Complex Hadamard matrices with a special structure II

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Hadamard Matrices, Sevilla May 29, **2025**

Hadamard matrices $\Rightarrow$ real quantum gates

Hadamard matrices are orthogonal (up to a rescaling)

as they consist of mutually orthogonal row and columns,

$$HH^* = N\mathbb{1} \;\Rightarrow\; H' := H/\sqrt{N} \text{ is unitary}$$

$N = 2$ Hadamard matrix $\Rightarrow$ one–qubit Hadamard gate

$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so that $H_2' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is orthogonal.

The most often used gate in Quantum Information Theory,
as (due to rotation of the Bloch sphere by π along the axis intermediate between directions x and z) it forms a **quantum superposition**

$$H_2|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

and

$$H_2|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

[in quantum information one denotes H_2 (no primes) for unitary]

Quantum computing

The basic building step is based on **Hadamard matrices**:

one qubit Hadamard matrix, H_2 (of size two)

multi–qubit Hadamard matrix, $H_{2^n} = H_2^{\otimes n}$ (of size $N = 2^n$)

Examples:

a) **two qubits**, $n = 2$

Note that $H_4|0,0\rangle = (H_2 \otimes H_2)|0\rangle \otimes |0\rangle = \frac{1}{2}[|00\rangle + |01\rangle + |10\rangle + |11\rangle]$

corresponds to the superposition: $0 \; + \; 1 \; + \; 2 \; + \; 3.$

b) n **qubits**: consider the n–qubit state

$$|\psi\rangle = H_2^{\otimes n}|0,\ldots 0\rangle \qquad (**)$$

which leads to the uniform superposition, $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$

and allows to process all $N = 2^n$ numbers **"in parallel"**.

Complex Hadamard matrices

Hadamard matrices of the Butson type

composed of q-th **roots of unity**; $H \in H(N,q)$ iff

$$HH^* = N\,\mathbb{1}\ , \qquad (H_{ij})^q = 1 \quad \text{for} \quad i,j = 1,\ldots N \tag{1}$$

Butson, 1962

special case: $q = 4$

$H \in H(N,4)$ iff $HH^* = N\,\mathbb{1}$ and $H_{ij} = \pm 1, \pm i$

(also called **complex** Hadamard matrices, **Turyn, 1970**)

Complex Hadamard matrices (*general case*)

$HH^* = N\,\mathbb{1}$ and $|H_{ij}| = 1$,

hence $H_{ij} = \exp(i\phi_{ij})$ with an **arbitrary complex** phase.

Complex Hadamard matrices do exist for any N !

example: the **Fourier matrix**

$$(F_N)_{jk} := \exp(ijk2\pi/N) \quad \text{with} \quad j,k = 0,1,\ldots,N-1. \qquad (2)$$

special case : $N = 4$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \in H(4,4) \qquad (3)$$

The **Fourier matrices** are constructed of N–th root of unity, so they are of the **Butson type**,

$$F_N \in H(N,N).$$

Equivalent Hadamard matrices

$$H' \sim H$$

iff there exist permutation matrices P_1 and P_2 and diagonal unitary matrices D_1 and D_2 such that

$$H' = D_1 P_1 \; H \; P_2 D_2 \; . \tag{4}$$

Dephased form of a Hadamard matrix

$$H_{1,j} = H_{j,1} = 1 \quad \text{for} \quad j = 1, \ldots, N. \tag{5}$$

Any complex Hadamard matrix can be brought to the dephased form by an equivalence relation.

example for $N = 3$, here $\alpha \in [0, 2\pi)$ while $w = \exp(i \cdot 2\pi/3)$, so $w^3 = 1$

$$F_3' = e^{i\alpha} \begin{bmatrix} w & 1 & w^2 \\ 1 & 1 & 1 \\ w^2 & 1 & w \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix} =: F_3 \; , \tag{6}$$

Classification of **Complex Hadamard matrices** I

$N = 2$

all complex Hadamard matrices are equivalent to the **real Hadamard (Fourier)** matrix

$$H_2 = F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{7}$$

$N = 3$

all complex Hadamard matrices are equivalent to the **Fourier matrix**

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix} , \quad w = e^{2\pi i/3}. \tag{8}$$

U. Haagerup, Orthogonal maximal abelian $*$-subalgebras of the $N \times N$ matrices and cyclic N–rots,
in Operator Algebras and Quantum Field Theory, **1996.**

Classification of **Complex Hadamard matrices** II

$N = 4$

Lemma (Haagerup). For $N = 4$ all complex Hadamard matrices are equivalent to one of the matrices from the following 1–d orbit, $w = i$

$$F_4^{(1)}(a) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w^1 \cdot \exp(i \cdot a) & w^2 & w^3 \cdot \exp(i \cdot a) \\ 1 & w^2 & 1 & w^2 \\ 1 & w^3 \cdot \exp(i \cdot a) & w^2 & w^1 \cdot \exp(i \cdot a) \end{bmatrix}, \quad a \in [0, \pi].$$

$N = 5$

All $N = 5$ complex Hadamard matrices are equivalent to the **Fourier matrix** F_5 (**Haagerup 1996**).

$N \geq 6$

Several orbits of Complex Hadamard matrices are known, but the problem of their complete classification remains **open!**

1-d family by Beauchamp & Nicoara, April 2006

$$B_6^{(1)}(y) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1/x & -y & y & 1/x \\ 1 & -x & 1 & y & 1/z & -1/t \\ 1 & -1/y & 1/y & -1 & -1/t & 1/t \\ 1 & 1/y & z & -t & 1 & -1/x \\ 1 & x & -t & t & -x & -1 \end{bmatrix}$$

where $y = \exp(i\ s)$ is a free parameter and

$$x(y) = \frac{1 + 2y + y^2 \pm \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{1 + 2y - y^2}$$

$$z(y) = \frac{1 + 2y - y^2}{y(-1 + 2y + y^2)}; \quad t(y) = xyz$$

W. Bruzda discovered this family independenty in May 2006

For **Complex Hadamard matrices** of size $N = 2, \dots 16$

see online **Catalog** at

http://chaos.if.uj.edu.pl/∼karol/hadamard

(improved engine by **Wojciech Bruzda**, some new data...)

If you know about **new complex Hadamard** matrices

(*or you found a* **misprint** *in the catalogue,*
or you wrote a great paper on this topic
which should be mentioned in the Catalogue)

please let know Wojtek (and me)

Wawel castle in Cracow

Composed systems & entangled states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

Two–qubit system: $2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$ distinguished by the fact that reduced states are **maximally mixed**,

e.g. $\rho_A = \mathrm{Tr}_B|\varphi^+\rangle\langle\varphi^+| = \frac{1}{2}\mathbb{1}_2$.

Maximally entangled states of $d \times d$ system

Define bi-partite pure state by a matrix of coefficients,
$|\psi\rangle = \sum_{i,j=1}^{d} \Gamma_{ij}|i,j\rangle$.

Then reduced state $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi| = \Gamma\Gamma^\dagger$.

It represents a **maximally entangled** state if $\rho_A = \Gamma\Gamma^\dagger = \mathbb{1}_d/d$, which is the case if the matrix $U = \sqrt{d}\Gamma$ of size d is **unitary**.

Multipartite entangled states

$k-$ uniform state of n subsystems

Consider a state of n subsystems with d levels each, $|\psi\rangle \in \mathcal{H}_d^{\otimes n}$.
Such a state is called k-**uniform** if for any choice of part X consisting of k subsystems out of n the partial trace over the part $\bar{X}$ consisting of remaining $n-k$ subsystems is maximally mixed,

$$\mathrm{Tr}_{\bar{X}} |\psi\rangle\langle\psi| = \frac{1}{d^k} \mathbb{1}_{d^k}. \tag{9}$$

Examples

a) 2–qubit state $|00\rangle + |01\rangle + |10\rangle - |11\rangle$ is **1–uniform** (Bell-like)

(as the coefficient matrix $\Gamma = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is Hadamard !)

b) 3–qubit state $|GHZ\rangle = (|000\rangle + |111\rangle)$ is **1–uniform**

c) there are no 2–**uniform** states of 4 qubits,

but they exist for larger systems...

Hadamard matrices & quantum states

A Hadamard matrix $H_8 = H_2^{\otimes 3}$ of order $N = 8$ implies

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}$$

This '**orthogonal array**'
allows us to construct a 2–**uniform state** of 7 qubits:

$$\begin{aligned} |\Phi_7\rangle \quad = \quad & |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + \\ & |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle. \end{aligned}$$

a **simplex** state $|\Phi_7\rangle$

Examples of 2–uniform states obtained form H_{12}

8 **qubits**

$$|\Phi_8\rangle = |00000000\rangle + |00011101\rangle + |10001110\rangle + |01000111\rangle + |10100011\rangle + |11010001\rangle + |01101000\rangle + |10110100\rangle + |11011010\rangle + |11101101\rangle + |01110110\rangle + |00111011\rangle.$$

9 **qubits**

$$|\Phi_9\rangle = |000000000\rangle + |100011101\rangle + |010001110\rangle + |101000111\rangle + |110100011\rangle + |011010001\rangle + |101101000\rangle + |110110100\rangle + |111011010\rangle + |011101101\rangle + |001110110\rangle + |000111011\rangle.$$

10 **qubits** (*note what Hadamard matrices are good for* :)

$$|\Phi_{10}\rangle = |0000000000\rangle + |0100011101\rangle + |1010001110\rangle + |1101000111\rangle + |0110100011\rangle + |1011010001\rangle + |1101101000\rangle + |1110110100\rangle + |0111011010\rangle + |0011101101\rangle + |0001110110\rangle + |1000111011\rangle,$$

Cracow

and Tatra mountains in the background

Combinatorial designes

$\Longrightarrow$ An introduction to *"Quantum Combinatorics"*

A classical example:

Take 4 **aces**, 4 **kings**, 4 **queens** and 4 **jacks**
and arrange them into an 4×4 array, such that

a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

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b) - in every row and column there is only a **single** card of each rank

$A\spadesuit$	$K\diamondsuit$	$Q\heartsuit$	$J\clubsuit$
$K\heartsuit$	$A\clubsuit$	$J\spadesuit$	$Q\diamondsuit$
$Q\clubsuit$	$J\heartsuit$	$A\diamondsuit$	$K\spadesuit$
$J\diamondsuit$	$Q\spadesuit$	$K\clubsuit$	$A\heartsuit$

Two **mutually orthogonal Latin squares** of size $N = 4$

2 **MOLS(4)** $=$ **Graeco–Latin** square !

Mutually ortogonal Latin Squares (MOLS)

♣) $N = 2$. There are no orthogonal Latin Square
(for **2** aces and **2** kings the problem has no solution)

♡) $N = 3, 4, 5$ (and any **power of prime**) $\Longrightarrow$ there exist $(N - 1)$ MOLS.

♠) $N = 6$. Only a **single** Latin Square exists (No OLS!).

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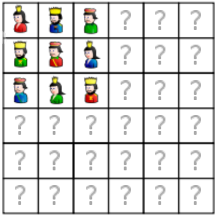
♠) $N = 6$. Only a **single** Latin Square exists (No OLS!).

Euler's problem: **36** officers of six different ranks from six different units come for a **military parade** Arrange them in a square such that: in each row / each column all uniforms are different.



No solution exists ! (conjectured by **Euler**), proof by:
Gaston Terry "Le Probléme de 36 Officiers". *Compte Rendu* **(1901)**.

Mutually ortogonal Latin Squares (MOLS)

An apparent solution of the $N = 6$ **Euler's** problem of **36 officers**.

a difference between math and physics: "imperfect realization"

highly entangled states for higher dimensions

A pair of **orthogonal Latin squares** of size 3,

$A\spadesuit$	$K\clubsuit$	$Q\diamondsuit$			$0,0$	$1,2$	$2,1$
$K\diamondsuit$	$Q\spadesuit$	$A\clubsuit$	$=$		$1,1$	$2,0$	$0,2$
$Q\clubsuit$	$A\diamondsuit$	$K\spadesuit$			$2,2$	$0,1$	$1,0$

yields a **2–uniform** state of **4 qutrits**: (Abslolutely maximally entangled)

$$
\begin{aligned}
|\Psi_3^4\rangle \quad = \quad & |0000\rangle + |0112\rangle + |0221\rangle + \\
& |1011\rangle + |1120\rangle + |1202\rangle + \\
& |2022\rangle + |2101\rangle + |2210\rangle .
\end{aligned}
$$

Corresponding **Quantum Code**:
$$
\begin{aligned}
|0\rangle &\rightarrow |\tilde{0}\rangle := |000\rangle + |112\rangle + |221\rangle \\
|1\rangle &\rightarrow |\tilde{1}\rangle := |011\rangle + |120\rangle + |202\rangle \\
|2\rangle &\rightarrow |\tilde{2}\rangle := |022\rangle + |101\rangle + |210\rangle
\end{aligned}
$$

k–uniform states and k–unitary matrices

Consider a **2–uniform** state of four parties A, B, C, D with d levels each,
$$|\psi\rangle = \sum_{i,j,l,m=1}^{d} \Gamma_{ijlm}|i,j,l,m\rangle$$

It is **maximally entangled** with respect to all **three** partitions:
$$AB|CD \text{ and } AC|BD \text{ and } AD|BC.$$

Let $\rho_{ABCD} = |\psi\rangle\langle\psi|$. Hence its three reductions are **maximally mixed**,
$\rho_{AB} = \mathrm{Tr}_{CD}\rho_{ABCD} = \rho_{AC} = \mathrm{Tr}_{BD}\rho_{ABCD} = \rho_{AD} = \mathrm{Tr}_{BC}\rho_{ABCD} = \mathbb{1}_{d^2}/d^2$

Thus matrices $U_{\mu,\nu}$ of order d^2 obtained by reshaping the tensor $d\Gamma_{ijkl}$ are **unitary** for three reorderings:

a) $\mu,\nu = ij, lm$, b) $\mu,\nu = im, jl$, c) $\mu,\nu = il, jm$.

Such a tensor Γ is called **perfect**.

Corresponding **unitary matrix** U of order d^2 is called **two–unitary**
if reordered matrices U^{R_1} and U^{R_2} remain **unitary.**

Unitary matrix U of order d^k with analogous property is called **k–unitary**

Exemplary multiunitary matrices

Fact: there are no 2-unitary matrices of order 4 (**Higuchi, Sudbery** 2001)

Two–unitary permutation matrix of size $9 = 3^2$
associated to 2 **MOLS(3)** and **2–uniform** state $|\Psi_3^4\rangle$ of 4 qutrits

$$U = U_{\substack{ij \\ ml}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \in U(9)$$

Furthermore, also two reordered matrices
(by partial transposition and reshuffling) remain **unitary**:

$$U^{T_1} = U_{\substack{il \\ mj}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in U(9)$$

$$U^{R} = U_{\substack{im \\ jl}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \in U(9)$$

Are there multiunitary Hadamard matrices?

no   for $d = 2$ and $N = d^2 = 4$ (*to many constraints!*)
Yes   for $d = 3$ and $N = d^2 = 9$ and for $d = 2$ and $N = d^3 = 8$
Example: 3–unitary real Hadamard matrix of size $N = 2^3 = 8$
associated to the **3–uniform** state $|\Psi_2^6\rangle$ of 6 qubits

$$H_{\substack{ijk\\lmn}} = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix} \in H_8$$

This unitary matrix remains **unitary** after any of $\frac{1}{2}\binom{6}{3} = 10$ reorderings related to different decomposition of the hypercube with $8^2 = 2^6 = 64$ entries.

$$H_{\substack{ijk\\lnm}} = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix} \in \mathcal{H}_8$$

$$H_{\substack{ijk\\nlm}} = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix} \in \mathcal{H}_8$$

Multi–unitary Hadamard (and other unitary) matrices

Let H be a Hadamard matrix of size $N = d^k$

It is called **multi–unitary**

if the corresponding tensor (of size d with $2k$ indices) is **perfect**,

which means that all its $\frac{1}{2}\binom{2k}{k}$ **reorderings** also form a **Hadamard matrix**

any such Hadamard matrix generates strongly **entangled** quantum state!

conference Hadamard Budapest 2017: Open problems

To be done: Identify and classify

a) multi–unitary **real Hadamard matrices**
is there a 2-unitary real Hadamard matrix H_{36} ?

b) multi–unitary **complex Hadamard matrices**
is there a 2-unitary complex Hadamard matrix H_{36} ?

2-unitary complex Hadamards of order d^2, $d \geq 3$, $d \neq 6$

two unitary permutations P_{d^2} of order d^2

1. Any pair of 2MOLS(d) generates such a 2-unitary permutation matrix P_{d^2}
2. Take a tensor product of two Fourier matrices, $F_d \otimes F_d \in B(d^2, d)$, which is a Butson class matrix,
3. This represents a local unitary operation, so it does not change the entanglement of the operation,
4. Thus the matrix $H = P_{d^2}(F_d \otimes F_d)$ does the job: it is a two-unitary **complex Hadamard** matrix and represents an **Absolutely Maximally Entangled (AME) state** in $\mathcal{H}_d^{\otimes 4}$.

If d is a multiple of 4 and there exist a real Hadamard H_d of size d then $P_{d^2}(H_d \otimes H_d)$ forms a **2-unitary real** Hadamard matrix

Bruzda, Rajchel-Mieldzioć, K.Ż (2024)

$d = 6 \quad N = d^2 = 36$ & Entangled officers of Euler

2021 Golden solution of the quantum version of the problem of 36 **officers of Euler**: a 2-unitary matrix U_{36} with ratio of amplitudes equal to the golden mean and phases multiples of $\omega_{20} = \exp(i2\pi/20)$
Rather, Burchardt, Bruzda, Rajchel, Lakshminarayan, K.Ż.

2023 alternative non-equivalent solutions by **Rather, Ramadas, Kodiyalam, and Lakshminarayan**, (computer aided search)

2024 three solutions by **Rather** based on *biunimodular vectors* (Führ and Rzeszotnik 2015)

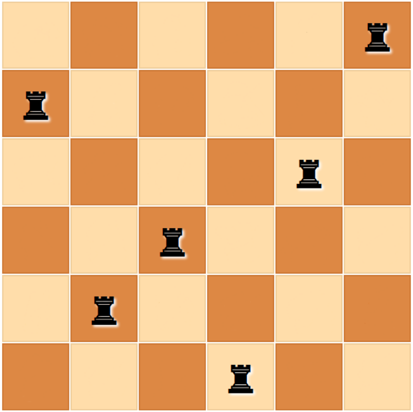
2025 hand made, artisanal solution by **Gross** and **Goedicke**

Proposition 1. There exists a 2-unitary *complex Hadamard matrix* of order 36 of Butson-type, $H_{36} \in B(36, 6)$ as all phases are multiples of $\omega_6 = \exp(i2\pi/6)$, **Bruzda, K. Ż** (2024)

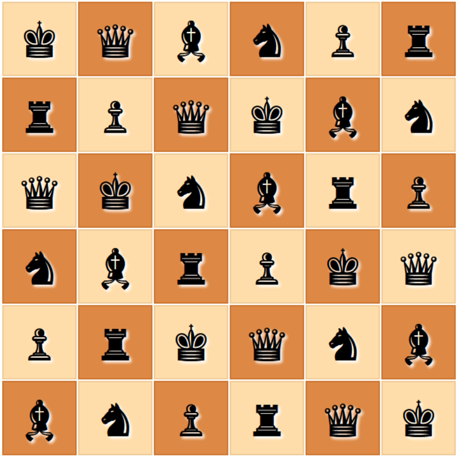
36 officers of Euler revisited

introductory exercise

Step i) Place **six rooks** on a chessboard of size six,
in such a way that no figure attacks any other:



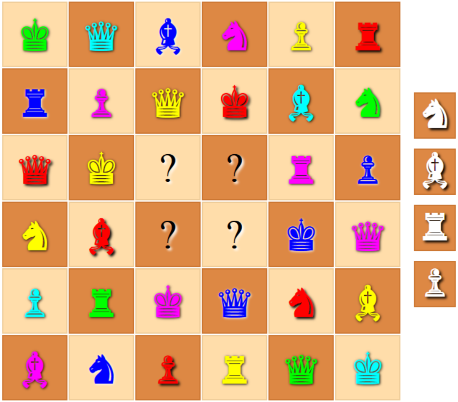
Step ii) Take six pieces of five other figures
and place them onto the board in an analogous way:



Latin Square of order six

Step iii) Color them into six colors,

so that all colors, in each row and column are different...

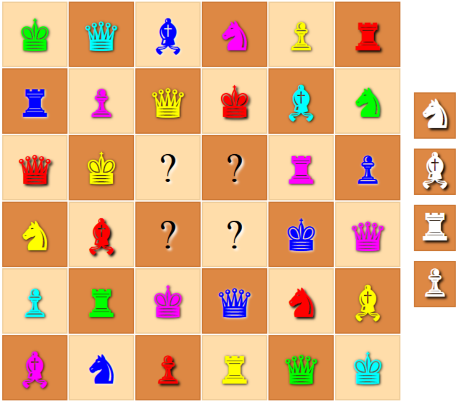


Place the remaining four figures, two of them in cyan and two in green, so that all the rules of Euler are fulfilled

36 officers of Euler, step three $\Rightarrow$ no go !

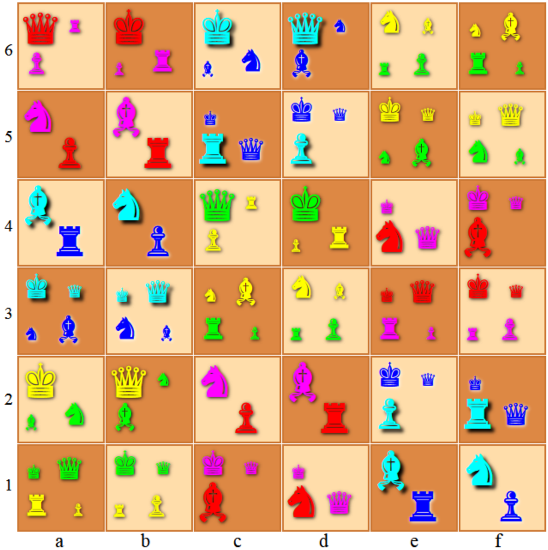
Step iii) Color them into six colors, $d = 6 = 2 * 3$,

so that all colors, in each row and column are different...

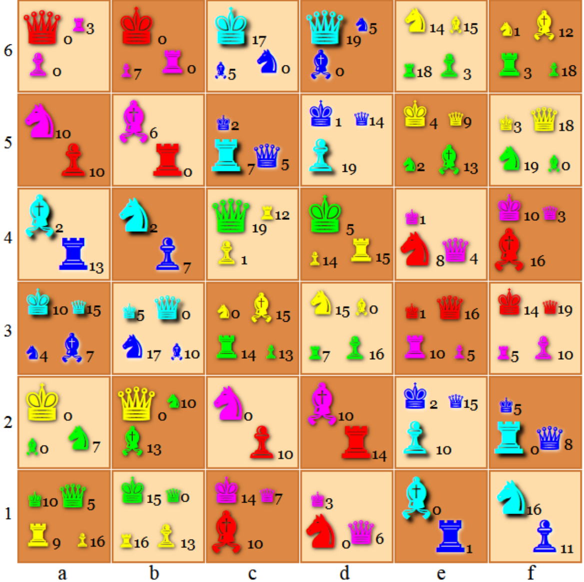


Place the remaining four figures, two of them in cyan and two in green, so that all the rules of Euler are fulfilled – **this is not doable !** **G. Tarry**

No classical OLS(6). **But a quantum solution exists !**



Quantum solution of *36 entangled officers* of **Euler**. Size of the figures represents moduli of superpositions, Is field $c2$ equal to $a5$?



Quantum solution of *36 entangled officers* of **Euler**. Size of the figures represents moduli of superpositions, index k denotes the complex **phase** $\exp(i\pi k/20)$, e.g. field c2) denotes $|$♞$\rangle - |$♟$\rangle$ and is orthogonal to a5).

Full solution of the problem of 36 **entangled officers of Euler**
encoded in the chessboard of size 6 looks like this...
(each state $|\psi_{ij}\rangle$ determines a single row of a 2-unitary matrix U_{36})

$$
\begin{aligned}
|\psi_{00}\rangle &= c|10\rangle + a\omega^3|43\rangle + b|53\rangle = c|\text{♛}\rangle + a\omega^3|\text{♜}\rangle + b|\text{♟}\rangle \\
|\psi_{01}\rangle &= c|00\rangle + b|43\rangle + a\omega^7|53\rangle = c|\text{♚}\rangle + b|\text{♜}\rangle + a\omega^7|\text{♟}\rangle \\
|\psi_{02}\rangle &= c\omega^{17}|01\rangle + b|24\rangle + a\omega^5|34\rangle = c\omega^{17}|\text{♚}\rangle + b|\text{♞}\rangle + a\omega^5|\text{♝}\rangle \\
|\psi_{10}\rangle &= c\omega^{10}|23\rangle + c\omega^{10}|50\rangle = c\omega^{10}|\text{♞}\rangle + c\omega^{10}|\text{♟}\rangle \\
|\psi_{11}\rangle &= c\omega^{6}|33\rangle + c|40\rangle = c\omega^{6}|\text{♝}\rangle + c|\text{♜}\rangle \\
|\psi_{12}\rangle &= a\omega^2|04\rangle + b\omega^5|14\rangle + c\omega^7|41\rangle = a\omega^2|\text{♚}\rangle + b\omega^5|\text{♛}\rangle + c\omega^7|\text{♜}\rangle \\
\dots &= \dots \\
|\psi_{55}\rangle &= c\omega^{16}|21\rangle + c\omega^{11}|54\rangle = c\omega^{16}|\text{♞}\rangle + c\omega^{11}|\text{♟}\rangle,
\end{aligned}
$$

where $\omega = \exp(i\pi k/20)$, and $a^2 + b^2 = c^2 = 1/2$,
while the ratio of the two sizes of the figures is equals to the
golden mean, $b/a = (1+\sqrt{5})/2 = \varphi$.

It is easy to check that this constellation satisfies the desired conditions **a'), b'), c')** specified above and it deservs an appelation **golden square**.

NUMERICAL SEARCH

$$U_0 \mapsto U_0^{\mathrm{R}} \mapsto \left(U_0^{\mathrm{R}}\right)^{\Gamma} := U_0^{\Gamma\mathrm{R}} \mapsto U_1$$

$\tilde{P} =$

11	22	33	44	55	66
23	14	45	36	61	52
32	41	64	53	16	25
46	35	51	62	24	13
54	63	26	15	42	31
65	56	12	21	33	44

=

A♠	K♣	Q♦	J♥	10✿	9✱
K♦	A♥	J✿	Q✱	9♠	10♣
Q♣	J♠	9♥	10♦	A✱	K✿
J✱	Q✿	10♠	9♣	K♥	A♦
10♥	9♦	K✱	A✿	J♣	Q♠
9✿	10✱	A♣	K♠	Q♦	J♥

$e_p(\tilde{P}) = \frac{314}{315} \simeq .9968$

$e_p(\tilde{P}) \to \frac{419}{420} \simeq .9976$

$e_p(\widetilde{P_s}) = \frac{104}{105} \simeq .9905$

$e_p(\tilde{P}e^{iH\varepsilon}) \to .9991$

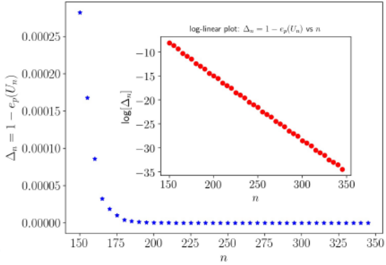
$e_p(\widetilde{P_S}e^{iH\varepsilon}) \to 1$

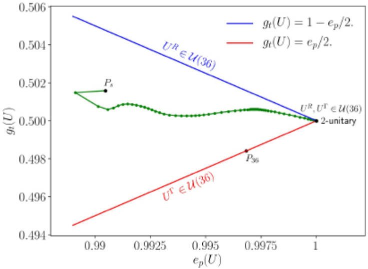
$\tilde{P}_s =$

11	22	33	44	55	66
23	14	45	36	61	52
32	41	64	53	16	25
46	35	51	62	24	13
64	56	26	15	43	31
55	63	12	21	42	34

=

A♠	K♣	Q♦	J♥	10✿	9✱
K♦	A♥	J✿	Q✱	9♠	10♣
Q♣	J♠	9♥	10♦	A✱	K✿
J✱	Q✿	10♠	9♣	K♥	A♦
9♥	10✱	K✱	A✿	J♦	Q♠
10✿	9♦	A♣	K♠	J♣	Q♥

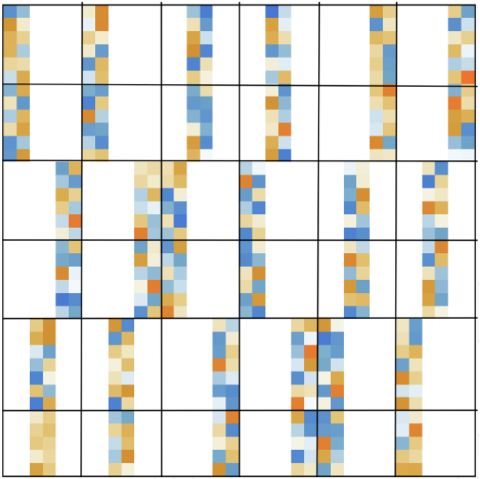




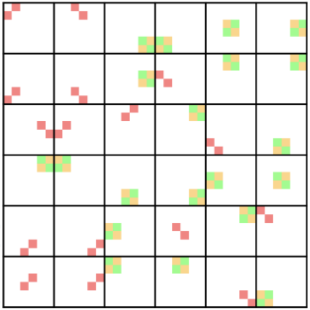
Numerical Cleaning

$$\left(U_6^{(1)} \otimes U_6^{(2)}\right) U_{36} \left(U_6^{(4)} \otimes U_6^{(3)}\right)$$

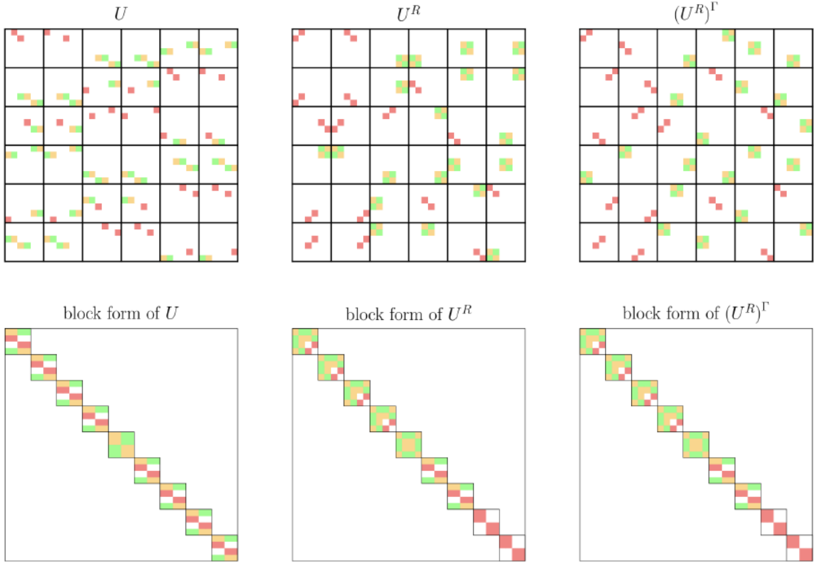
$$\left(\cancel{U_6} \otimes U_2^{\otimes 3}\right) U_{36} \left(U_2^{\otimes 3} \otimes U_2^{\otimes 3}\right)$$



$$\begin{pmatrix} 0.75 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 \\ 0 & 0 & 0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.75 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0.75 & 0 & 0 \\ 0.25 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0 \\ 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 \end{pmatrix}$$



Solution Found



Solution Found

(1,1)	(2,2)	(1,2)	(2,1)	
$a\,\omega^{10}$	a	$b\,\omega^{15}$	$b\,\omega^{5}$	(6,3)
		c	c	(1,1)
c	c			(5,6)
$b\,\omega^{10}$	b	$a\,\omega^{5}$	$a\,\omega^{15}$	(4,2)

(1,3)	(2,4)	(1,4)	(2,3)	
$a\,\omega^{2}$	$a\,\omega^{14}$	$b\,\omega$	$b\,\omega^{5}$	(2,5)
		$c\,\omega^{5}$	$c\,\omega^{19}$	(3,3)
$c\,\omega^{17}$	$c\,\omega^{19}$			(1,2)
$b\,\omega^{14}$	$b\,\omega^{6}$	$a\,\omega^{3}$	$a\,\omega^{7}$	(6,4)

(1,5)	(2,6)	(1,6)	(2,5)	
$a\,\omega$	$a\,\omega^{19}$	$b\,\omega^{14}$	$b\,\omega^{16}$	(4,1)
$a\,\omega$	$a\,\omega^{3}$	$b\,\omega^{10}$	$b\,\omega^{4}$	(3,4)
$b\,\omega^{4}$	$b\,\omega^{18}$	$a\,\omega^{3}$	$a\,\omega^{9}$	(2,6)
$b\,\omega^{2}$	$b\,\omega^{8}$	$a\,\omega^{5}$	$a\,\omega^{15}$	(5,5)

(3,1)	(4,2)	(3,2)	(4,1)	
$a\,\omega^{4}$	$a\,\omega^{10}$	$b\,\omega^{17}$	$b\,\omega^{7}$	(4,5)
		$c\,\omega^{2}$	$c\,\omega^{2}$	(3,2)
$c\,\omega^{10}$	$c\,\omega^{6}$			(2,4)
$b\,\omega^{7}$	$b\,\omega^{13}$	$a\,\omega^{10}$	a	(5,3)

(3,3)	(4,4)	(3,4)	(4,3)	
a	a	$b\,\omega^{15}$	$b\,\omega^{15}$	(4,6)
		c	$c\,\omega^{10}$	(6,1)
c	$c\,\omega^{10}$			(5,4)
b	b	$a\,\omega^{5}$	$a\,\omega^{5}$	(1,5)

(3,5)	(4,6)	(3,6)	(4,5)	
$a\,\omega^{2}$	a	$b\,\omega^{19}$	$b\,\omega^{13}$	(2,3)
		$c\,\omega^{16}$	c	(6,2)
$c\,\omega^{8}$	$c\,\omega^{16}$			(3,1)
$b\,\omega^{14}$	$b\,\omega^{12}$	$a\,\omega$	$a\,\omega^{15}$	(1,6)

(5,1)	(6,2)	(5,2)	(6,1)	
$a\,\omega^{3}$	$a\,\omega^{7}$	b	b	(1,4)
		c	$c\,\omega^{10}$	(2,1)
$c\,\omega^{13}$	$c\,\omega^{7}$			(3,5)
$b\,\omega^{9}$	$b\,\omega^{13}$	$a\omega^{16}$	$a\omega^{16}$	(6,6)

(5,3)	(6,4)	(5,4)	(6,3)	
$a\,\omega^{12}$	$a\,\omega^{14}$	$b\,\omega^{15}$	$b\,\omega$	(3,6)
		$c\,\omega^{14}$	$c\,\omega^{10}$	(5,1)
$c\,\omega^{7}$	$c\,\omega^{19}$			(2,2)
$b\,\omega^{14}$	$b\,\omega^{16}$	$a\omega^{7}$	$a\omega^{13}$	(4,3)

(5,5)	(6,6)	(5,6)	(6,5)	
$a\,\omega^{18}$	$a\,\omega^{18}$	$b\,\omega^{3}$	$b\,\omega^{3}$	(1,3)
		c	$c\,\omega^{10}$	(5,2)
$c\,\omega$	$c\,\omega^{11}$			(6,5)
$b\,\omega^{10}$	$b\,\omega^{10}$	$a\omega^{5}$	$a\omega^{5}$	(4,4)

AME(4,6) state

$$\frac{1}{6}\sum_{i,j,k,\ell=1}^{d} t_{i,j,k,\ell}\,|i\rangle\,|j\rangle\,|k\rangle\,|\ell\rangle$$

$$a = \frac{1}{\sqrt{2}(\omega+\overline{\omega})} = \frac{1}{\sqrt{5+\sqrt{5}}}$$

$$b = \frac{1}{\sqrt{2}(\omega^3+\overline{\omega}^3)} = \sqrt{\frac{5+\sqrt{5}}{20}}$$

$$c = \frac{1}{\sqrt{2}}$$

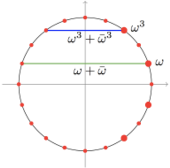
$$\omega = exp(i\,\pi/20)$$

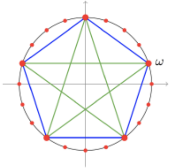
Pythagoras theorem

$$a^2 + b^2 = c^2 = \frac{1}{2}$$

Golden ratio

$$b/c = \varphi = = \frac{1+\sqrt{5}}{2}$$





with a kind invitation to **Cracow, Poland**

Quantum Euler problem: Zoo of non-equivalent solutions

no **classical** **solution**: no 2-unitary permutation matrix P_{36}
desired **quantum** **solution**: find any 2-unitary matrix U_{36}

2021 original "Golden solution" based on golden mean, **RBBRLŻ**

2023 alternative non-equivalent solutions (**Lakshminarayan** et al.)

2024 three new solutions by **Rather**

2025 hand made, artisanal solution by **Gross** and **Goedicke**

2024 *complex Hadamard* solution of Butson-type, $H_{36} \in B(36, 6)$
yields entire 19 dimensional family

$$H_{36}(\mathbf{a}) = \exp\left(\frac{i2\pi}{6}B\right) \odot \exp\left(\frac{i2\pi}{6}A\right)$$

where $A = A(\mathbf{a})$ denotes the matrix of 19 linear parameters
$\mathbf{a} = (a_1, a_1, \dots a_{19})$, $\odot$ stands for Hadamard product
and the original **complex Hadamard** solution found
is given by the phase matrix B,

$$B = \begin{bmatrix}
1 & 5 & 5 & 5 & 1 & 3 & 4 & 2 & 4 & 4 & 2 & 4 & 4 & 2 & 0 & 0 & 0 & 2 & 1 & 5 & 5 & 5 & 1 & 3 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 2 & 4 & 2 & 0 \\
5 & 5 & 5 & 1 & 3 & 1 & 1 & 3 & 3 & 1 & 3 & 3 & 0 & 4 & 4 & 4 & 0 & 2 & 2 & 2 & 2 & 4 & 0 & 4 & 0 & 2 & 2 & 0 & 2 & 2 & 1 & 3 & 5 & 3 & 1 & 1 \\
3 & 3 & 5 & 1 & 5 & 3 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 2 & 4 & 2 & 3 & 3 & 5 & 1 & 5 & 3 & 2 & 2 & 0 & 2 & 2 & 0 & 2 & 4 & 2 & 0 & 0 & 0 \\
1 & 3 & 5 & 3 & 1 & 1 & 3 & 1 & 3 & 3 & 1 & 3 & 2 & 2 & 4 & 0 & 4 & 2 & 4 & 0 & 2 & 0 & 4 & 4 & 2 & 0 & 2 & 2 & 0 & 2 & 3 & 1 & 5 & 5 & 5 & 1 \\
5 & 1 & 5 & 3 & 3 & 3 & 2 & 4 & 4 & 2 & 4 & 4 & 2 & 4 & 0 & 4 & 2 & 2 & 5 & 1 & 5 & 3 & 3 & 3 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 2 & 2 & 2 & 4 & 0 \\
3 & 1 & 5 & 5 & 5 & 1 & 5 & 5 & 3 & 5 & 5 & 3 & 4 & 0 & 4 & 2 & 2 & 2 & 0 & 4 & 2 & 2 & 2 & 4 & 4 & 4 & 2 & 4 & 4 & 2 & 5 & 5 & 5 & 1 & 3 & 1 \\
4 & 2 & 4 & 1 & 5 & 1 & 5 & 3 & 1 & 4 & 4 & 0 & 3 & 1 & 1 & 4 & 0 & 2 & 0 & 4 & 0 & 3 & 1 & 3 & 1 & 1 & 3 & 2 & 0 & 4 & 3 & 1 & 1 & 4 & 0 & 2 \\
2 & 4 & 4 & 5 & 1 & 1 & 2 & 0 & 0 & 3 & 5 & 1 & 5 & 5 & 5 & 4 & 0 & 4 & 1 & 3 & 3 & 4 & 0 & 0 & 3 & 5 & 1 & 2 & 0 & 0 & 2 & 2 & 2 & 1 & 3 & 1 \\
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0 & 4 & 0 & 3 & 1 & 3 & 0 & 0 & 2 & 1 & 5 & 3 & 3 & 5 & 1 & 2 & 0 & 0 & 5 & 3 & 5 & 2 & 0 & 2 & 1 & 5 & 3 & 0 & 0 & 2 & 0 & 2 & 4 & 5 & 3 & 3 \\
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1 & 1 & 3 & 5 & 3 & 1 & 5 & 1 & 3 & 1 & 5 & 5 & 2 & 0 & 2 & 2 & 0 & 2 & 2 & 0 & 4 & 4 & 4 & 0 & 2 & 4 & 0 & 4 & 2 & 2 & 3 & 1 & 3 & 3 & 1 & 3 \\
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1 & 3 & 1 & 5 & 5 & 5 & 5 & 3 & 1 & 1 & 1 & 3 & 2 & 2 & 0 & 2 & 2 & 0 & 2 & 2 & 2 & 4 & 0 & 4 & 2 & 0 & 4 & 4 & 4 & 0 & 3 & 3 & 1 & 3 & 3 & 1 \\
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5 & 2 & 5 & 4 & 3 & 4 & 3 & 2 & 5 & 0 & 5 & 2 & 4 & 3 & 2 & 3 & 4 & 1 & 2 & 5 & 2 & 1 & 0 & 1 & 4 & 3 & 0 & 1 & 0 & 3 & 3 & 4 & 1 & 4 & 3 & 2 \\
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2 & 3 & 2 & 2 & 3 & 2 & 4 & 5 & 0 & 0 & 3 & 2 & 3 & 2 & 1 & 5 & 0 & 3 & 3 & 4 & 3 & 3 & 4 & 3 & 3 & 0 & 5 & 1 & 2 & 3 & 0 & 5 & 4 & 2 & 3 & 0 \\
2 & 1 & 4 & 2 & 1 & 4 & 3 & 4 & 1 & 1 & 0 & 5 & 5 & 4 & 5 & 3 & 0 & 3 & 0 & 5 & 2 & 0 & 5 & 2 & 1 & 0 & 5 & 3 & 4 & 1 & 5 & 4 & 5 & 3 & 0 & 3 \\
0 & 3 & 4 & 0 & 3 & 4 & 2 & 5 & 2 & 4 & 3 & 4 & 1 & 2 & 3 & 3 & 0 & 5 & 1 & 4 & 5 & 1 & 4 & 5 & 1 & 0 & 1 & 5 & 2 & 5 & 4 & 5 & 0 & 0 & 3 & 2 \\
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5 & 2 & 1 & 0 & 1 & 2 & 2 & 3 & 0 & 3 & 2 & 1 & 4 & 5 & 4 & 1 & 2 & 1 & 3 & 4 & 5 & 2 & 5 & 4 & 2 & 3 & 0 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 4 & 3 \\
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\end{bmatrix}.$$

$B_{jm} \in [0,1,2,3,4,5]$ represent phases, $H_{jm} = exp(i\ B_{jm} 2\pi/6)$

reminisces from **Cracow**: New campus of Jagiellonian University

Concluding Remarks

- 2-unitary matrices of order d^2 (unitarity is preserved after partial tranpose and reshuffling) represent state $|AME(4, d)\rangle$
- There exit $2 - unitary$ complex Hadamard matrices of order d^2 for $d \neq 2$ and $d \neq 6$ related to OLS(d).
- **Theorem**. Absolutely maximally entangled states $|AME(4, 6)\rangle$ of 4 subsystems with 6 levels each **do** exist ! (2021).
  This implies **existence** of
  1. solution of the quantum analogue of the 36 officers problem of **Euler**,
  2. existence of 2-unitary gate U_{36} with maximal **entangling power**
  3. **perfect tensor** $t_{ijk\ell}$ with 4 indices, each running from 1 to 6, to be applied for tensor networks and bulk/boundary correspondence,
  4. nonadditive **quantum error correction code** $((3, 6, 2))_6$ which allows one to encode a single quhex in three quhexes
- for $d = 6$ there exist also 2-unitary *complex Hadamard* matrix of Butson-type, $H_{36} \in B(36, 6)$.

Question: is there a **real** 2-unitary **Hadamard** matrix of order 36?

Many thanks for inviting us to **Sevilla** !

With hard work and long practice one can develop skills to an astonishing level – enough to dazzle an audience with beautiful, yet utterly impractical feats