Proof: Let $1 \leq r \leq p^k + 1$ and Ψ be a union of r sets of the partition $\Omega_1(\ \Omega_2, \Theta_1, \Theta_2\ )$. The non-principal characters of $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ are $\chi_{u,v}(x, y) = \epsilon_p^{\mathrm{Tr}_m(ux+vy)}$, $(u, v) \neq (0, 0)$.

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- If Ψ contains $U^*(V^*, W^*, T^*)$, then

$$\chi_{u,v}(\Psi) = \begin{cases} p^m - (r-1)p^{m-k} - 1 & \text{or} \\ -(r-1)p^{m-k} - 1 & . \end{cases}$$

Lemma (Ma, 1994): Let G be an abelian group of order v. Suppose that D is a κ-subset of G which satisfies $-D = D$ and $0 \notin D$. Then D is a $(v, \kappa, \lambda, \mu)$ partial difference set if and only if for each non-principal character χ of G we have

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Theorem (Anbar, K., Meidl, 2024): Let Ω_1 $(\Omega_2, \Theta_1, \Theta_2)$ be a bent partition. Then any union of sets from $\Omega_1(\Omega_2, \Theta_1, \Theta_2)$ is a regular partial difference set.

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$$\chi_{u,v}(\Psi) = \begin{cases} p^m - (r-1)p^{m-k} - 1 & \text{or} \\ -(r-1)p^{m-k} - 1 & . \end{cases}$$

- If Ψ does not contain $U^*(V^*, W^*, T^*)$, then

$$\chi_{u,v}(\Psi) = \begin{cases} p^m - rp^{m-k} & \text{or} \\ -rp^{m-k} & . \end{cases}$$

Vectorial dual-bent functions

Recall: Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_m^{(p)}$ be a vectorial bent function. Then the component functions $F_a(x) =< a, F(x) >_m$ of F are p-ary bent functions for all $a \in \mathbb{V}_m^{(p)} \setminus \{0\}$.

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$$\{F_a : a \in \mathbb{V}_m^{(p)} \setminus \{0\}\} = \{\langle a, F\rangle_m : a \in \mathbb{V}_m^{(p)} \setminus \{0\}\}$$

can be seen as an m-dimensional vector space of bent functions over $\mathbb{F}_p$ together with the 0-function.

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$$\{F_a^* : a \in \mathbb{V}_m^{(p)} \setminus \{0\}\} = \{\langle a, F\rangle_m^* : a \in \mathbb{V}_m^{(p)} \setminus \{0\}\}$$

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In general

$$\langle a_1, F\rangle_m^* + \langle a_2, F\rangle_m^* \neq \langle a_1 + a_2, F\rangle_m^*.$$

Recall: Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_m^{(p)}$ be a vectorial bent function. Then the component functions $F_a(x) =< a, F(x) >_m$ of F are p-ary bent functions for all $a \in \mathbb{V}_m^{(p)} \setminus \{0\}$. In this case, the set

$$\{F_a : a \in \mathbb{V}_m^{(p)} \setminus \{0\}\} = \{\langle a, F \rangle_m : a \in \mathbb{V}_m^{(p)} \setminus \{0\}\}$$

can be seen as an m-dimensional vector space of bent functions over $\mathbb{F}_p$ together with the 0-function.

Definition (Çeşmelioğlu, Meidl, Pott, 2018): A vectorial bent function $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_m^{(p)}$ is called vectorial dual-bent if the set

$$\{(F_a)^* : a \in \mathbb{V}_m^{(p)} \setminus \{0\}\} = \{\langle a, F \rangle_m^* : a \in \mathbb{V}_m^{(p)} \setminus \{0\}\}$$

of the duals of the component functions of F also forms an m-dimensional vector space of bent functions over $\mathbb{F}_p$ together with the 0-function.

Definition (Wang, Fu, Wei, 2023): Let k, n be positive integers with n even, $k \leq n/2$.

A vectorial dual-bent function $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$ satisfies Condition A if

I) every component function $F_\alpha = \langle \alpha, F \rangle_k$, $\alpha \in \mathbb{V}_k^{(p)} \setminus \{0\}$, of F is regular (Type I), or
every component function F_α is weakly regular but not regular (Type II),

II) For all $\alpha, \beta \in \mathbb{V}_k^{(p)} \setminus \{0\}$ with $\alpha + \beta \neq 0$,
$(F_\alpha)^* + (F_\beta)^* = (F_\alpha + F_\beta)^*$

A bent partition $\Omega = \{A_c \ : \ c \in \mathbb{V}_k^{(p)}\}$ of $\mathbb{V}_n^{(p)}$ of depth p^k satisfies Condition C if

I) $\mathbb{F}_p^* A_c = A_c$ for all $c \in \mathbb{V}_k^{(p)}$,

II) every bent function which is obtained from Ω is regular (Type I), or
every bent function which is obtained from Ω is weakly regular but not regular (Type II).

Theorem (Wang, Fu, Wei, 2023): Let k, n be positive integers with n even, $k \leq n/2$.
Let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$, and D_F^c be the preimage set of c for every $c \in \mathbb{V}_k^{(p)}$.

I) If p is odd, then the following are equivalent.
   i) F is a vectorial dual-bent function satisfying Condition A.
   ii) $\Omega = \{D_F^c \ : \ c \in \mathbb{V}_k^{(p)}\}$ is a bent partition of $\mathbb{V}_n^{(p)}$ of depth p^k satisfying Condition C.

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II) If $p = 2$, then (i) implies (ii).

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II) If $p = 2$, then (i) implies (ii).

Remark: The converse of the statement does not hold for $p = 2$.

Latin square type partial difference set packings

- A partial difference set is called (n, s) Latin square type if $(v, \kappa, \lambda, \mu) = (n^2, s(n-1), n+s^2-3s, s^2-s)$.

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Definition (Jedwab, Li, 2022): Let

- $t > 1$ and $c > 0$ be integers,
- G be an abelian group, $|G| = t^2c^2$,
- $U \leq G$, $|U| = tc$.

A (c, t) LP-packing in G relative to U is a collection $\{P_1, \ldots, P_t\}$ of subsets of G such that

I) P_i are pairwise disjoint regular (tc, c) Latin square type partial difference sets in G for $1 \leq i \leq t$,

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II) $\bigcup_{i=1}^{t} P_i = G \setminus U$.

Example: Let $\{U_0, U_1, \ldots, U_{p^m}\}$ be a spread of $\mathbb{V}_{2m}^{(p)}$ and $U_i^* = U_i \setminus \{0\}$. Then $\{U_1^*, \ldots, U_{p^m}^*\}$ is a $(1, p^m)$ LP-packing in $\mathbb{V}_{2m}^{(p)}$ relative to U_0.

Proposition (Alkan, Anbar, K., Meidl, 2025): For an odd prime p, an even integer n and an integer $k \leq n/2$ with $p^k > 3$, let $F : \mathbb{V}_n^{(p)} \to \mathbb{V}_k^{(p)}$, and for every $c \in \mathbb{V}_k^{(p)}$, let D_F^c be the preimage set of c. Then the following are equivalent.

I) F is a vectorial dual-bent function satisfying Condition A, which is constant c_0 on an $(n/2)$-dimensional subspace U.

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I) F is a vectorial dual-bent function satisfying Condition A, which is constant c_0 on an $(n/2)$-dimensional subspace U.

II) $\Omega_F = \{U, D_F^{c_0} \setminus U, D_F^c \ : \ c \in \mathbb{V}_k^{(p)}, c \neq c_0\}$ is a normal bent partition of $\mathbb{V}_n^{(p)}$ with $\mathbb{F}_p^* D_F^c = D_F^c$, $c \in \mathbb{V}_k^{(p)}$, and $\mathbb{F}_p^*(D_F^{c_0} \setminus U) = D_F^{c_0} \setminus U$.

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III) $\Omega_F = \{U, D_F^{c_0} \setminus U, D_F^c \ : \ c \in \mathbb{V}_k^{(p)}, c \neq c_0\}$ induces a $(p^{n/2}, p^{n/2-k})$ LP-packing in $\mathbb{V}_n^{(p)}$ relative to U with $\mathbb{F}_p^* D_F^c = D_F^c$, $c \in \mathbb{V}_k^{(p)}$, and $\mathbb{F}_p^*(D_F^{c_0} \setminus U) = D_F^{c_0} \setminus U$.

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I) F is a vectorial dual-bent function satisfying Condition A, which is constant c_0 on an $(n/2)$-dimensional subspace U.

II) $\Omega_F = \{U, D_F^{c_0} \setminus U, D_F^c \ : \ c \in \mathbb{V}_k^{(p)}, c \neq c_0\}$ is a normal bent partition of $\mathbb{V}_n^{(p)}$ with $\mathbb{F}_p^* D_F^c = D_F^c$, $c \in \mathbb{V}_k^{(p)}$, and $\mathbb{F}_p^*(D_F^{c_0} \setminus U) = D_F^{c_0} \setminus U$.

III) $\Omega_F = \{U, D_F^{c_0} \setminus U, D_F^c \ : \ c \in \mathbb{V}_k^{(p)}, c \neq c_0\}$ induces a $(p^{n/2}, p^{n/2-k})$ LP-packing in $\mathbb{V}_n^{(p)}$ relative to U with $\mathbb{F}_p^* D_F^c = D_F^c$, $c \in \mathbb{V}_k^{(p)}$, and $\mathbb{F}_p^*(D_F^{c_0} \setminus U) = D_F^{c_0} \setminus U$.

IV) $\Omega_F = \{U, D_F^{c_0} \setminus U, D_F^c \ : \ c \in \mathbb{V}_k^{(p)}, c \neq c_0\}$ induces a $(p^k + 1)$-class amorphic association scheme on $\mathbb{V}_n^{(p)}$.

Direct sum constructions

Theorem (Alkan, Anbar, K., Meidl, 2025): Let $F : \mathbb{V}_n^{(p)} \to \mathbb{F}_{p^k}$ and $G : \mathbb{V}_s^{(p)} \to \mathbb{F}_{p^k}$ be vectorial dual-bent functions satisfying Condition A. Consider $H : \mathbb{V}_n^{(p)} \times \mathbb{V}_s^{(p)} \to \mathbb{F}_{p^k}$ given by $H(x, y) = F(x) + G(y)$. Then

- H is a vectorial dual-bent function satisfying Condition A,
- $\Omega_H = \{C_j : j \in \mathbb{F}_{p^k}\}$ is a bent partition of $\mathbb{V}_n^{(p)} \times \mathbb{V}_s^{(p)}$ satisfying Condition C,

$$C_j = \bigcup_{i \in \mathbb{F}_{p^k}} (A_i \times B_{j-i}),$$

$\Omega_F = \{A_i : i \in \mathbb{F}_{p^k}\}$, $\Omega_G = \{B_i : i \in \mathbb{F}_{p^k}\}$ are bent partitions of $\mathbb{V}_n^{(p)}$, $\mathbb{V}_s^{(p)}$ satisfying Condition C and the indices are determined in $\mathbb{F}_{p^k}$.

Theorem (Alkan, Anbar, K., Meidl, 2025) Let $F : \mathbb{V}_n^{(p)} \to \mathbb{F}_{p^k}$, $G : \mathbb{V}_s^{(p)} \to \mathbb{F}_{p^k}$ be vectorial dual-bent functions satisfying Condition A.

- $F(x) = \beta$ if $x \in U_1 \leq \mathbb{V}_n^{(p)}$, $\dim(U_1) = n/2$.
- $G(x) = \gamma$ if $x \in U_2 \leq \mathbb{V}_s^{(p)}$, $\dim(U_2) = s/2$.

Consider $H : \mathbb{V}_n^{(p)} \times \mathbb{V}_s^{(p)} \to \mathbb{F}_{p^k}$ given by $H(x, y) = F(x) + G(y)$.

- H is a vectorial dual-bent function satisfying Condition A,
- $H(x, y) = \gamma + \beta$ for $(x, y) \in U_1 \times U_2$,
- $\Omega_H = \{U_1 \times U_2, C_j \, : \, j \in \mathbb{F}_{p^k}\}$ is a normal bent partition of $\mathbb{V}_n^{(p)} \times \mathbb{V}_s^{(p)}$,

$$C_j = (A_{j-\gamma} \times U_2) \cup (U_1 \times B_{j-\beta}) \cup \bigcup_{i \in \mathbb{F}_{p^k}} (A_i \times B_{j-i}),$$

$\Omega_F = \{U_1, A_i \, : \, i \in \mathbb{F}_{p^k}\}$, $\Omega_G = \{U_2, B_i \, : \, i \in \mathbb{F}_{p^k}\}$ are the normal bent partitions of $\mathbb{V}_n^{(p)}$ and $\mathbb{V}_s^{(p)}$ obtained from F and G, respectively, and the indices are determined in $\mathbb{F}_{p^k}$.

Remark: The normal bent partition Ω_H gives rise to a $(p^{(n+s)/2-k}, p^k)$ LP-packing in $\mathbb{V}_n \times \mathbb{V}_s$ relative to $U_1 \times U_2$, which is a direct sum of the $(p^{n/2-k}, p^k)$ LP-packing in $\mathbb{V}_n^{(p)}$ relative to U_1 and $(p^{s/2-k}, p^k)$ LP-packing in $\mathbb{V}_s^{(p)}$ relative to U_2, obtained from normal bent partitions Ω_F and Ω_G, respectively.

Generalized Maiorana-McFarland construction

Theorem (Çeşmelioğlu, Meidl, Pott, 2013)
Let s and n be positive integers with $s < n$, and let $\{F^{(z)} : \mathbb{F}_{p^n} \to \mathbb{F}_p \mid z \in \mathbb{F}_{p^s}\}$ be a set of p-ary bent functions. Consider the function $H : \mathbb{F}_{p^n} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s} \to \mathbb{F}_p$ given by

$$H(x, y, z) = F^{(z)}(x) + \mathrm{Tr}_1^s(yz).$$

Then H is a p-ary bent function.
Theorem (Alkan, Anbar, K., Meidl, 2025):
Let n, s be an integers, $s < n$ and let k be a divisor of s. Let $\{F^{(z)} : \mathbb{F}_{p^n} \to \mathbb{F}_{p^k} \mid z \in \mathbb{F}_{p^s}\}$ be a set of vectorial bent functions. Let $H : \mathbb{F}_{p^n} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s} \to \mathbb{F}_{p^k}$ defined by

$$H(x, y, z) = F^{(z)}(x) + \mathrm{Tr}_k^s(y\pi(z)),$$

where π is a permutation of $\mathbb{F}_{p^s}$ with $\pi(0) = 0$.
Then H is a vectorial bent function.

Recall A vectorial generalized Maiorana-McFarland function $H : \mathbb{F}_{p^n} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s} \to \mathbb{F}_{p^k}$ is given by

$$H(x, y, z) = F^{(z)}(x) + \mathrm{Tr}_k^s(y\pi(z)),$$

where π is a permutation of $\mathbb{F}_{p^s}$ with $\pi(0) = 0$.

Corollary (Alkan, Anbar, K., Meidl, 2025):

Let n be even and $H : \mathbb{F}_{p^n} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s} \to \mathbb{F}_{p^k}$ be a vectorial generalized Maiorana-McFarland bent function. Then the following statements hold.

I) Suppose that the functions $F^{(z)}$ are vectorial dual-bent functions of the same type that satisfy Condition A and
- $F^{(\alpha z)} = F^{(z)}$,
- $\pi^{-1}(z/\alpha) = \alpha\,\pi^{-1}(z)$ for all nonzero $\alpha \in \mathbb{F}_{p^k}$, $z \in \mathbb{F}_{p^s}$.

Then
- H is a vectorial dual-bent function satisfying Condition A.
- $\Omega_H = \{C_j : j \in \mathbb{F}_{p^k}\}$ is a bent partition of $\mathbb{F}_{p^n} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s}$ satisfying Condition C.

Recall A vectorial generalized Maiorana-McFarland function $H : \mathbb{F}_{p^n} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s} \to \mathbb{F}_{p^k}$ is given by

$$H(x, y, z) = F^{(z)}(x) + \mathrm{Tr}_k^s(y\pi(z)),$$

where π is a permutation of $\mathbb{F}_{p^s}$ with $\pi(0) = 0$.

II) Suppose that the functions $F^{(z)}$ are vectorial dual-bent functions of the same type that satisfy Condition A and

- $F^{(\alpha z)} = F^{(z)}$ for all $z \in \mathbb{F}_{p^k}$,
- $\pi^{-1}(z/\alpha) = \alpha\,\pi^{-1}(z)$ for all nonzero $\alpha \in \mathbb{F}_{p^s}$,

and

- $F^{(0)}(x) = 0$ for all $x \in U$, where $U \leq \mathbb{V}_n^{(p)}$, $\dim(U) = n/2$.

Then

- $H(x, y, z) = 0$ for all $(x, y, z) \in U \times \mathbb{F}_{p^s} \times \{0\} \leq \mathbb{F}_{p^n} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s}$, $\dim(U \times \mathbb{F}_{p^s} \times \{0\}) = (n/2 + s)$,
- Ω_H is a normal bent partition of $\mathbb{F}_{p^n} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s}$,
- Ω_H gives rise to a $(p^{n/2+s-k}, p^k)$ LP-packing in $\mathbb{F}_{p^n} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s}$ relative to the subspace $U \times \mathbb{F}_{p^s} \times \{0\}$.

A secondary construction by Wang, Fu, Wei

Theorem (Wang, Fu, Wei, 2023): Let m, n be integers, n even, $m \leq n/2$ and let $k < m$ be a divisor of m. For every $i \in \mathbb{F}_{p^k}$, let $F(i; x)$ be a vectorial dual-bent function from $\mathbb{V}_n^{(p)}$ to $\mathbb{F}_{p^k}$ satisfying Condition A, and suppose that all $F(i; x)$ are of the same type. Let $\alpha, \beta \in \mathbb{F}_{p^m}$ be linearly independent over $\mathbb{F}_{p^k}$, let R be a permutation of $\mathbb{F}_{p^m}$ with $R(0) = 0$. Then $H : \mathbb{V}_n^{(p)} \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$

$$H(x, y, z) = F(\mathrm{Tr}_k^m(\alpha R(yz^{p^m-2})); x) + \mathrm{Tr}_k^m(\beta R(yz^{p^m-2}))$$

is a vectorial dual-bent function satisfying Condition A.

Theorem (Alkan, Anbar, K., Meidl, 2025): Let n be an even integer, m, k be integers with $s \leq m \leq n/2$, and let k be a divisor of s. For every $i \in \mathbb{F}_{p^k}$, let $F(i;x)$ be a vectorial dual-bent function from $\mathbb{V}_n^{(p)}$ to $\mathbb{F}_{p^k}$ satisfying Condition A, and suppose that all $F(i;x)$ are of the same type. Let $e : \mathbb{V}_{2m}^{(p)} \to \mathbb{F}_{p^s}$ be a vectorial dual-bent function satisfying Condition A, and let $\alpha, \beta \in \mathbb{F}_{p^s}$ be linearly independent over $\mathbb{F}_{p^k}$. Then $H : \mathbb{V}_n^{(p)} \times \mathbb{F}_{p^s} \times \mathbb{F}_{p^s} \to \mathbb{F}_{p^k}$

$$H(x, y) = F(\mathrm{Tr}_k^s(\alpha e(y)); x) + \mathrm{Tr}_k^s(\beta e(y))$$

is a vectorial dual-bent function satisfying Condition A.

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Theorem (Anbar, Fu, K., Meidl, Wang, Wei, 2025):
Let k, m be positive integers with $k \mid m$, $k \geq 2$, a, b be integers with $a \equiv b \equiv p^l \mod (p^k - 1)$ and $\gcd(a, p^m - 1) = \gcd(b, p^m - 1) = 1$.
Let $M : \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ with $M(cx) = cM(x), c \in \mathbb{F}_{p^k}^*$.
Consider $F : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ be given by

$$F(x, y) = \mathrm{Tr}_k^m(yx^{-a}) + M(x^{-b}).$$

Let D_F^c be the preimage set of $c \in \mathbb{F}_{p^k}$ and $U = \{(0, y) : y \in \mathbb{F}_{p^m}\}$. Then

I) $\Omega_F = \{U, D_F^0 \setminus U,\ D_F^c : c \in \mathbb{F}_{p^k}^*\}$ is a normal bent partition.

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Remark: The function $G(x, y) = \mathrm{Tr}_k^m(yx^{-a})$ is a vectorial dual-bent function satisfying Condition A, therefore, from the preimage sets one can obtain an LP-packing.

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Let $M : \mathbb{F}_{p^m} \to \mathbb{F}_{p^k}$ with $M(cx) = cM(x), c \in \mathbb{F}_{p^k}^*$.
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Remark: The function $G(x, y) = \mathrm{Tr}_k^m(yx^{-a})$ is a vectorial dual-bent function satisfying Condition A, therefore, from the preimage sets one can obtain an LP-packing.

- The sets in Ω_F are not the shifts of the partial difference sets in Ω_G.
- The p-ary bent functions obtained from Ω_F and the p-ary bent functions obtained from Ω_G are not equivalent.

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Open Problems:

- (Kantor, 1983) The number of pairwise inequivalent bent functions obtained from Desarguesian spread grows exponentially with n.
  - How many pairwise inequivalent bent functions can be obtained from generalized semifield spreads?
- Let Ω_1 and Ω_2 be bent partitions obtained from vectorial bent functions $F, \tilde{F}$ and $G, \tilde{G}$, respectively.
  Consider the vectorial bent functions $H(x, y) = F(x) + G(y)$ and $\tilde{H}(x, y) = \tilde{F}(x) + \tilde{G}(y)$, and the associated bent partitions Ω_H and $\Omega_{\tilde{H}}$. Are the bent partitions Γ_H and $\Gamma_{\tilde{H}}$ inequivalent?

Open Problems:

- There exist normal bent partitions where the sets are not partial difference sets.
  - What are these sets? Do they correspond to combinatorial objects?
  - What can be said about the corresponding Cayley graphs? Do they have interesting properties?
- Suppose that Ω_1 and Ω_2 are not equivalent bent partitions obtained from two semifields. Are the amorphic association schemes obtained from Ω_1 and Ω_2 are isomorphic?