Partial Difference Sets: Broadening the Scope of the Denniston Family

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Joint work with James Davis, Sophie Huczynska, Laura Johnson, John Polhill

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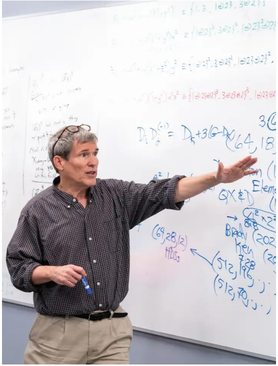

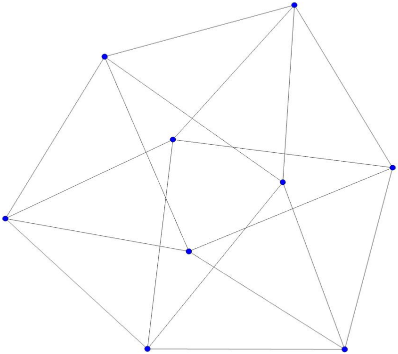

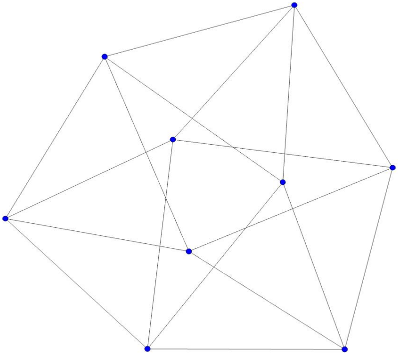

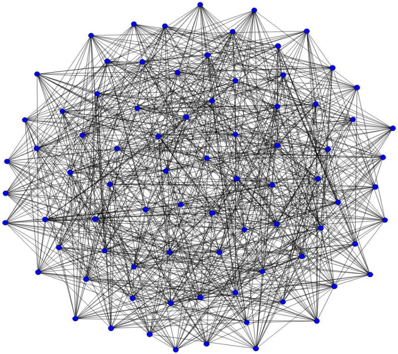

Let $ax^2 + bxy + cy^2$ be an irreducible quadratic form over $\mathbb{F}_{2^m}$. Let H be an order 2^r subgroup of $(\mathbb{F}_{2^m}, +)$, with $1 \le r \le m-1$. Consider a subset

$$D = \{(x, y) \mid ax^2 + bxy + cy^2 \in H\} \subset \mathbb{F}_{2^m} \times \mathbb{F}_{2^m},$$

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Note that $a+bd+cd^2 \neq 0$ and x^2 induces a permutation over $\mathbb{F}_{2^m}$ (due to $\mathbb{F}_{2^m}$ has characteristic 2), $|D \cap L| = 2^r$.

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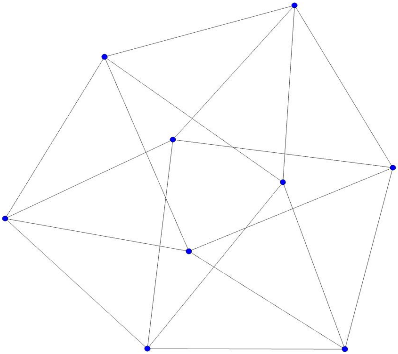
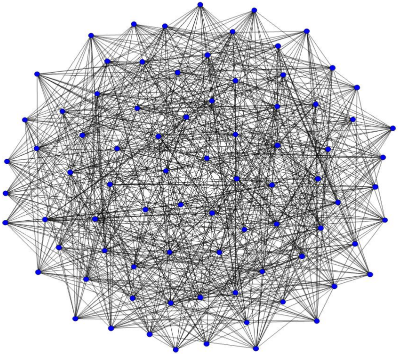
Two-intersection sets in $\mathrm{PG}(2, q)$ are well-appreciated structures, which lead to two-weight codes (coding theory), strongly regular graphs (graph theory), partial difference sets (design theory).





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An (n, d)-arc in $PG(2, q)$ is *nontrivial* if $1 < d < q$.

$ax^2 + bxy + cy^2$ irreducible over $\mathbb{F}_{2^m}$; H subgroup of $(\mathbb{F}_{2^m}, +)$ with $|H| = 2^r$, $1 \leq r \leq m-1$, we have

$$D' = \{[1 : x : y] : ax^2 + bxy + cy^2 \in H\}$$

satisfying $|D'| = 1 + (2^m + 1)(2^r - 1)$ and every line in $\mathrm{PG}(2, 2^m)$ intersecting D' in either 0 or 2^r points.

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Therefore, D' is a $(1 + (2^m + 1)(2^r - 1), 2^r)$-arc in $\mathrm{PG}(2, 2^m)$.

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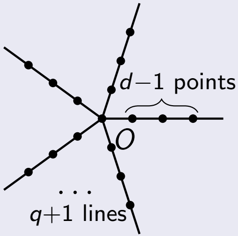

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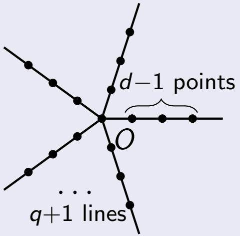

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Denniston's maximal arc (Denniston, 1969)

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is a $(1 + (2^m + 1)(2^r - 1), 2^r)$ maximal arc in $\mathrm{PG}(2, q)$ with $1 \le r < m$.

Expand

$$D' = \{[1 : x : y] : ax^2 + bxy + cy^2 \in H\}$$

into

$$\overline{D'} = \{z(1, x, y) : ax^2 + bxy + cy^2 \in H, z \in \mathbb{F}_{2^m}^*\} \subset (\mathbb{F}_{2^m}^3, +),$$

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For each nonzero element $g \in \mathbb{F}_{2^m}^3$,

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$\overline{D'}$ is a $(2^{3m}, (2^{m+r} - 2^m + 2^r)(2^m - 1), 2^m - 2^r + (2^{m+r} - 2^m + 2^r)(2^r - 2), (2^{m+r} - 2^m + 2^r)(2^r - 1))$ *partial difference sets* in $\mathbb{F}_2^{3m}$, which are called Denniston partial difference set (in characteristic 2).

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The Denniston family stands out as a family of PDSs that is distinct from the two classical families with intrinsic connection to maximal arcs.

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The nonexistence of maximal arcs in $\mathrm{PG}(2, q)$ with odd prime power q *seemingly* indicates a negative answer.

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How to reconcile these constructions with the nonexistence result of Ball, Blokhuis, and Mazzocca, that no maximal arcs in $PG(2, q)$ with odd prime power q exist?

Reconciliation (Bao, Xiang, Zhao, 2025)

For a prime power q and $1 \leq r \leq m-1$,

an (n, q^r) Denniston maximal arc in $\mathrm{PG}(2, q^m)$ $\Leftrightarrow$ a Denniston PDS S in $\mathbb{F}_q^{3m}$ which is $\mathbb{F}_{q^m}^*$ invariant: $\{\alpha x \mid x \in S\} = S$ for each $\alpha \in \mathbb{F}_{q^m}^*$

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The nonexistence of Denniston maximal arcs does not imply the nonexistence of Denniston PDSs, but only the nonexistence of Denniston PDSs with an additional $\mathbb{F}_{q^m}^*$ invariant property

Remark

Bao, Xiang, and Zhao (2025) established the existence of Denniston PDSs in elementary abelian groups for all possible parameters. It seems that the existence of Denniston PDSs in odd characteristic, after being open for more than five decades, has finally been resolved.

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De Winter (2025) observed that, in a largely unnoticed paper, Bierbrauer and Edel (1997) constructed two-weight $[\frac{q^r(q^m-1)(q^m-q^{m-r}+1)}{q-1}, 3m]_q$ code with weights $q^{m+r-1}(q^m-q^{m-r}+1)$ and $q^{2m+r-1}-q^{2m-1}$. The construction gave Denniston PDSs in both even and odd characteristic, sharing the same parameter as Bao, Xiang, and Zhao (2025).

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It remains unclear whether these two constructions are equivalent or not.

Hence, as a matter of fact, the nonexistence of Denniston maximal arcs in odd characteristic and the existence Denniston PDSs in odd characteristic have both been confirmed in the same year of 1997!

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Luckily, our construction includes Ott's as a special case.

Theorem (Li, Davis, Huczynska, Johnson, and Polhill, 2025+)

Let $s \geq 1$ and $q = p^s$ be a power of prime p. Let m and ℓ be positive integers. For each $0 \leq r \leq m$, there exists a $(v, k_r, \lambda_r, \mu_r)$ generalized Denniston PDS D_r in the elementary abelian p-group $G = \mathbb{Z}_p^{sm(2\ell+1)} \cong (\mathbb{F}_q^{m\ell} \times \mathbb{F}_q^{m(\ell+1)}, +)$, where

$$\begin{aligned}
v =& q^{m(2\ell+1)}, \\
k_r =& \frac{(q^r-1)(q^{m\ell}-1)(q^{m(\ell+1)}-1)}{q^m-1} + q^{m\ell} - 1, \\
\lambda_r =& \frac{(q^r-1)(q^{m(\ell+1)}-1)}{q^m-1}\left(\frac{(q^r-1)(q^{m\ell}-1)}{q^m-1} - 1\right) + q^{m\ell} - 2, \\
\mu_r =& \frac{(q^r-1)(q^{m\ell}-1)}{q^m-1}\left(\frac{(q^r-1)(q^{m(\ell+1)}-1)}{q^m-1} + 1\right).
\end{aligned}$$

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(7) Applying $\ell = 1$ and $1 \leq r \leq m - 1$, we recover the construction by Bao, Xiang, and Zhao (2025).

Proof techniques

Our proof relies heavily on the elegant machinery established in Momihara and Xiang (2014), which presented a meticulous manipulation of Gauss sums.

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Open Question

Our construction in group $(\mathbb{F}_q^{m\ell} \times \mathbb{F}_q^{m(\ell+1)}, +)$ relies crucially on the fact that $\ell + 1 - \ell = 1$. Would it be possible to ease or lift this restriction?

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