# On Hadamard matrices of order $4n^2$ , *n* odd

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May 26, 2025



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Here is a skew-type (skew) Hadamard matrix of order 4:

 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ - & 1 & 1 & - \\ - & - & 1 & 1 \\ - & 1 & - & 1 \end{bmatrix},$ 

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a regular:

Here is a skew-type (skew) Hadamard matrix of order 4:



a regular: Row and column sums are  $2 \mod 4$ ,

$$\begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix},$$

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a hybrid:

Here is a skew-type (skew) Hadamard matrix of order 4:



a regular: Row and column sums are  $2 \mod 4$ ,

$$\begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix},$$

a hybrid: skew, row sums:  $2 \mod 4$ , column sums:  $0 \mod 4$ ,

$$\begin{bmatrix} 1 & 1 & - & 1 \\ - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ - & - & - & 1 \end{bmatrix}$$

# A recursive method

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#### Mukhopadhaya in JCTA(1978) introduced a recursive method

Mukhopadhaya in JCTA(1978) introduced a recursive method that was extended by Seberry in J. Austral. Math.Soc.(1980):

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$$D = \begin{bmatrix} a & b & b & b \\ \bar{b} & a & b & \bar{b} \\ \bar{b} & \bar{b} & a & b \\ \bar{b} & b & \bar{b} & a \end{bmatrix},$$

$$D = \begin{bmatrix} a & b & b & b \\ \overline{b} & a & b & \overline{b} \\ \overline{b} & \overline{b} & a & b \\ \overline{b} & b & \overline{b} & a \end{bmatrix},$$
  
repeatedly changing *a* to 
$$\begin{bmatrix} b & b & b \\ b & b & b \\ b & b & b \end{bmatrix}$$

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repeatedly changing a to 
$$\begin{bmatrix} b & b & b \\ b & b & b \\ b & b & b \end{bmatrix}$$
 and b to 
$$\begin{bmatrix} a & b & \overline{b} \\ \overline{b} & a & b \\ b & \overline{b} & \overline{a} \end{bmatrix}$$
 results in an OD(4(3<sup>n</sup>); 3<sup>n</sup>, 3<sup>n+1</sup>),

$$D = \begin{bmatrix} a & b & b & b \\ \overline{b} & a & b & \overline{b} \\ \overline{b} & \overline{b} & a & b \\ \overline{b} & \overline{b} & \overline{b} & a \\ \overline{b} & \overline{b} & \overline{b} & \overline{b} \end{bmatrix},$$
  
repeatedly changing *a* to 
$$\begin{bmatrix} b & b & b \\ \overline{b} & b & b \\ \overline{b} & b & b \\ \overline{b} & \overline{b} & \overline{b} \end{bmatrix}$$
 and *b* to 
$$\begin{bmatrix} a & b & \overline{b} \\ \overline{b} & a & b \\ \overline{b} & \overline{b} & \overline{a} \end{bmatrix}$$
 results  
n an OD(4(3<sup>n</sup>); 3<sup>n</sup>, 3<sup>n+1</sup>), and therefore a Hadamard matrix of  
order 4(3<sup>n</sup>) for each  $n \in \mathbb{N}$ .

## By changing

$$D = \begin{bmatrix} a & b & b & b \\ \bar{b} & a & b & \bar{b} \\ \bar{b} & \bar{b} & a & b \\ \bar{b} & b & \bar{b} & a \end{bmatrix},$$

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#### By changing

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makes the corresponding Hadamard matrices of order  $4(3^{2n})$ **absolutely regular** for each *n*.

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makes the corresponding Hadamard matrices of order  $4(3^{2n})$ absolutely regular for each *n*.

Ionin-Kharaghani DCC(2003): Whenever  $q = (2h - 1)^2$ ,  $h = \pm 3^n$ , is a prime power, for every positive integer *m*, there is a symmetric design with parameters

$$\left(4h^{2}(\frac{q^{m+1}-1}{q-1}),(2h^{2}-h)q^{m},(h^{2}-h)q^{m}\right).$$

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Before I lose your interest!

Before I lose your interest! There are many hybrid Hadamard matrices of order 36.

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Before I lose your interest! There are many hybrid Hadamard matrices of order 36. Here is a Skew-type Hadamard matrix of order 36 with row sums  $\pm 6$ .

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# A new class of Hadamard matrices

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## skew-regular Hadamard matrices

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## skew-regular Hadamard matrices

#### Definition

An **absolutely regular** Hadamard matrix is a Hadamard matrix for which the **absolute value** of row sums is constant.

#### Definition

A **skew-regular** Hadamard matrix is a **skew-type** Hadamard matrix which is absolutely regular.

## Theorem (BKST 2025+)

Let H be a skew-regular Hadamard matrix of order  $4m^2$ ; then m is odd.

## Proof.

Note that each column sum of H is 2m + 2 modulo 4 and that negating any row of H changes every column sum by exactly two modulo 4.

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This rather simple-looking theorem has strong implications.

## First an application

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• If there is a skew-regular Hadamard matrix of order  $4m^2$ , then there is a regular Hadamard matrix of order  $4m^2(4m^2-1)^{2n}$  for each *n*.
## First an application

- If there is a skew-regular Hadamard matrix of order  $4m^2$ , then there is a regular Hadamard matrix of order  $4m^2(4m^2-1)^{2n}$  for each *n*.
- If there is a skew-regular Hadamard matrix of order  $4k^2$ . Then potentially whenever  $q = (2h 1)^2$ ,  $h = \pm (4k^2 1)^n$ , is a prime power, for every positive integer *m*, there is a symmetric design with parameters

$$\left(4h^{2}(\frac{q^{m+1}-1}{q-1}),(2h^{2}-h)q^{m},(h^{2}-h)q^{m}\right).$$

• A Hadamard matrix is skew-regular if and only if it is equivalent to a regular and a skew-type Hadamard matrix.

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- A Hadamard matrix is skew-regular if and only if it is equivalent to a regular and a skew-type Hadamard matrix.
- In order 16n<sup>2</sup>, the set of skew-type Hadamard matrices is disjoint from the set of regular matrices.

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- In order 16n<sup>2</sup>, the set of skew-type Hadamard matrices is disjoint from the set of regular matrices.
- In order 16, a skew-regular Hadamard matrix belongs to Hall class 4 or 5.

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• No skew-type Hadamard matrix of order 16n<sup>2</sup> has a **bent** sequence.

- A Hadamard matrix is skew-regular if and only if it is equivalent to a regular and a skew-type Hadamard matrix.
- In order 16n<sup>2</sup>, the set of skew-type Hadamard matrices is disjoint from the set of regular matrices.
- In order 16, a skew-regular Hadamard matrix belongs to Hall class 4 or 5.

- No skew-type Hadamard matrix of order 16n<sup>2</sup> has a **bent** sequence.
- No skew-type Hadamard matrix of order  $16n^2$  has an **unbiased mate**.



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Theorem (Araya, Harada, Kharaghani, Mohammadian, Tayfeh-Rezaie (DCC 2024)

There are at least 157132 SH-inequivalent skew-type Hadamard matrices of order 36.

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**Remark:** This new fact that every skew-type H(36) is equivalent to regular ones and all known H(36)s are potentially regular

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There are at least 157132 SH-inequivalent skew-regular Hadamard matrices of order 36.

**Remark:** This new fact that every skew-type H(36) is equivalent to regular ones and all known H(36)s are potentially regular made me consult with AI.

A conversation with Google Gemini and Microsoft Copilot

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Hadi: Is it true that all Hadamard matrices of order 36 are equivalent to regular ones?

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Yes, according to current mathematical knowledge, all Hadamard matrices of order 36 are equivalent to a regular Hadamard matrix,

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Hadi: Is it true that all Hadamard matrices of order 36 are equivalent to regular ones? Gemini:

Yes, according to current mathematical knowledge, all Hadamard matrices of order 36 are equivalent to a regular Hadamard matrix, meaning that any Hadamard matrix of order 36 can be transformed into a regular one by operations like row/column permutations and negations.

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Copilot:

Yes, it is true that every Hadamard matrix of order 36 is equivalent to a regular Hadamard matrix. This means that all Hadamard matrices of this order can be transformed into a regular form through equivalence operations.

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Yes, according to current mathematical knowledge, all Hadamard matrices of order 36 are equivalent to a regular Hadamard matrix, meaning that any Hadamard matrix of order 36 can be transformed into a regular one by operations like row/column permutations and negations.

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Yes, it is true that every Hadamard matrix of order 36 is equivalent to a regular Hadamard matrix. This means that all Hadamard matrices of this order can be transformed into a regular form through equivalence operations.

Is it time to try to prove every H(36) is equivalent to a regular one?

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#### Definition

# Let H be a quaternary complex Hadamard matrix of order n (QH(n)).

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#### Definition

Let *H* be a quaternary complex Hadamard matrix of order *n* (QH(n)). A solution to the equation  $HX = \sqrt{n}Y$ , *X*, *Y* in  $\{\pm 1, \pm i\}^n$  is called a *bent sequence*.

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#### Lemma

Let X be a bent sequence for the QH(n),

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#### Lemma

Let X be a bent sequence for the QH(n), and D the diagonal matrix with entries of X on its diagonal, then the quaternary Hadamard matrix HD is regular.

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Let X be a bent sequence for the QH(n), and D the diagonal matrix with entries of X on its diagonal, then the quaternary Hadamard matrix HD is regular.

#### Proof.

Let  $\mathbf{J}$  be the column vector with all one entries,

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#### Lemma

Let X be a bent sequence for the QH(n), and D the diagonal matrix with entries of X on its diagonal, then the quaternary Hadamard matrix HD is regular.

#### Proof.

Let **J** be the column vector with all one entries, then the row sums  $HD\mathbf{J} = HX = \sqrt{n}Y$  are of absolute value  $\sqrt{n}$ .

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It was necessary to use all related literature before proving that any Hadamard matrix of order 36 is regular.

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A recent paper by **Patric Östergård, and William Paavola, [J. Combin. Des., 2021] shows that:** 

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There are 1,955,625 inequivalent classes of quaternary Hadamard matrices of order 18.

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- There are 1,955,625 inequivalent classes of quaternary Hadamard matrices of order 18.
- First step: Check the 1,955,625 matrices for regularity.

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First step: Check the 1,955,625 matrices for regularity.

An email on Sat, Feb 15, 2025, at 2:40 PM, Zaitsev, Vlad, had the subject: **NOT Regular** 

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There are 1,955,625 inequivalent classes of quaternary Hadamard matrices of order 18.

First step: Check the 1,955,625 matrices for regularity.

An email on Sat, Feb 15, 2025, at 2:40 PM, Zaitsev, Vlad, had the subject: **NOT Regular** 

And it said: I think this is very interesting! The Hadamard matrix of order 36 coming from the complex matrix also had NO regularizing vectors!

It was necessary to use all related literature before proving that any Hadamard matrix of order 36 is regular.

A recent paper by **Patric Östergård, and William Paavola, [J. Combin. Des., 2021] shows that:** 

There are 1,955,625 inequivalent classes of quaternary Hadamard matrices of order 18.

First step: Check the 1,955,625 matrices for regularity.

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# 28 nonregular QH(18)

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# 28 nonregular QH(18)

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Kharaghani, Tayfeh-Rezaie, Zaitsev [2025+]:
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Kharaghani, Tayfeh-Rezaie, Zaitsev [2025+]: There are 28 non-regular and 1,955,597 regular quaternary complex Hadamard matrices of order 18.

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• First class consisting of eight inequivalent H36 with a maximum excess of 208.

Kharaghani, Tayfeh-Rezaie, Zaitsev [2025+]: There are 28 non-regular and 1,955,597 regular quaternary complex Hadamard matrices of order 18. The 28 non-regular QH(18) leads to two classes of Hadamard matrices of order 36 (H36).

- First class consisting of eight inequivalent H36 with a maximum excess of 208.
- The second class consisting of eight inequivalent H36 with a maximum excess of 204.

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We have seen that every skew-type H(36) is also skew-regular.

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Answer: No! We have tested five out of the nine known skew-type H(100) arising from Good matrices and none were equivalent to a regular one.

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#### Question 2: Is there a skew-regular H(100)?

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#### Question 2: Is there a skew-regular H(100)?

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#### Question 2: Is there a skew-regular H(100)?

Answer: Yes!

Question 3: Is it true that for any odd integer *n*, there is a skew-regular  $H(4n^2)$ ?

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Answer: I don't know!

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Answer: No! We have tested five out of the nine known skew-type H(100) arising from Good matrices and none were equivalent to a regular one.

Question 2: Is there a skew-regular H(100)?

Answer: Yes!

Question 3: Is it true that for any odd integer *n*, there is a skew-regular  $H(4n^2)$ ?

Answer: I don't know!

And I leave that for you to show.

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# Key Words

- Hadamard matrix.
- Quaternary Hadamard matrix.
- Skew-type Hadamard matrix.
- Regular Hadamard matrix.
- Skew-regular Hadamard matrix.
- Equivalence.
- Excess of a Hadamard matrix.
- Maximum Excess.
- Bent sequence.

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Happy Birthday Dane

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Happy Birthday Dane

# Thank you Andrés, and all the organizers of such a wonderful conference

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