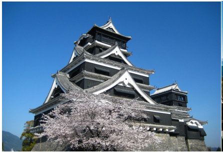
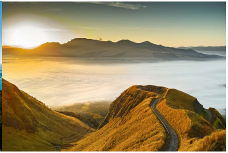

Kumamoto castle


Amakusa Sea


Aso Mountain

Non-commutative association schemes having divisible design graphs as relations from pseudo-cyclic association schemes

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Outline

1. Symmetric group divisible design
   - GDDs from Hadamard matrices
   - GDDs from balanced generalized weighing matrices
2. Divisible design graph
   - BGW matrices from pseudo-cyclic association schemes
   - DD graphs from pseudo-cyclic association schemes
3. Isomorphism
   - Isomorphism between non-commutative association schemes
   - Transformation of pseudo-cyclic association schemes

Keywords: Group divisible design; Hadamard matrix, Balanced generalized weighing matrix; Divisible design graph; Pseudo-cyclic association scheme; Cyclotomic scheme; Conference graph; Godsil-McKay switching

Group divisible designs from Hadamard matrices

Replace $+1$ by I_2 and -1 by $J_2 - I_2$ in a Hadamard matrix.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

- $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for any two distinct rows $\mathbf{x}, \mathbf{y}$ from the same group.
- $\langle \mathbf{x}, \mathbf{y} \rangle = 2$ for any two rows $\mathbf{x}, \mathbf{y}$ from distinct groups.

Group divisible design

Definition: GD design

V: a finite set of $v = mn > 0$ elements
$\mathcal{G}$: a partition of V into m subsets (called **groups**) of size n
$\mathcal{B}$: a set of b subsets (called **blocks**) of size k of V

$(V, \mathcal{G}, \mathcal{B})$ is called a **group divisible (GD) design** with parameters $(m, n, k, \lambda_1, \lambda_2)$ if

(1) every pair of distinct elements of V in the same group occurs in exactly λ_1 blocks, and

(2) every pair of distinct elements of V from distinct groups occurs in exactly λ_2 blocks.

We are concerned with only **symmetric** GD designs satisfying $v = b$.

The construction of GDDs using Hadamard matrices was generalized by Gibbons-Mathon (1987) [1] and De Launey (1987) [2] using generalized Bhaskar Rao designs (or balanced generalized weighing matrices for symmetric GD design).

Definition: BGW matrix

A **balanced generalized weighing (BGW) matrix** with parameters (v, k, λ) over a group G is a square matrix $M = (m_{i,j})$ of order v with entries from $G \cup \{0\}$ s.t.

(i) every row of M contains exactly k nonzero entries, and

(ii) for any distinct $i, h \in \{1, 2, \ldots, v\}$, every element of G is contained exactly λ times in the multiset $\{m_{i,j} m_{h,j}^{-1} \mid 1 \leq j \leq v, m_{i,j}, m_{h,j} \neq 0\}$.

[1] P. B. Gibbons, R. Mathon, Construction methods for Bhaskar Rao and related designs, *J. Austral. Math. Soc. Ser. A* **42** (1987) 5–30.

[2] W. De Launey, $(0, G)$-*Designs and Applications*, PhD thesis, Univ. of Sydney, (1987).

GDDs from BGW matrices

Theorem

Let M be a BGW matrix with parameters (v, k, λ) over G of order g. Replace the elements of G by the corresponding $g \times g$ permutation matrices and 0-entry by the $g \times g$ null matrix in M. Then, the resulting matrix M' is an incidence matrix of a symmetric GD design with parameters $(v, g, k, 0, \lambda)$.

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \Rightarrow M' = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

Note that M is symmetric but M' is not necessarily symmetric.

Divisible design graph

Let N be an incidence matrix of symmetric GD design with parameters $(m, n, k, \lambda_1, \lambda_2)$. If N is symmetric and off-diagonal, N can be viewed as an adjacency matrix of a k-regular graph, called a **divisible design (DD) graph** with parameters $(m, n, k, \lambda_1, \lambda_2)$, whose vertex-set can be partitioned into m classes of size n s.t.

1. any two vertices from the same class have exactly λ_1 common neighbors; and
2. any two vertices from different classes have exactly λ_2 common neighbors.

DD graphs were introduced by Haemers-Kharaghani-Meulenberg [3].

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[3]W. H. Haemers, H. Kharaghani, M. A. Meulenberg, Divisible design graphs, *J. Combin. Theory, Ser. A* **118** (2011) 978–992.

Association schemes

Definition: Association scheme (AS)

A **d-class association scheme** on a finite set X is a partition of $X \times X$ into subsets $R_0, R_1, \ldots, R_d$, called **relations**, s.t.

(1) $R_0 = \{(x, x) \mid x \in X\}$,

(2) $R_i^\top := \{(y, x) \mid (x, y) \in R_i\} \in \{R_0, R_1, \ldots, R_d\}$ for any i,

(3) for all $i, j, k \in \{0, 1, \ldots, d\}$, there is an integer $p_{i,j}^k$ s.t. for all $(x, y) \in R_k$, $|\{z \in X \mid (x, z) \in R_i$ and $(z, y) \in R_j\}| = p_{i,j}^k$.

Let A_i be the adjacency matrix of R_i. By (3), for $i, j \in \{0, 1, \ldots, d\}$,

$$A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k$$

for some positive integers $p_{i,j}^k$, called **intersection numbers** of the AS.

We denote the AS by $(X, \{R_i\}_{i=0}^d)$ or $(X, \{A_i\}_{i=0}^d)$.

The first eigenmatrix

If $A_iA_j = A_jA_i$ for all i, j, $(X, \{A_i\}_{i=0}^d)$ is said be **commutative**. In this case, $A_0, A_1, \ldots, A_d$ form a basis of a commutative algebra, called the **Bose-Mesner algebra**, generated by $A_0, A_1, \ldots, A_d$ over $\mathbb{C}$. In particular, if $A_i = A_i^\top$ for all i, $(X, \{A_i\}_{i=0}^d)$ is said be **symmetric**.

$E_0 = \frac{1}{|X|}J, E_1, \ldots, E_d$: the unique primitive idempotents of the Bose-Mesner algebra of $(X, \{R_i\}_{i=0}^d)$, which form a basis of the algebra. Define $P = (P_j(i))_{0\leq i,j\leq d}$, called the **first eigenmatrix**, satisfying

$$(A_0, A_1, \ldots, A_d) = (E_0, E_1, \ldots, E_d)P.$$

Note that $P_i(j)$ is an eigenvalue of A_i as $A_iE_j = P_i(j)E_j$.

The integers $k_i = P_i(0)$, $0 \leq i \leq d$, and $m_i = \text{rank}\, E_i$, $0 \leq i \leq d$, are called **valencies** and **multiplicities**, respectively.

Pseudo-cyclic association scheme

A d-class AS having the nontrivial multiplicities $m_1 = m_2 = \cdots = m_d$ is called **pseudo-cyclic**.

Lemma

The nontrivial valencies of a pseudo-cyclic AS are all same, which coincide with the nontrivial multiplicity.

Example (Cyclotomic scheme)

Let $X = \mathbb{F}_q$ be the finite field of order q and C_0 be a multiplicative subgroup of index d with $-1 \in C_0$. Furthermore, let $C_0, C_1, \ldots, C_{d-1}$ be the cosets of C_0. Define

$$(x, y) \in R_{i+1} \text{ iff } x - y \in C_i.$$

Then, $(X, \{R_i\}_{i=0}^{d})$ with R_0 the diagonal relation is a pseudo-cyclic AS.

BGW matrices from pseudo-cyclic association schemes

Proposition

Let $\mathcal{A} = (X, \{R_i\}_{i=0}^{d})$ be a d-class symmetric pseudo-cyclic AS with common valency k, whose first eigenmatrix has the form

$$P = \begin{pmatrix} 1 & k\mathbf{1}_d^\top \\ \mathbf{1}_d & P' \end{pmatrix}$$

for some square circulant matrix P' of size d, where the columns of P' are labeled by $A_1, A_2, \ldots, A_d$ in this order. Let $G = \langle \omega \rangle$ be a cyclic group of order d with identity 1. Then,

$$M_{\mathcal{A}} = \begin{pmatrix} 0 & \mathbf{1}_v^\top \\ \mathbf{1}_v & \sum_{i=1}^{d} \omega^{i-1} A_i \end{pmatrix}$$

is a BGW matrix with parameters $(v + 1, v, (v - 1)/d)$ over G.

Key property

We call P' the **principal part** of P. Assume that P' is circulant. Then,

$$\sum_{i=1}^{d} A_i A_{i+t} = \begin{cases} (k-1)J_v + (v-k)I_v & \text{if } t = 0, \\ k(J_v - I_v) & \text{if } t \geq 1, \end{cases}$$

where $i + t$ takes the value in $\{1, 2, \ldots, d\}$ computed modulo d.

The principal parts P' of the following classes of ASs are circulant [4].

- 2 or 3-class pseudo-cyclic ASs
- Cyclotomic schemes
- Strongly regular decompositions of Latin or negative Latin square type

[4] M. Muzychuk, I. Ponomarenko, On pseudocyclic association schemes, *Ars Math. Contemp.* **5** (2012) 1–25.

DD graphs from pseudo-cyclic association schemes

$\mathcal{A} = (X, \{A_i\}_{i=0}^{d})$: a symmetric pseudo-cyclic AS s.t. P' is circulant
C: the circulant matrix of order d with the first row $(0, 1, 0, 0, \ldots, 0)$
R: the back diagonal matrix of order d
Define for $t = 0, 1, \ldots, d-1$, the $(0, 1)$-matrix $B_{0,t}$ to be

$$B_{0,t} = \begin{pmatrix} C^t & O_{d,dv} \\ O_{dv,d} & C^{-t} \otimes I_v \end{pmatrix}.$$

Furthermore, define for $t = 0, 1, \ldots, d-1$, the $(0, 1)$-matrix $B_{1,t}$ to be

$$B_{1,t} = \begin{pmatrix} O_d & C^t \otimes \mathbf{1}_v^\top \\ C^{-t} \otimes \mathbf{1}_v & \sum_{i=1}^{d} C^i R \otimes A_{i+t} \end{pmatrix}.$$

Theorem 1 (M.-Suda)

$\{B_{i,t} \mid i = 0, 1, 0 \le t \le d-1\}$ forms a $(2d-1)$-class non-commutative AS.
In particular, $B_{1,t}$, $0 \le t \le d-1$, are DD graphs with parameters $(d(v+1), v, 0, k)$.

Denote the resulting AS by $\mathcal{D}_{\mathcal{A}}$ for the assumed AS $\mathcal{A}$.

Example ($B_{0,t}$'s in the $d = 4$ case)

Let $d = 4$. Then, $B_{i,t}$, $i = 0, 1$, $t = 0, 1, 2, 3$, are given as follows.

$$B_{0,0} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & O_{4,4v} & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ \hline & & & & I_v & O_v & O_v & O_v \\ & O_{4v,4} & & & O_v & I_v & O_v & O_v \\ & & & & O_v & O_v & I_v & O_v \\ & & & & O_v & O_v & O_v & I_v \end{array}\right), B_{0,1} = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & O_{4,4v} & & \\ 0 & 0 & 0 & 1 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ \hline & & & & O_v & O_v & O_v & I_v \\ & O_{4v,4} & & & I_v & O_v & O_v & O_v \\ & & & & O_v & I_v & O_v & O_v \\ & & & & O_v & O_v & I_v & O_v \end{array}\right),$$

$$B_{0,2} = \left(\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 1 & & O_{4,4v} & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ \hline & & & & O_v & O_v & I_v & O_v \\ & O_{4v,4} & & & O_v & O_v & O_v & I_v \\ & & & & I_v & O_v & O_v & O_v \\ & & & & O_v & I_v & O_v & O_v \end{array}\right), B_{0,3} = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & & & & \\ 1 & 0 & 0 & 0 & & O_{4,4v} & & \\ 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ \hline & & & & O_v & I_v & O_v & O_v \\ & O_{4v,4} & & & O_v & O_v & I_v & O_v \\ & & & & O_v & O_v & O_v & I_v \\ & & & & I_v & O_v & O_v & O_v \end{array}\right).$$

Example ($B_{1,t}$'s in the $d = 4$ case)

$$B_{1,0} = \left(\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\multirow{4}{*}{O_4}} & \mathbf{1}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top \\ &&&& \mathbf{0}_v^\top & \mathbf{1}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top \\ &&&& \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{1}_v^\top & \mathbf{0}_v^\top \\ &&&& \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{1}_v^\top \\ \hline \mathbf{1}_v & \mathbf{0}_v & \mathbf{0}_v & \mathbf{0}_v & A_3 & A_2 & A_1 & A_4 \\ \mathbf{0}_v & \mathbf{1}_v & \mathbf{0}_v & \mathbf{0}_v & A_2 & A_1 & A_4 & A_3 \\ \mathbf{0}_v & \mathbf{0}_v & \mathbf{1}_v & \mathbf{0}_v & A_1 & A_4 & A_3 & A_2 \\ \mathbf{0}_v & \mathbf{0}_v & \mathbf{0}_v & \mathbf{1}_v & A_4 & A_3 & A_2 & A_1 \end{array}\right), B_{1,1} = \left(\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\multirow{4}{*}{O_4}} & \mathbf{0}_v^\top & \mathbf{1}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top \\ &&&& \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{1}_v^\top & \mathbf{0}_v^\top \\ &&&& \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{1}_v^\top \\ &&&& \mathbf{1}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top \\ \hline \mathbf{0}_v & \mathbf{0}_v & \mathbf{0}_v & \mathbf{1}_v & A_4 & A_3 & A_2 & A_1 \\ \mathbf{1}_v & \mathbf{0}_v & \mathbf{0}_v & \mathbf{0}_v & A_3 & A_2 & A_1 & A_4 \\ \mathbf{0}_v & \mathbf{1}_v & \mathbf{0}_v & \mathbf{0}_v & A_2 & A_1 & A_4 & A_3 \\ \mathbf{0}_v & \mathbf{0}_v & \mathbf{1}_v & \mathbf{0}_v & A_1 & A_4 & A_3 & A_2 \end{array}\right)$$

$$B_{1,2} = \left(\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\multirow{4}{*}{O_4}} & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{1}_v^\top & \mathbf{0}_v^\top \\ &&&& \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{1}_v^\top \\ &&&& \mathbf{1}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top \\ &&&& \mathbf{0}_v^\top & \mathbf{1}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top \\ \hline \mathbf{0}_v & \mathbf{0}_v & \mathbf{1}_v & \mathbf{0}_v & A_1 & A_4 & A_3 & A_2 \\ \mathbf{0}_v & \mathbf{0}_v & \mathbf{0}_v & \mathbf{1}_v & A_4 & A_3 & A_2 & A_1 \\ \mathbf{1}_v & \mathbf{0}_v & \mathbf{0}_v & \mathbf{0}_v & A_3 & A_2 & A_1 & A_4 \\ \mathbf{0}_v & \mathbf{1}_v & \mathbf{0}_v & \mathbf{0}_v & A_2 & A_1 & A_4 & A_3 \end{array}\right), B_{1,3} = \left(\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\multirow{4}{*}{O_4}} & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{1}_v^\top \\ &&&& \mathbf{1}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top \\ &&&& \mathbf{0}_v^\top & \mathbf{1}_v^\top & \mathbf{0}_v^\top & \mathbf{0}_v^\top \\ &&&& \mathbf{0}_v^\top & \mathbf{0}_v^\top & \mathbf{1}_v^\top & \mathbf{0}_v^\top \\ \hline \mathbf{0}_v & \mathbf{1}_v & \mathbf{0}_v & \mathbf{0}_v & A_2 & A_1 & A_4 & A_3 \\ \mathbf{0}_v & \mathbf{0}_v & \mathbf{1}_v & \mathbf{0}_v & A_1 & A_4 & A_3 & A_2 \\ \mathbf{0}_v & \mathbf{0}_v & \mathbf{0}_v & \mathbf{1}_v & A_4 & A_3 & A_2 & A_1 \\ \mathbf{1}_v & \mathbf{0}_v & \mathbf{0}_v & \mathbf{0}_v & A_3 & A_2 & A_1 & A_4 \end{array}\right)$$

Normalized BGW matrices ⇒ Association schemes?

Let $\mathcal{A} = (X, \{A_i\}_{i=0}^d)$ be a d-class pseudo-cyclic symmetric AS s.t. the principal part of its first eigenmatrix is circulant. Recall that

$$M_{\mathcal{A}} = \begin{pmatrix} 0 & \mathbf{1}_v^\top \\ \mathbf{1}_v & \sum_{i=1}^d \omega^{i-1} A_i \end{pmatrix}$$

is a BGW matrix with parameters $(v+1, v, (v-1)/d)$ over $G = \langle \omega \rangle$.

For $a \in X$, let $M_{\mathcal{A}}^{(a)}$ be the BGW matrix obtained by re-normalizing $M_{\mathcal{A}}$ so that its ath row and ath column have only the identity of G or 0. Then, removing the ath row and ath column, we have a symmetric matrix of order v of the form $\sum_{i=1}^d \omega^{i-1} A_i^{(a)}$ for some $(0,1)$-matrices $A_i^{(a)}$, $i = 1, 2, \ldots, d$.

Problem

Does $\mathcal{A}^{(a)} = \{A_i^{(a)} \mid i = 1, 2, \ldots, d\} \cup \{I_v\}$ form an AS?

Example

$M_{\mathcal{A}}$ is a BGW matrix with parameters $(8, 7, 2)$ over $G = \langle \omega \rangle$ obtained from the 3-class cyclotomic scheme $\mathcal{A} = (\mathbb{F}_7, \{A_i\}_{i=0}^3)$. Let $a = 0 \in \mathbb{F}_7$.

$$\sum_{i=1,2,3} i \cdot A_i = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 & 0 \end{pmatrix} \Rightarrow M_{\mathcal{A}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 & \omega^2 & \omega & 1 \\ 1 & 1 & 0 & 1 & \omega & \omega^2 & \omega^2 & \omega \\ 1 & \omega & 1 & 0 & 1 & \omega & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega & 1 & 0 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & 0 & 1 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 & 0 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\sum_{i=1,2,3} i \cdot A_i^{(a)} = \begin{pmatrix} 0 & 1 & 3 & 2 & 2 & 3 & 1 \\ 1 & 0 & 3 & 3 & 1 & 2 & 2 \\ 3 & 3 & 0 & 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 0 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 & 0 & 1 & 3 \\ 3 & 2 & 1 & 2 & 1 & 0 & 3 \\ 1 & 2 & 2 & 1 & 3 & 3 & 0 \end{pmatrix} \Leftarrow M_{\mathcal{A}}^{(a)} = \begin{pmatrix} 0 & 1 & 1 & \omega^2 & \omega & \omega & \omega^2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & \omega^2 & \omega^2 & 1 & \omega & \omega \\ \omega^2 & 1 & \omega^2 & 0 & 1 & \omega & 1 & \omega \\ \omega & 1 & \omega^2 & 1 & 0 & \omega^2 & \omega & 1 \\ \omega & 1 & 1 & \omega & \omega^2 & 0 & 1 & \omega^2 \\ \omega^2 & 1 & \omega & 1 & \omega & 1 & 0 & \omega^2 \\ 1 & 1 & \omega & \omega & 1 & \omega^2 & \omega^2 & 0 \end{pmatrix}$$

Isomorphism between new non-commutative ASs

We call that two ASs $(X, \{A_i\}_{i=0}^d)$ and $(X, \{A'_i\}_{i=0}^d)$ are **isomorphic** if $\{PA_iP^\top \mid 0 \le i \le d\} = \{A'_i \mid 0 \le i \le d\}$ for some permutation matrix P.

Theorem 2 (M.-Suda)

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two d-class symmetric pseudo-cyclic ASs of order v s.t. the principal parts of their first eigenmatrices are circulant. If $\mathcal{D}_{\mathcal{A}_1}$ and $\mathcal{D}_{\mathcal{A}_2}$ are isomorphic, $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic or there exists $a \in X$ s.t. $\mathcal{A}_1^{(a)}$ is an AS isomorphic to $\mathcal{A}_2$.

$$\mathcal{A}^{(a)} := \{A_i^{(a)} \mid i = 1, 2, \ldots, d\} \cup \{I_v\}$$

The $d = 2$ case

Let $A_i(a) := \{x \in X \mid A_i(x,a) = 1\}$ for $a \in X$.

When $d = 2$, $A_i^{(a)}$, $i = 1, 2$, are given as follows (by suitable permutations of rows and columns):

$$A_1^{(a)} \sim \begin{pmatrix} 0 & 1_k^\top & 0_k^\top \\ 1_k & A_{1|1,1} & A_{2|1,2} \\ 0_k & A_{2|2,1} & A_{1|2,2} \end{pmatrix}, A_2^{(a)} \sim \begin{pmatrix} 0 & 0_k^\top & 1_k^\top \\ 0_k & A_{2|1,1} & A_{1|1,2} \\ 1_k & A_{1|2,1} & A_{2|2,2} \end{pmatrix}$$

while A_i, $i = 1, 2$, have the forms

$$A_1 \sim \begin{pmatrix} 0 & 1_k^\top & 0_k^\top \\ 1_k & A_{1|1,1} & A_{1|1,2} \\ 0_k & A_{1|2,1} & A_{1|2,2} \end{pmatrix}, A_2 \sim \begin{pmatrix} 0 & 0_k^\top & 1_k^\top \\ 0_k & A_{2|1,1} & A_{2|1,2} \\ 1_k & A_{2|2,1} & A_{2|2,2} \end{pmatrix},$$

where $A_{i|s,t}$ denote the submatrix of A_i obtained by restricting its rows to $A_s(a)$ and columns to $A_t(a)$.

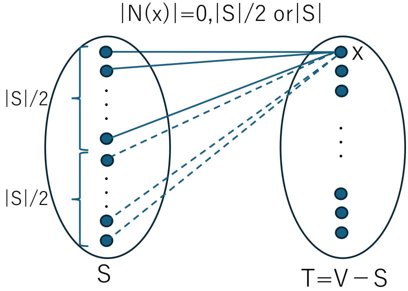
Godsil-McKay switching

Proposition (Godsil-McKay switching)

Let Γ be a graph. Assume that $V(\Gamma)$ has a subset S inducing a regular subgraph of Γ, and that each vertex in $T = V(\Gamma) \setminus S$ has $0, |S|/2$ or $|S|$ neighbors in S.

For each $x \in T$ having $|S|/2$ neighbors in S, delete the corresponding $|S|/2$ edges and join x instead to the $|S|/2$ other vertices in S.

Then, the resulting graph is cospectral to Γ.



The $d = 2$ case

Recall that $A_i(a) := \{x \in X \mid A_i(x, a) = 1\}$ for $a \in X$.

Remark

When $d = 2$, since each $\Gamma_i = (X, A_i)$, $i = 1, 2$, is a conference graph, $S = X \setminus (A_i(a) \cup \{a\})$ satisfies that $|S| = (v-1)/2$ and each vertex in $T = A_i(a) \cup \{a\}$ has 0 or $|S|/2$ neighbors in S in Γ_i.

By applying Godsil-McKay switching to S in Γ_i, we have a cospectral graph, that is, the graph having $A_i^{(a)}$ as its adjacency matrix.

Hence, $\mathcal{A}^{(a)} = \{A_1^{(a)}, A_2^{(a)}, I_v\}$ forms a 2-class AS having the same first eigenmatrix with $\mathcal{A} = (X, \{A_i\}_{i=0}^2)$.

A counter-example in the $d = 3$ case

There is an example of a 3-class symmetric pseudo-cyclic AS $\mathcal{A} = (X, \{A_i\}_{i=0}^{3})$ with 28 points s.t. $\mathcal{A}^{(a)} = \{A_i^{(a)} \mid i = 1, 2, 3\} \cup \{I_v\}$ is not an AS for some $a \in X$ [5].

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# No. 74
[ [ 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3 ],
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  [ 1, 2, 2, 3, 1, 3, 0, 2, 1, 2, 1, 3, 1, 3, 2, 3, 1, 1, 2, 2, 1, 3, 3, 1, 3, 3, 2, 2 ],
  [ 1, 2, 3, 2, 3, 1, 2, 0, 2, 1, 3, 1, 3, 1, 3, 2, 1, 1, 2, 1, 2, 3, 3, 1, 2, 2, 3, 3 ],
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  [ 2, 1, 2, 1, 1, 2, 3, 1, 3, 3, 3, 0, 3, 1, 1, 3, 2, 3, 2, 2, 2, 3, 2, 1, 1, 1, 3, 2 ],
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  [ 2, 2, 1, 1, 1, 3, 2, 3, 1, 3, 3, 1, 3, 2, 0, 3, 1, 2, 3, 3, 1, 2, 1, 3, 2, 2, 1, 2 ],
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  [ 2, 3, 3, 1, 3, 2, 1, 1, 2, 1, 2, 3, 3, 3, 2, 1, 1, 0, 3, 2, 1, 3, 2, 2, 3, 2, 1, 1 ],
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  [ 3, 3, 3, 1, 1, 2, 2, 3, 3, 2, 1, 2, 2, 3, 2, 2, 3, 1, 1, 3, 2, 3, 1, 1, 2, 1, 1, 0 ] ]
```

[5] I. Miyamoto, A. Hanaki, Classification of association schemes with small vertices, `http://math.shinshu-u.ac.jp/~hanaki/as`.

Cyclotomic schemes

Theorem 3 (M.-Suda)

Let $\mathcal{A} = (\mathbb{F}_q, \{A_i\}_{i=0}^d)$ be a d-class symmetric cyclotomic AS. Then, for any $a \in \mathbb{F}_q$, $\mathcal{A}^{(a)} = \{A_i^{(a)} \mid i = 1, 2, \ldots, d\} \cup \{I_v\}$ forms a d-class symmetric **pseudo-cyclic** AS having the same intersection numbers with $\mathcal{A}$.

The problem is reduced to computing

$$\sum_{j=0}^{d-1} \left|\{x \in \mathbb{F}_q \mid x \in C_j \cap (C_{j+h} + c) \cap (C_{j+i} + d)\}\right|$$

for some $c, d \in \mathbb{F}_q$.

Problems

Problem

Find a class of d-class pseudo-cyclic AS $\mathcal{A}$ such that $\mathcal{A}^{(a)} = \{A_i^{(a)} \mid i = 1, 2, \ldots, d\} \cup \{I_v\}$ are again pseudo-cyclic ASs for $d > 2$.

Problem

Determine whether a symmetric cyclotomic AS $\mathcal{A}$ is isomorphic to $\mathcal{A}^{(a)} = \{A_i^{(a)} \mid i = 1, 2, \ldots, d\} \cup \{I_v\}$, $a \in X$.

Problem

How about the case when $\mathcal{A}$ is nonsymmetric?

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Problem

How about the case when $\mathcal{A}$ is nonsymmetric?

Thank you very much for for your attention!