A new perspective on cocyclic development

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Outline

Complex Hadamard matrices

Theorem (Hadamard)

Let M be a matrix with complex entries of modulus 1. The following are equivalent.

- *The rows/columns of M are orthogonal (with respect to the standard Hermitian inner product).*
- *The determinant of M is maximal among all matrices with entries of unit modulus.*
- $MM^* = nI_n$.
- *All eigenvalues of M have modulus of norm $\sqrt{n}$.*
- *M is a (complex) Hadamard matrix (CHM).*

Subsets of CHMs have attracted attention: Real, Butson, QUH...
Connections to block designs & finite geometries, discrete Fourier analysis & signal processing, quantum information theory...

Monomial matrices and automorphisms

- A matrix is **monomial** if each row and column has a single non-zero entry.
- A pair of monomial matrices (P, Q) is an *automorphism* of H if

$$PMQ^* = M\,.$$

- The set of all automorphisms forms a *group* denoted $\mathrm{Aut}(M)$. If we restrict everything to k^{th} roots, then the group is finite.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

Motivation: Paley matrices and their symmetries

- Legendre: $\left(\frac{a}{p}\right) := a^{\frac{p-1}{2}} \mod p$, which is 1 if a is a non-zero quadratic residue. Define $Q = \left[\left(\frac{a-b}{p}\right)\right]_{a,b\in\mathbb{F}_p}$.
- Gauss: $QQ^\top = pI - J$ and $Q^\top = -Q$, for $p \equiv 3 \mod 4$.
- Schur-Paley:

$$M = \begin{pmatrix} 1 & 1 \\ -1 & Q+I \end{pmatrix}$$

  is an example of a Hadamard matrix, i.e. $HH^\top = (p+1)I_{p+1}$.
- de Launey-Stafford: The (full) automorphism group of H is $2.PSL_2(p) \cong SL_2(p)$ for $p > 11$.
- Flannery-de Launey: The Paley I matrix of order $4t$ is cocyclic over D_{2t}.

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- Monomial representations are induced from linear characters of subgroups of an abstract group G.
- Given a monomial representation ρ of G, want to find M such that $\rho(g)M = M\rho(g)$. This is the centraliser algebra. Schur's Lemma.
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- Hadamard matrices correspond to solutions $\mathcal{C}v = \lambda$ where v has all entries of norm 1 and λ has all entries of norm n, this requires Gröbner bases in general.
- Lifting problem: $1 \to \mathbb{C}^* \to \Gamma \to G \to 1$ such that Γ' contains non-trivial scalars is solved by the Schur multiplier, which is the cohomology group $H^2(G, \mathbb{C}^*)$.
- Cocyclic development comes from a central extension of the left-regular representation.

Motivation: strongly regular graphs

Definition

The **rank** of a permutation group is the number of orbits of a point stabiliser.

Proposition (D.G. Higman)

The following are equivalent for a (transitive) permutation group G.

- *G is a permutation group of rank* 3 *of even order.*
- *The point stabiliser* G_α *has three orbits (including* $\{\alpha\}$*).*
- *The corresponding permutation representation* ρ *has three (distinct) irreducible constituents.*
- *The algebra of matrices commuting with all* $\rho(g)$ *has a basis of* $\{0, 1\}$ *matrices, a generator of which gives a srg.*
- *G acts edge-transitively on a strongly regular graph.*

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- Let $G_\alpha \leqslant G$ and χ be a linear character of G_α. Let T be a transversal of G_α in G. The monomial representation of G associated to χ is

$$\chi \uparrow_{G_\alpha}^{G} (g) = \left[\chi^{+}(t_i g t_j^{-1})\right]_{t_i, t_j \in T} .$$

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- **Reading the formulas is optional.**

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- Write $t_i \circ g = t_j$ if and only if $G_\alpha t_i g = G_\alpha t_j$.
- The rows and columns of elements $\mu(g)$ and of the centraliser may be indexed by T. Write $m(t_i^{-1}, t_k)$ for the element in row t_i^{-1} and column t_k.
- For M in the centraliser, comparing $\mu(g)M$ and $M\mu(g)$ gives the identity

$$U(t_i^{-1}, t_j) = U((t_i \circ g)^{-1}, t_j \circ g)\chi(t_i g(t_i \circ g^{-1}))^{-1}\chi(t_j g(t - j \circ g)^{-1}),$$

and varying $g \in G$ gives the values of all cells across an orbital.

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- **Non-orientable**: if there exist h_1 and h_2 with $G_\alpha g h_1 = G_\alpha g h_2$ and $\chi(h_1) \neq \chi(h_2)$ that basis element becomes identically zero for the monomial representation.

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- **Higman** attempted a combinatorial construction analogous to srgs, called **rainbows**, but it didn't take off.
- **We can compute monomial representations, a basis for the centraliser algebra, and the character table for that algebra.**

- Transitive permutation representations of G are in bijection with characters of subgroups of G.
- **Assume π is multiplicity free.** The character table C records the eigenvalues of a basis for the centraliser algebra, and is computable via character sums.
- CHMs determined by norm conditions on entries and eigenvalues; solve $Cv = \lambda$ where entries of v have norm 1 and entries of λ have norm n.
- Each entry of λ is $\pm\sqrt{n}$. Finitely many cases to check (in practice $-n \leqslant \mathrm{tr}(M) \leqslant n$ eliminates many possibilities).

Finding Hadamard matrices in centraliser algebras

We illustrate with an example. Let $G \leqslant S_{16}$ be the group

$$\langle \sigma = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16), \\ \tau = (2,3,5,9,16)(4,7,13,8,15)(6,11,12,10,14) \rangle.$$

This group is a Frobenius group of order 80, with an elementary abelian subgroup of order 16 and a point stabiliser H of order 5.

Let ρ be the permutation representation induced by the trivial character χ of H. The associated centraliser algebra is commutative and spanned by the identity matrix, and three matrices of constant row-sum 5.

The character table of the centraliser algebra is

	M_1	M_2	M_3	M_4
	1	5	5	5
	1	-3	1	1
	1	1	-3	1
	1	1	1	-3

The character table of the centraliser algebra is

$$\begin{pmatrix} M_1 & M_2 & M_3 & M_4 \\ \hline 1 & 5 & 5 & 5 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} v \\ \hline 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \hline -4 \\ -4 \\ 4 \\ 4 \end{pmatrix}$$

The following $\{\pm 1\}$-linear combination of basis matrices

$$M = M_1 + M_2 - M_3 - M_4$$

is Hadamard matrix, because its eigenvalues are all of absolute value 4, by virtue of which its determinant achieves the Hadamard bound.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - \\
1 & 1 & - & - & - & - & 1 & - & - & - & 1 & 1 & 1 & - & - & - \\
1 & - & 1 & - & - & - & 1 & 1 & - & - & - & - & - & 1 & 1 & - \\
1 & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & - & - & - & 1 \\
1 & - & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & - & - & - & 1 & - & - & - & 1 & 1 & - & - & - & 1 & 1 \\
- & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & - & - & - & - & 1 \\
- & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & - & - \\
- & - & - & 1 & 1 & - & 1 & - & 1 & - & 1 & - & - & - & 1 & - \\
- & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - \\
- & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & - \\
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- & 1 & - & - & 1 & - & - & - & 1 & - & - & - & 1 & - & 1 & 1 \\
- & - & 1 & - & 1 & - & - & - & - & - & 1 & 1 & - & 1 & - & 1 \\
- & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & - \\
- & - & - & 1 & - & 1 & 1 & - & - & - & - & - & 1 & 1 & - & 1
\end{bmatrix}
$$

$$M = M_1 + M_2 - M_3 - M_4$$

- Monomial representations of G are in bijection with linear characters of subgroups of G.
- **Assume ρ is multiplicity free.** The character table M is computable and records the eigenvalues of a basis for the centraliser algebra.
- CHMs determined by norm conditions on entries and eigenvalues; solve $Mv = \lambda$ where entries of v have norm 1 and entries of λ have norm n.
- Norm conditions are not algebraic, but $N(x + iy) = x^2 + y^2$, so we get a system of (quadratic) polynomial equations in twice as many variables.
- Geometrically, intersection of two torii. Gröbner bases are the standard computational tool for solving such systems.
- In practice, depends super-exponentially on number of variables and on field of definition.

Example: A CHM and ETF from M_{11}

- On LHS: H is a subgroup of order 55 and c is the root of some quadratic equation.
- On the RHS: H is a subgroup of order 72 and index 110.
- In both cases, the character is real but non-trivial.

$$\mathcal{H} = H_1 - H_2 + H_3 + H_4 + H_5 - H_6\,, \quad \mathcal{X} = 5X_1 - X_2 + X_3 - X_4$$

d	H_1	H_2	H_3	H_4	H_5	H_6	$\mathcal{H}$
55	1	-1	-1	-1	-5	7	-12
45	1	-1	3	3	-1	-5	12
16	1	-1	c-5	-c	10	-5	12
16	1	-1	-c	c-5	10	-5	12
11	1	11	-1	-1	-5	-5	-12
1	1	11	11	11	55	55	12

d	X_1	X_2	X_3	X_4	$\mathcal{X}$
10	1	1	18	0	22
11	1	-1	0	-24	30
44	1	-1	0	6	0
45	1	1	-4	0	0

An example

- The Schur cover of $\mathrm{PSL}_2(p)$ is $\mathrm{SL}_2(p)$ for $p \equiv 3 \mod 4$ and $p > 11$.
- A subgroup of index $p+1$ in $SL_2(p)$ is conjugate to
$$H = \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p \}$$
- A transversal of H in G is given by
$$T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \mid a \in \mathbb{F}_q \right\},$$
(The second rows in bijection with points on the projective line.)
-
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ac^{-1}d - b & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & c^{-1}d \end{pmatrix}$$
- Let ρ be the representation induced from the non-trivial linear character of H to G. This is monomial of degree $p+1$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ac^{-1}d - b & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & c^{-1}d \end{pmatrix}$$

- Take $t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to be a representative of the non-trivial double-coset of H in G.
- Let M be a matrix in the centraliser of ρ, with rows and columns labelled by T, and $M[I, t] = 1$.
- By Higman,

$$M(g, tg) = M(1, t)\chi_H(g)^{-1}\chi_H(tg)\,.$$

where χ_H is the determinant of the h-part of a group element.

$$g = \begin{pmatrix} (i-j)^{-1} & j(i-j)^{-1}) \\ 1 & i \end{pmatrix} = \begin{pmatrix} 1 & (i-j)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix},\ tg = \ldots$$

It follows that $M(g, tg) = \chi_H(g)\chi_H(tg) = \chi(1)\chi(i-j)$.
- So the centraliser of ρ is spanned by I and the Paley matrix.

Input: a permutation group, G.
Output: CHMs with Γ as automorphism group where Γ is a monomial cover of G.

n	Schur cover of	Stab	$[x_1, \ldots, x_r]$
7	$C_7 \rtimes C_3$	C_3	$[1, 1, \frac{-3+i\sqrt{7}}{4}]$
7	$C_7 \rtimes C_3$	C_3	$[1, 1, \frac{(\sqrt{3}+i)(\sqrt{7}-3i)}{8}]$
11	$C_{11} \rtimes C_5$	C_5	$[1, 1, \frac{-5+i\sqrt{11}}{6}]$
11	$C_{11} \rtimes C_5$	C_5	$[1, 1, \frac{x^2}{3}]^*$
27	$3^3 \rtimes S_4$	S_4	$[1, \zeta_3, 1, \zeta_3^2]$
35	A_7	$(S_3 \times S_4) \cap A_7$	$[1, \frac{15-i\sqrt{31}}{16}, \frac{15-i\sqrt{31}}{16}, -1]$
35	A_7	$(S_3 \times S_4) \cap A_7$	$[1, 1, 1, \frac{17+i\sqrt{35}}{18}]$
49	$C_7^2 \rtimes (D_4 \times C_3)$	$D_4 \times C_3$	$[1, \frac{1-i\sqrt{35}}{6}, \frac{1-i\sqrt{35}}{6}, \frac{1+i\sqrt{35}}{6}]$
63	$G_2(2)'$	$2^{2+1+2} \rtimes C_3$	$[1, 1, 1, \frac{-31+i\sqrt{63}}{32}]$
63	$G_2(2)'$	$2^{2+1+2} \rtimes C_3$	$[1, \frac{29+i\sqrt{59}}{30}, \frac{29+i\sqrt{59}}{30}, -1]$

$*$ x satisfies $x^8 - x^7 - 2x^6 + 5x^5 + x^4 + 15x^3 - 18x^2 - 27x + 81$

Cocyclic development

- Let G be a group and A an abelian group; a 2-cocycle is a function $\psi : G \times G \to A$ satisfying

$$\psi(g,h)\psi(gh,k) = \psi(g,hk)\psi(h,k)$$

- Cocycles (up to equivalence) classify central extensions (up to equivalence):

$$1 \to A \to \Gamma \to G \to 1\,.$$

- Recall that H is a **cocyclic Hadamard matrix** if and only if (up to equivalence)

$$H = [\psi(g,h)]_{g,h \in G}$$

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- Introduced by Warwick de Launey and Kathy Horadam, developed by Dane Flannery and studied extensively in Sevilla!

- Suppose that Γ is a central extension of C_2 by G with cocycle ψ; and that $H = [\psi(g, h)]_{g,h\in G}$.
- Flannery: Then Γ is a *centrally regular* subgroup of $\mathrm{Aut}(H)$.
- Conversely, let χ be the non-trivial character of the central C_2, and ρ the induced monomial representation. This is a signed version of the left-regular representation.

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- In fact, it is the Hadamard product of $[\psi(g, h)]_{g,h \in G}$ with the left regular representation.
- Its centraliser is the Hadamard product of $[\psi(g, h)]_{g,h \in G}$ with the **right** regular representation.

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- The centraliser admits a basis of matrices $[\psi(g, h)(-1)^{\delta_x(gh)}]_{g,h \in G}$.
- In bijection with the elements of G; a Hadamard matrix corresponds to choosing a transversal of C_2 in Γ (which is a RDS).

The home of cocyclic Hadamard matrices

- Alvarez, Armario, Frau, Gudiel, Güemes, & Osuna: *On D_{4t} cocyclic Hadamard matrices*, 2015.
- Explicit cocycle matrix $M_\psi = [\psi(g,h)]_{g,h \in D_{4t}}$, block-nega-back-circulant.

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- Recall BACDE basis for the centraliser of the RRR: $M_x = [\psi(g,h)(-1)^{\delta_x(gh)}]_{g,h \in G}$.
- $\overline{M}_x = [\psi(g,h)(-1)^{\delta_x(gh)\delta_x(g)\delta_x(h)}]_{g,h \in G}$ is both a coboundary and the normalisation of M_x.

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- A basis for the coboundaries is given by **normalising** the right-regular representation of G.

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- A basis for the coboundaries is given by **normalising** the right-regular representation of G.
- AAFGGO give necessary and sufficient conditions for a **Hadamard product** $M_\psi \prod_{d\in D} \overline{M}_d$ to be a cocyclic Hadamard matrix. Cocyclic test: all non-initial rows sum to zero; with specifics of D_{4t}.

The home of cocyclic Hadamard matrices

- Alvarez, Armario, Frau, Gudiel, Güemes, & Osuna: *On D_{4t} cocyclic Hadamard matrices*, 2015.
- Explicit cocycle matrix $M_\psi = [\psi(g,h)]_{g,h\in D_{4t}}$, block-nega-back-circulant.
- Recall BACDE basis for the centraliser of the RRR: $M_x = [\psi(g,h)(-1)^{\delta_x(gh)}]_{g,h\in G}$.
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- **Question:** Is this equivalent to the relative difference set characterisation of CHMs? Multipliers?

Go raibh maith agaibh as ucht éisteacht. Lá breithe shona do Dane agus Rob.

Gracias por escuchar. Feliz cumpleaños a Dane y Rob.

Good on ya, eh, buddy?