Aalto University
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Classification of weighing matrices

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Joint work with Mikhail Ganzhinov

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Weighing matrix

Definition

A weighing matrix of order n and weight k, denoted by $W(n, k)$ is an $n \times n$ matrix with entries $0, 1, -1$, such that,

$$\mathbf{W}\mathbf{W}^T = k\,\mathbf{I}_n \quad ,$$

where $1 \leq k \leq n$.

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Example: a $W(8, 4)$

0	0	0	0	1	1	1	1
0	0	1	1	0	0	1	-1
0	1	0	1	0	1	-1	0
0	1	-1	0	-1	0	1	0
1	0	0	1	0	-1	0	1
1	0	-1	0	1	0	0	-1
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1	-1	0	0	-1	1	0	0

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1	0	0	1	0	-1	0	1
1	0	-1	0	1	0	0	-1
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0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\
0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\
1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\
1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0
\end{matrix}$$

- rows (columns) are mutually orthogonal
- when $k = n$, the matrix is a Hadamard matrix.
- when $k = n - 1$, the matrix is equivalent to a conference matrix.

Equivalence of weighing matrices

These operations produce a weighing matrix when applied to any weighing matrix:

1. Permuting the order of rows,
2. Permuting the order of columns,
3. Multiplying a row by -1,
4. Multiplying a column -1.

Equivalence of weighing matrices

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Weighing matrices are equivalent if they are essentially the same in the following sense:

Definition

Weighing matrices A and B are equivalent, denoted by, $A \cong B$, if B can be generated from A by applying Operations 1, 2, 3, and 4.

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1. Existence
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   - In this work we give complete classification of orders 16,18,19,20 and 21.
   - Also we give complete classification of some cases in orders 22,24,28.

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   - In this work we give complete classification of orders 16,18,19,20 and 21.
   - Also we give complete classification of some cases in orders 22,24,28.
3. Exhaustive computer search
   - A new computational method: the criss-cross search.

Computational methods

1. Orderly generation of matrices

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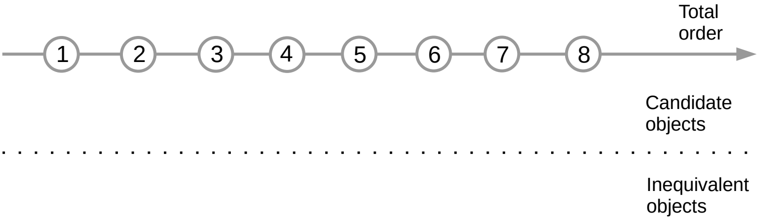
3. Pruning of isomorphic matrices and substructures
4. Criss-cross search (NEW!)
5. Cliquer search

Orderly generation

- Objects are generated in some predefined total order.
- Example:
  - colors depict equivalence classes



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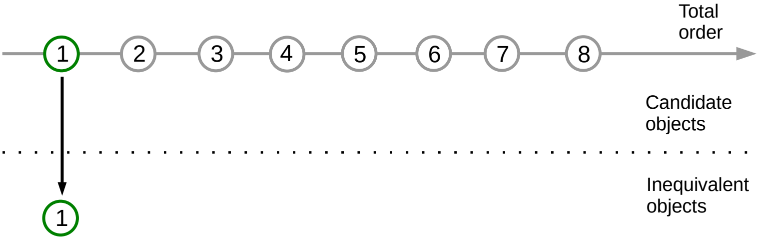

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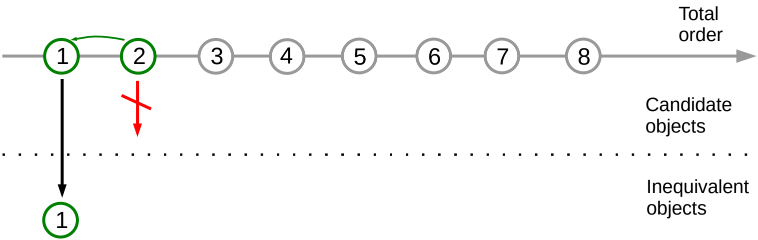

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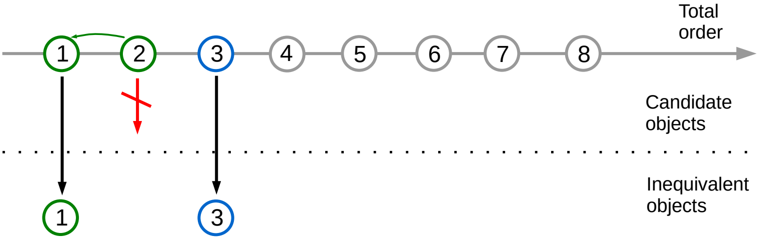

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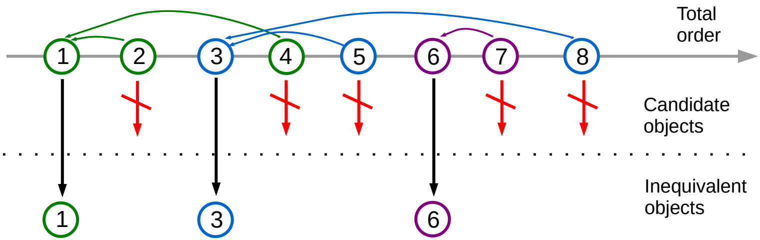

Computational methods: basic idea

Append rows (or columns) to a rectangular weighing matrix until it becomes a square matrix. Example: $W(8,4)$:

$$\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{matrix} \quad \in \mathrm{R2}$$

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$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{array} \quad \in \mathrm{R3}$$

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$$\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{matrix} \quad \in \text{R4}$$

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Append rows (or columns) to a rectangular weighing matrix until it becomes a square matrix. Example: $W(8,4)$:

$$\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \end{matrix} \quad \in \mathrm{R8}$$

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Append rows (or columns) to a rectangular weighing matrix until it becomes a square matrix. Example: $W(8,4)$:

$$
\begin{matrix}
0 & 0 & . & . & . & . & . & . \\
0 & 0 & . & . & . & . & . & . \\
0 & 1 & . & . & . & . & . & . \\
0 & 1 & . & . & . & . & . & . \\
1 & 0 & . & . & . & . & . & . \\
1 & 0 & . & . & . & . & . & . \\
1 & 1 & . & . & . & . & . & . \\
1 & -1 & . & . & . & . & . & .
\end{matrix}
\quad \in \mathrm{C2}
$$

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Append rows (or columns) to a rectangular weighing matrix until it becomes a square matrix. Example: $W(8,4)$:

$$\begin{array}{cccccccc} 0 & 0 & 0 & . & . & . & . & . \\ 0 & 0 & 1 & . & . & . & . & . \\ 0 & 1 & 0 & . & . & . & . & . \\ 0 & 1 & -1 & . & . & . & . & . \\ 1 & 0 & 0 & . & . & . & . & . \\ 1 & 0 & -1 & . & . & . & . & . \\ 1 & 1 & 1 & . & . & . & . & . \\ 1 & -1 & 0 & . & . & . & . & . \end{array} \quad \in \mathrm{C3}$$

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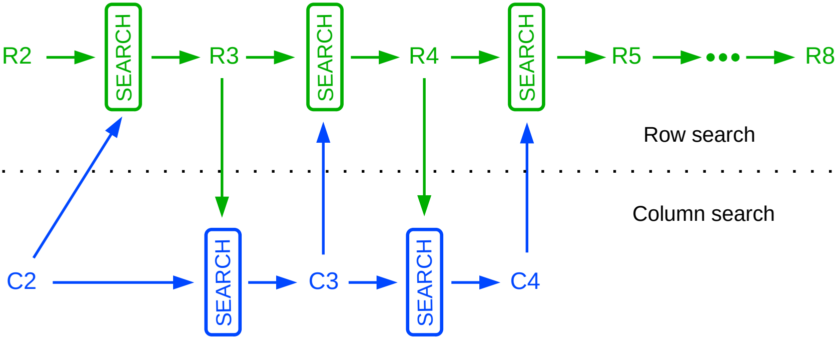
Append rows (or columns) to a rectangular weighing matrix until it becomes a square matrix. Example: $W(8, 4)$:

$$\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \end{matrix} \qquad \in \mathrm{C8}$$

Criss-cross search



Criss-cross search (cont.)

Let C be an $n \times k$ and R $m \times n$ rectangular weighing matrices:

C/R	C/R	R	R	R	R	R	R
C/R	C/R	R	R	R	R	R	R
C/R	C/R	R	R	R	R	R	R
C	C	.	.	.	.	.	.
C	C	.	.	.	.	.	.
C	C	.	.	.	.	.	.
C	C	.	.	.	.	.	.
C	C	.	.	.	.	.	.

C and R are matching row and column search matrices.

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C/R	C/R	R	R	R	R	R	R
C	C	.	.	.	.	.	.
C	C	.	.	.	.	.	.
C	C	.	.	.	.	.	.
C	C	.	.	.	.	.	.
C	C	.	.	.	.	.	.

C and R are matching row and column search matrices.

An equivalence defining operation f (row/column permutation, row/column multiplication) is compatible with C if

$$f(R) \geq R \quad , \text{for all } R.$$

Computational methods: clique search

Check if a $k \times n$ rectangular matrix $\mathbf{A}$ could be extended to a full weighing matrix $\mathbf{W}$ of order n.

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Check if a $k \times n$ rectangular matrix **A** could be extended to a full weighing matrix **W** of order n.

- Find set C of all those rows that are orthogonal with all rows of A.

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- Form a graph G where each row in C is a node and there is an edge between the rows if they are orthogonal.

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- Form a graph G where each row in C is a node and there is an edge between the rows if they are orthogonal.
- Solve maximum clique problem in G with Cliquer software (Niskanen, Östergård)
- If the size of the maximum clique is $n - k$, we have found a weighing matrix.

Computational methods: example

How we computed $W(20, 12)$.

1. Created seeds with Criss-cross search.

Computational methods: example

How we computed $W(20, 12)$.

1. Created seeds with Criss-cross search.
   - seeds are 5×20 rectangular weighing matrices
   - single-threaded computation
   - CPU-time 4.36 hours
   - Real-time 4.38 hours
   - Found $4,055,445$ seeds

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   - CPU-time 4.36 hours
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   - Found $4, 055, 445$ seeds
2. Extended seeds to full matrices with Clique search.
   - Parallel computation on a computing cluster using 640 cores
   - CPU-time 38.3 years
   - Real-time 23.4 days
   - $3, 503, 212$ inequivalent matrices found

Results

Order	Weight	#	Ref.
16	1,2,3	1	[1]
	4	10	[1]
	5	4	[1]
	6	30	[3]
	7	55	
	8	4631	
	9	704	[3]
	10	743	670 in [3]
	11	43	
	12	279	[3]
	13	14	
	14	17	
	15	1	[1]
	16	5	[2]

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16	1,2,3	1	[1]
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	4	2	[1]
	9	2360	[3]

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17	1	1	[1]
	4	2	[1]
	9	2360	[3]
18	1	1	[1]
	2	1	[1]
	4	11	[1]
	5	5	[3]
	8	4817	
	9	11891	[3]
	10	70133	
	13	53	
	16	4	
	17	1	[1]

Results

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16	1,2,3	1	[1]
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	5	4	[1]
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	7	55	
	8	4631	
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	2	1	[1]
	4	11	[1]
	5	5	[3]
	8	4817	
	9	11891	[3]
	10	70133	
	13	53	
	16	4	
	17	1	[1]
19	1	1	[1]
	4	5	[1]
	9	189076	

Order	Weight	#	Ref.
20	1,2,3	1	[1]
	4	18	[1]
	5	7	[3]
	6	49	[3]
	7	159	
	8	18294	
	9	2924696	
	10	73418583	
	11	13506863	
	12	3503212	
	13	34312	
	14	6254	
	15	1351	
	16	2164	
	17	58	
	18	53	[3]
	19	2	[2]
	20	3	[2]

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	9	2924696	
	10	73418583	
	11	13506863	
	12	3503212	
	13	34312	
	14	6254	
	15	1351	
	16	2164	
	17	58	
	18	53	[3]
	19	2	[2]
	20	3	[2]

Order	Weight	#	Ref.
21	1	1	[1]
	4	1	[1]
	9	5756272	
	16	40	

Order	Weight	#	Ref.
22	1,2	1	[1]
	4	22	[1]
	5	9	[3]
	8	196986	
	9,10,13,16	?	
	17	1588	
	18	1621	
	20	6	

Order	Weight	#	Ref.
22	1,2	1	[1]
	4	22	[1]
	5	9	[3]
	8	196986	
	9,10,13,16	?	
	17	1588	
	18	1621	
	20	6	
23	1	1	[1]
	4	10	[1]
	9,16	?	

Order	Weight	#	Ref.
22	1,2	1	[1]
	4	22	[1]
	5	9	[3]
	8	196986	
	9,10,13,16	?	
	17	1588	
	18	1621	
	20	6	
23	1	1	[1]
	4	10	[1]
	9,16	?	

Order	Weight	#	Ref.
24	1,2,3	1	[1]
	4	33	[1]
	5	14	[3]
	6	190	[3]
	7	336	
	8-20	?	
	21	2493	
	22	1440	
	23	9	[5]
	24	60	[5]

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22	1,2	1	[1]
	4	22	[1]
	5	9	[3]
	8	196986	
	9,10,13,16	?	
	17	1588	
	18	1621	
	20	6	
23	1	1	[1]
	4	10	[1]
	9,16	?	

Order	Weight	#	Ref.
24	1,2,3	1	[1]
	4	33	[1]
	5	14	[3]
	6	190	[3]
	7	336	
	8-20	?	
	21	2493	
	22	1440	
	23	9	[5]
	24	60	[5]
25	1	1	[1]
	4	11	[1]
	9,16	?	

Order	Weight	#	Ref.
26	1,2	1	[1]
	4	39	[1]
	5	16	[3]
	8,9,10,13,16,17,18,20	?	
	25	4	[4]

Order	Weight	#	Ref.
26	1,2	1	[1]
	4	39	[1]
	5	16	[3]
	8,9,10,13,16,17,18,20	?	
	25	4	[4]
27	1	1	[1]
	4	18	[1]
	9,16	?	

Order	Weight	#	Ref.
26	1,2	1	[1]
	4	39	[1]
	5	16	[3]
	8,9,10,13,16,17,18,20	?	
	25	4	[4]
27	1	1	[1]
	4	18	[1]
	9,16	?	
28	1,2,3	1	[1]
	4	57	[1]
	5	22	[3]
	6	684	
	7-26	?	
	27	41	[5]
	28	487	[5]

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- We have classified 32 cases.
- These results are preliminary.
- Further performance improvements are possible.
- The work continues.

Thank you!

References

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