Inextendibility from Zero-Entanglement MUBs in $\mathbb{C}^6$

Rakesh Kumar[1]

Joint work with S. Sarangi, S. Maitra, and A. Kumar

[1]Indian Statistical Institute, Kolkata

8th Workshop on Design Theory, Hadamard Matrices and Applications (Hadamard 2025)
organized by
Institute of Mathematics of the University of Seville (IMUS), Seville, Spain
26–30 May, 2025

Introduction to Mutually Unbiased Bases (MUBs)

Definition: Two orthonormal bases $B_1 = \{a_1, a_2..., a_d\}$ and $B_2 = \{b_1, b_2, ...b_d\}$ in d dimensional Hilbert spaces are mutually unbiased if,

$$|\langle a_i, b_j \rangle| = \frac{1}{\sqrt{d}}; \text{ for every } 1 \leq i, j \leq d.$$

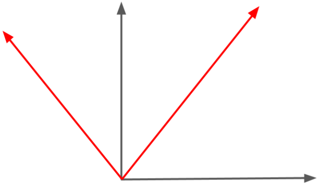
- A set $\{B_1, B_2, \ldots, B_m\}$ of orthonormal bases in C^d is called a set of mutually unbiased bases (a set of MUB) if each pair of bases B_i and B_j are mutually unbiased.

MUBs in dimension 2

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$$B_2 = \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\},$$

$$B_3 = \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$$



MUBs in dimension 3

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$B_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix},$$

$$B_4 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 \end{pmatrix}$$

Known Results:

$N(d) := \max\{n :$ there exist n MUBs of $C^d\}$.

Upper bound: $N(d) \leq d + 1$.

Lower bound: If $d = p_1^{k_1} p_2^{k_2} ... p_r^{k_r}$ such that $p_1^{k_1} < p_2^{k_2} < ... < p_r^{k_r}$; then

$$N(d) \geq p_1^{k_1} + 1.$$

- $N(p^k) = p^k + 1$ for all primes p.
- Some special constructions in specific dimensions beat lower bound.

  example: There are at least 6 MUBs in dimension $d = 26^2$.

  [Wockjan and Beth' 2004]

Open Problem

Determine $N(d)$ exactly for any d, not a prime power, or even just improve on the upper bound :

$$N(d) \leq d < d + 1$$

Zauner's conjecture: $N(6) = 3$

Preliminaries:

1. If $M_1, M_2, \ldots, M_k$ be a system of k MUBs in $\mathbb{C}^d$. Then for any unitary matrix U, the system $UM_1, UM_2, \ldots, UM_k$ is again a system of k MUBs in $\mathbb{C}^d$.
2. By corollary, if (M_1, M_2) are pair of MUB in $\mathbb{C}^d$ then $(I, M_1^{-1}M_2)$ are also pair of MUB in $\mathbb{C}^d$ and $M_1^{-1}M_2 = \frac{1}{\sqrt{d}}H$.
3. If tensor product of two unitary matrices is a Hadamard matrix, then both unitaries has to be Hadamard.
4. If $(M_i \otimes N_i, M_j \otimes N_j)$ are MUBs, this implies M_i and M_j are MUBs, and N_i and N_j are MUBs in corresponding dimensions. (follows from statement 3)

Main result

Theorem

There does not exist a unitary matrix U in the zero-entanglement subspace such that

$$\{M_1 \otimes N_1,\, M_2 \otimes N_2,\, M_3 \otimes N_3,\, U\}$$

are MUBs in $\mathbb{C}^6$, where $M_i \in U_2, N_i \in U_3$ and $U \in U_6$.

Proof Sketch

Lemma

A unitary matrix $U \in \mathbb{C}^{6\times 6}$ *in zero-entanglement sector can be represented in one of the following two forms:*

Form 1:

$$U = A \otimes B$$

where

$$A \in U_2, \quad B \in U_3.$$

Form 2:

$$U = A_1 \otimes B_1 + A_2 \otimes V_3 B_1$$

where

$$A_1 = \begin{bmatrix} |a_1\rangle & \mathbf{0} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mathbf{0} & |a_2\rangle \end{bmatrix}, \text{ and } \quad B_1, V_3 \in U_3.$$

Proof of Lemma

Let,

$$U = \begin{bmatrix} |\psi_1\rangle & |\psi_2\rangle & |\psi_3\rangle & |\psi_4\rangle & |\psi_5\rangle & |\psi_6\rangle \end{bmatrix};$$

such that

$$\langle \psi_i | \psi_j \rangle = \delta_{ij} \quad \text{and} \quad U^\dagger U = I.$$

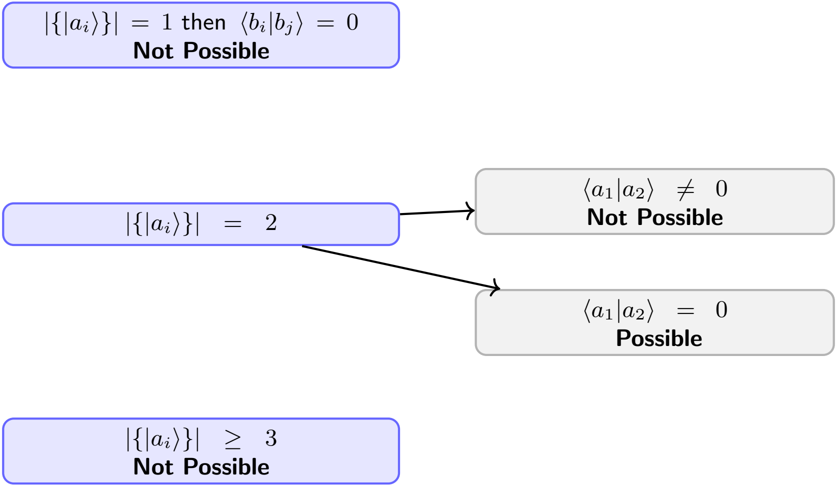
Since $|\psi_i\rangle = |a_i\rangle \otimes |b_i\rangle$, then

$$\langle \psi_i | \psi_j \rangle = \langle a_i | a_j \rangle \langle b_i | b_j \rangle = \delta_{ij} = 0 \quad \text{for } i \neq j.$$

Notation: For a set S, the cardinality $|S|$ represents the number of *distinct elements* in the set S.

We have only **three** cases:

Case Analysis Diagram (Proof Cont...)



Only Possibility (Proof Cont...)

$$|\{|a_i\rangle\}| = 2$$

$$|\{|b_i\rangle\}| = 3$$

Then, $U = A \otimes B$

$$|\{|b_i\rangle\}| = 6$$

Then, $U = A_1 \otimes B_1 + A_2 \otimes V_3 B_1$

Proof of Main result if $U = U_2 \otimes U_3$

$\{M_1 \otimes N_1, M_2 \otimes N_2, M_3 \otimes N_3, U_2 \otimes U_3\}$ are MUBs;

$$\text{So, } (U_2 \otimes U_3)^\dagger (M_i \otimes N_i) = \frac{1}{\sqrt{6}} H_i.$$

Let

$$U_2^\dagger M_i = A_i \quad \text{and} \quad U_3^\dagger N_i = B_i,$$

such that

$$A_i \otimes B_i = \frac{1}{\sqrt{6}} H_i.$$

Fact: If tensor product of two unitary matrices is a Hadamard matrix, then both unitaries has to be Hadamard.

Proof of Main result if $U = A_1 \otimes B_1 + A_2 \otimes B_2$

$$\{M_1 \otimes N_1, M_2 \otimes N_2, M_3 \otimes N_3, U\} \text{ are MUBs;}$$

$$\text{So,} \quad U^\dagger (M_i \otimes N_i) = \frac{1}{\sqrt{6}} H_i.$$

Since,

$$U = \begin{bmatrix} |a_1\rangle & \mathbf{0} \end{bmatrix} \otimes B_1 + \begin{bmatrix} \mathbf{0} & |a_2\rangle \end{bmatrix} \otimes B_2$$

Where,

$$A_1 = \begin{bmatrix} |a_1\rangle & \mathbf{0} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mathbf{0} & |a_2\rangle \end{bmatrix} = \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}.$$

Proof (Cont...)

$$U^\dagger(M_i \otimes N_i) = \left(\begin{bmatrix}|a_1\rangle & \mathbf{0}\end{bmatrix}^\dagger \otimes B_1^\dagger + \begin{bmatrix}\mathbf{0} & |a_2\rangle\end{bmatrix}^\dagger \otimes B_2^\dagger\right)(M_i \otimes N_i)$$

$$= \begin{bmatrix}|a_1\rangle & \mathbf{0}\end{bmatrix}^\dagger M_i \otimes B_1^\dagger N_i + \begin{bmatrix}\mathbf{0} & |a_2\rangle\end{bmatrix}^\dagger M_i \otimes B_2^\dagger N_i$$

Let,

$$B_1^\dagger N_i = V_1^i \text{ and } B_2^\dagger N_i = V_2^i.$$

Note:

- $$V_1^i, V_2^i \in U_3.$$
- If X and Y be unitary matrices such that

  $$X = \begin{bmatrix}|x_1\rangle & |x_2\rangle & \cdots & |x_d\rangle\end{bmatrix}, \quad Y = \begin{bmatrix}|y_1\rangle & |y_2\rangle & \cdots & |y_d\rangle\end{bmatrix}$$

  then,

  $$(X^\dagger Y)_{ij} = \langle x_i | y_j \rangle$$

  .

Proof(Cont...)

So,

$$U^\dagger(M_i \otimes N_i) = \begin{bmatrix} \langle a_1|m_1^i\rangle & \langle a_1|m_2^i\rangle \\ 0 & 0 \end{bmatrix} \otimes V_1^i + \begin{bmatrix} 0 & 0 \\ \langle a_2|m_1^i\rangle & \langle a_2|m_2^i\rangle \end{bmatrix} \otimes V_2^i$$

$$= \begin{bmatrix} \langle a_1|m_1^i\rangle \, V_1^i & \langle a_1|m_2^i\rangle \, V_1^i \\ \langle a_2|m_1^i\rangle \, V_2^i & \langle a_2|m_2^i\rangle \, V_2^i \end{bmatrix} = \frac{1}{\sqrt{6}} H_i.$$

This implies,

$$A^\dagger M_i = \begin{bmatrix} \langle a_1|m_1^i\rangle & \langle a_1|m_2^i\rangle \\ \langle a_2|m_1^i\rangle & \langle a_2|m_2^i\rangle \end{bmatrix} \text{ is } \frac{1}{\sqrt{2}} \text{ times a Hadamard matrix in d=2,}$$

which means A and M_i must be MUBs for all i.

Generic Result

Result: Consider generic composite dimension $d = p_1^{k_1} p_2^{k_2} ... p_r^{k_r}$ such that $p_1^{k_1} < p_2^{k_2} < ... < p_r^{k_r}$; then it is not possible to have more than $p_1^k + 1$ MUBs if all vectors come from the tensor product space.

Thank you !