Projection cubes of symmetric designs

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(joint work with Vedran Krčadinac and Mario Osvin Pavčević)

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What is a projection cube?

Definition

A (v, k, λ) *projection n-cube* is a matrix

$$C : \{1, \dots, v\}^n \to \mathbb{F}$$

with $\{0, 1\}$-entries such that all projections $\Pi_{xy}(C)$, $1 \le x < y \le n$ are symmetric (v, k, λ) designs. The set of all such matrices will be denoted $\mathcal{P}^n(v, k, \lambda)$.

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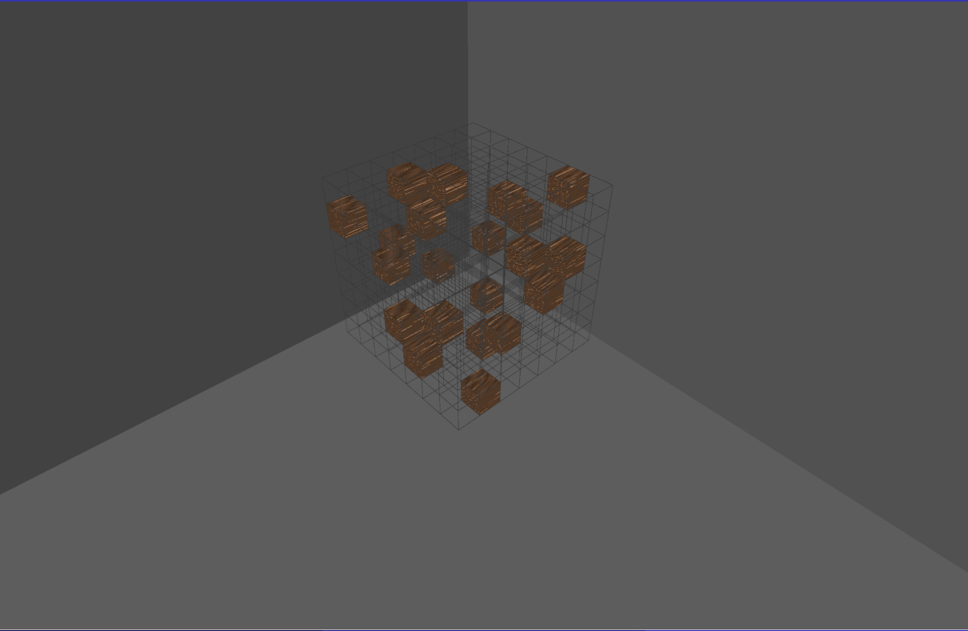
For $1 \leq x < y \leq n$, the projection $\Pi_{xy}(C)$ is defined as the 2-dimensional matrix with (i_x, i_y)-entry

$$\sum_{1 \leq i_1, \dots, \widehat{i_x}, \dots, \widehat{i_y}, \dots, i_n \leq v} C(i_1, \dots, i_n).$$

The sum is taken over all n-tuples $(i_1, \dots, i_n) \in \{1, \dots, v\}^n$ with fixed coordinates i_x and i_y.

Example: $(7, 3, 1)$ projection 3-cube



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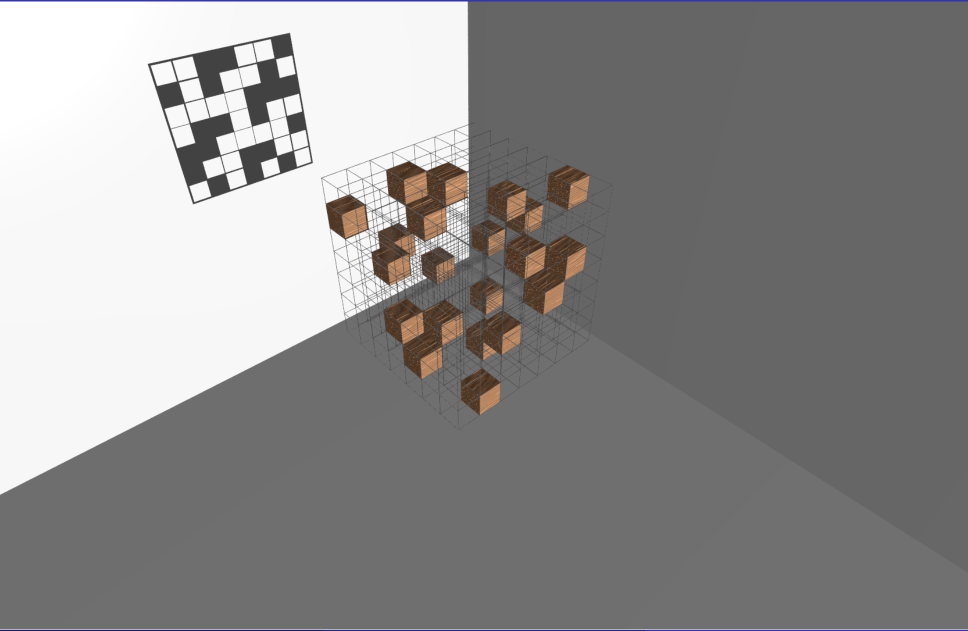

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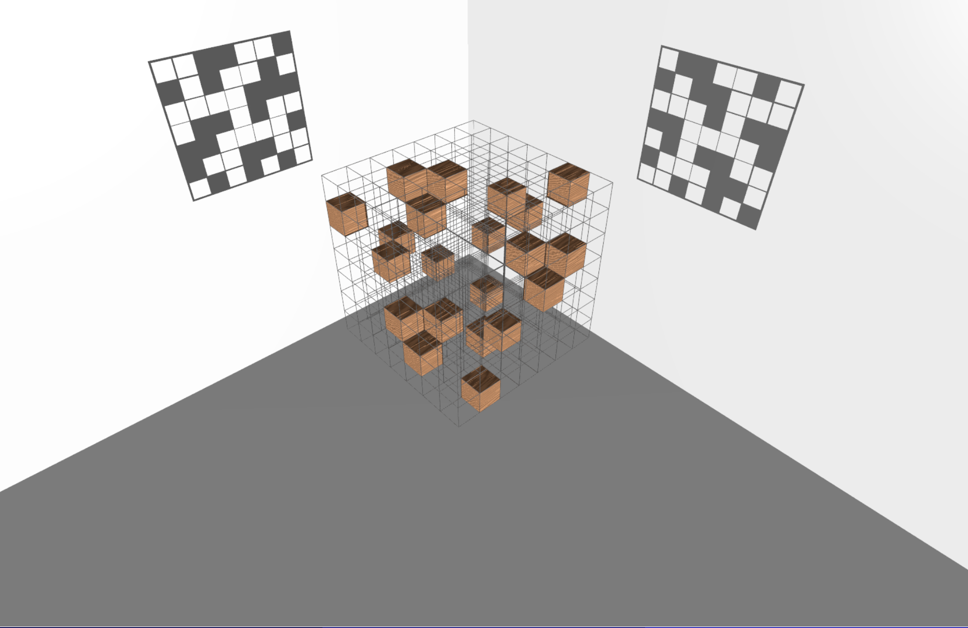

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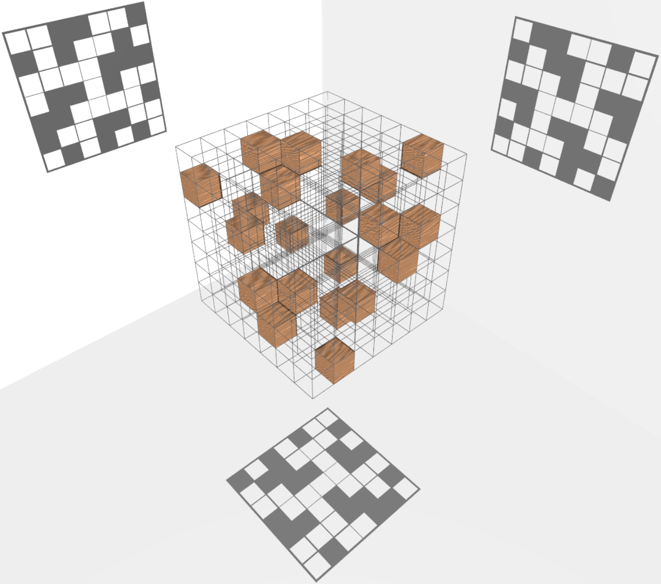

Projection cubes of symmetric designs

V. Krčadinac, L. Relić, *Projection cubes of symmetric designs*, to appear in Math. Comput. Sci. https://arxiv.org/abs/2411.06936

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Proposition

The number of incidences of $C \in \mathcal{P}^n(v, k, \lambda)$ *is* vk.

Practical representation

We can interpret $C : \{1, \ldots, v\}^n \to \{0, 1\}$ as a characteristic function and identify it with the set of n-tuples

$$\overline{C} = \{(i_1, \ldots, i_n) \in \{1, \ldots, v\}^n \mid C(i_1, \ldots, i_n) = 1\}$$

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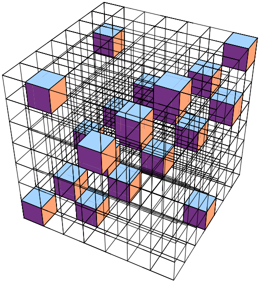

$(1,1,1)$	$(3,4,5)$	$(5,1,4)$
$(1,2,3)$	$(3,6,2)$	$(6,6,6)$
$(1,4,7)$	$(4,4,4)$	$(6,7,1)$
$(2,2,2)$	$(4,5,6)$	$(6,2,5)$
$(2,3,4)$	$(4,7,3)$	$(7,7,7)$
$(2,5,1)$	$(5,5,5)$	$(7,1,2)$
$(3,3,3)$	$(5,6,7)$	$(7,3,6)$

Practical representation

Proposition

Let $S \subseteq \{1, \dots, v\}^n$ be a subset of cardinality vk. There exists a $C \in \mathcal{P}^n(v, k, \lambda)$ such that $S = \overline{C}$ if and only if the following statements are true for all $1 \leq x < y \leq n$:

1. *for all $i \in \{1, \dots, v\}$, there are exactly k elements $j \in \{1, \dots, v\}$ such that $(i, j) \in \Pi_{xy}(S)$,*
2. *for all $j \in \{1, \dots, v\}$, there are exactly k elements $i \in \{1, \dots, v\}$ such that $(i, j) \in \Pi_{xy}(S)$,*
3. *for all $i, i' \in \{1, \dots, v\}$, $i \neq i'$, there are exactly λ elements $j \in \{1, \dots, v\}$ such that $(i, j) \in \Pi_{xy}(S)$ and $(i', j) \in \Pi_{xy}(S)$.*

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Corollary

If C is a (v, k, λ) projection n-cube, then $\overline{C}$ is an orthogonal array of size vk, degree n, order v, strength 1, and index k, i.e. an $OA(vk, n, v, 1)$.

Basic properties

Theorem

If a (v, k, λ) projection n-cube with $k \geq 2$ exists, then

$$n \leq \frac{v(v+1)}{2}.$$

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If a (v, k, λ) projection n-cube with $k \geq 2$ exists, then

$$\cancel{n \leq \frac{v(v+1)}{2}}.$$

Better bound obtained from coding theory:

$$n \leq \frac{vk-1}{k-1}.$$

Thanks to the anonymous referee!

Complete classification of $\mathcal{P}^n(v, k, \lambda)$ cubes up to equivalence

...by computer calculations using orthogonal array representation

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Complete classification of $\mathcal{P}^n(v, k, \lambda)$ cubes up to equivalence

...by computer calculations using orthogonal array representation

- $\rightsquigarrow$ extending incidence pairs to triplets and increasing the dimension
- $\rightsquigarrow$ eliminating equivalent copies

	n								
(v, k, λ)	2	3	4	5	6	7	8	9	10
$(3, 2, 1)$	1	2	1	1	0	0	0	0	0
$(7, 3, 1)$	1	13	20	4	3	2	0	0	0
$(7, 4, 2)$	1	877	884	74	19	9	6	5	0

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$$\nu(3, 2, 1) = 5, \quad \nu(7, 3, 1) = 7, \quad \nu(7, 4, 2) = 9$$

($\nu(v, k, \lambda)$ = largest integer n such that (v, k, λ) projection n-cubes exist)

Higher-dimensional difference sets

A (v, k, λ) difference set in G is a subset $D \subseteq G$ of size k such that every element $g \in G \setminus \{0\}$ can be written as $g = d_1 - d_2$ for exaclty λ choices of $d_1, d_2 \in D$.

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Definition

An *n-dimensional* (v, k, λ) *difference set* in G is a set of n-tuples $D \subseteq G^n$ of size k such that $\{d_x - d_y \mid d \in D\} \subseteq G$ are (v, k, λ) difference sets for all $1 \leq x < y \leq n$.

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Proposition

If D is an n-dimensional (v, k, λ) difference set in G, the development $\mathrm{dev}\, D = \{(d_1 + g, \dots, d_n + g) \mid g \in G,\, d \in D\}$ is the representation $\overline{C} \subseteq G^n$ of a projection cube $C \in \mathcal{P}^n(v, k, \lambda)$.

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If an n-dimensional (v, k, λ) difference set $D \subseteq G^n$ exists, then $n \leq v$.

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Theorem

Let D be an n-dimensional (v, k, λ) difference set in G. Then the projection cube $\overline{C} = \mathrm{dev}\, D$ has an autotopy group isomorphic to G acting sharply transitively on each coordinate.

Theorem

Let $C \in \mathcal{P}^n(v, k, \lambda)$ be a projection cube with an autotopy group G acting sharply transitively on each coordinate. Then there is an n-dimensional (v, k, λ) difference set D in G such that $\overline{C}$ is equivalent with $\mathrm{dev}\, D$.

Constructions – Paley difference sets

For a prime power $q \equiv 3 \pmod 4$, the squares in $\mathbb{F}_q^*$ constitute a $(q, (q-1)/2, (q-3)/4)$ difference set in $(\mathbb{F}_q, +)$, as do the non-squares.

Theorem (Higher-dimensional Paley difference sets)

If $q \equiv 3 \pmod 4$ is a prime power, then there exists a q-dimensional difference set with parameters $(q, (q-1)/2, (q-3)/4)$ in the additive group of $\mathbb{F}_q$.

7-dimensional $(7, 3, 1)$ difference set in $\mathbb{Z}_7$:

$$D_5 = \{(0,1,3,2,6,4,5),\ (0,2,6,4,5,1,3),\ (0,4,5,1,3,2,6)\}.$$

Constructions

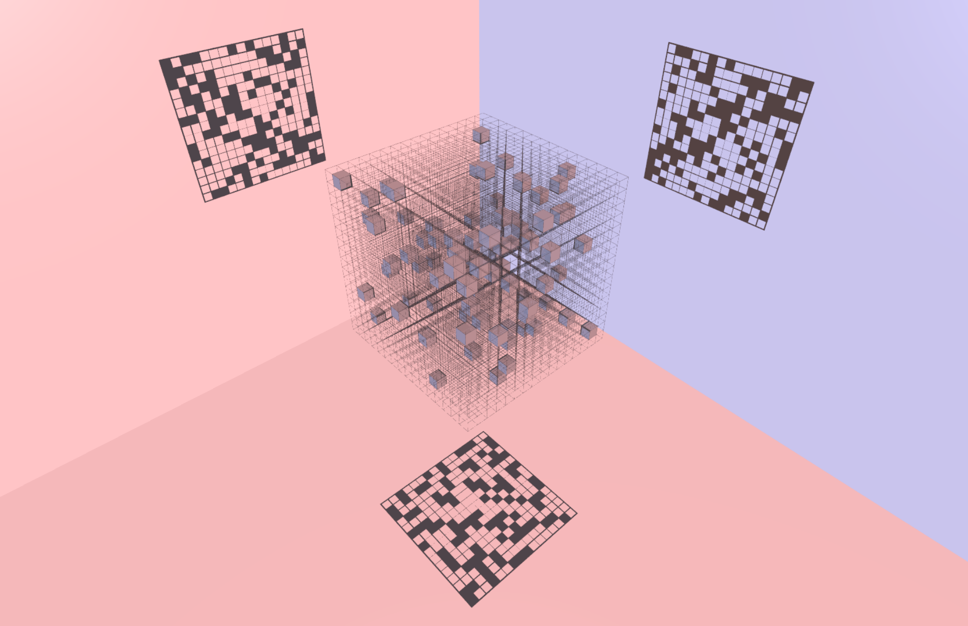
Theorem (Higher-dimensional cyclotomic difference sets)

If q is a prime power such that the 4th powers in $\mathbb{F}_q$ make a $(q,(q-1)/4,(q-5)/16)$ difference set, or the 8th powers in $\mathbb{F}_q$ make a $(q,(q-1)/8,(q-9)/64)$ difference set, then there exists a q-dimensional difference set with the same parameters.

Theorem (Higher-dimensional twin prime power difference sets)

If q and $q+2$ are odd prime powers, then there exists a q-dimensional difference set in $G=\mathbb{F}_q \times \mathbb{F}_{q+2}$ with parameters $(4m-1,2m-1,m-1)$ for $m=(q+1)^2/4$.

$(16, 6, 2)$ projection 3-cubes



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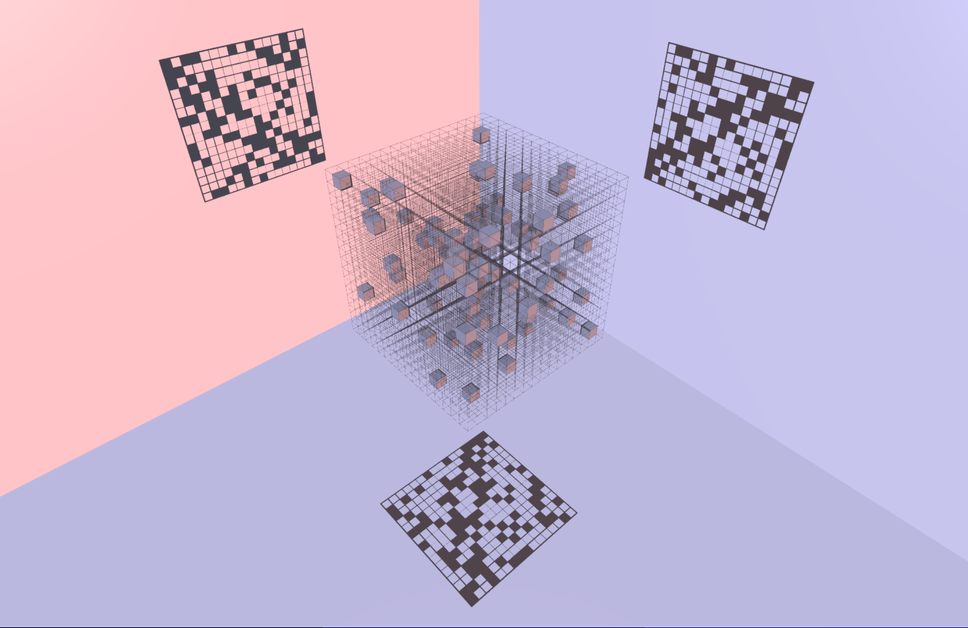

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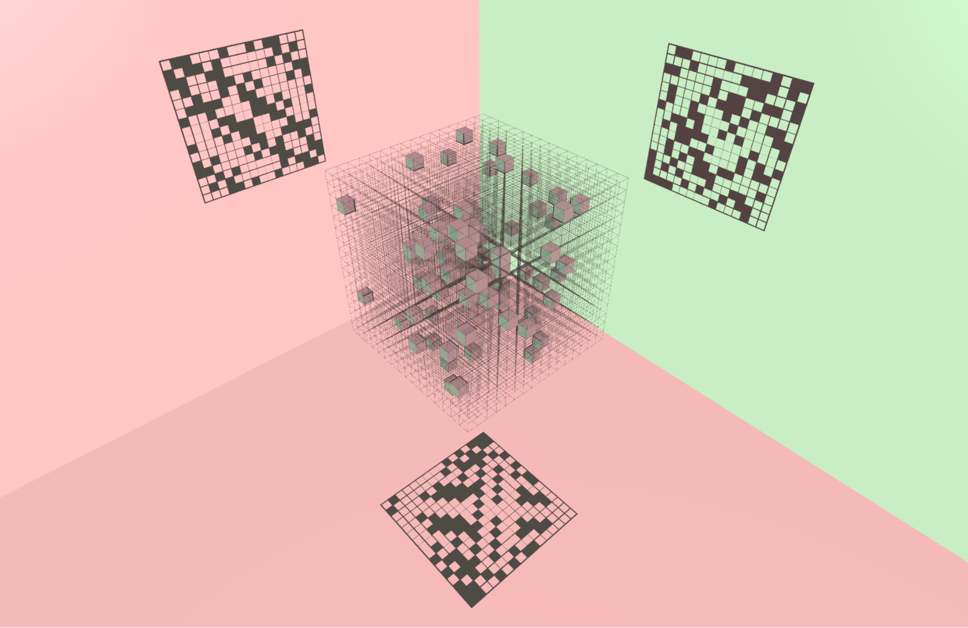

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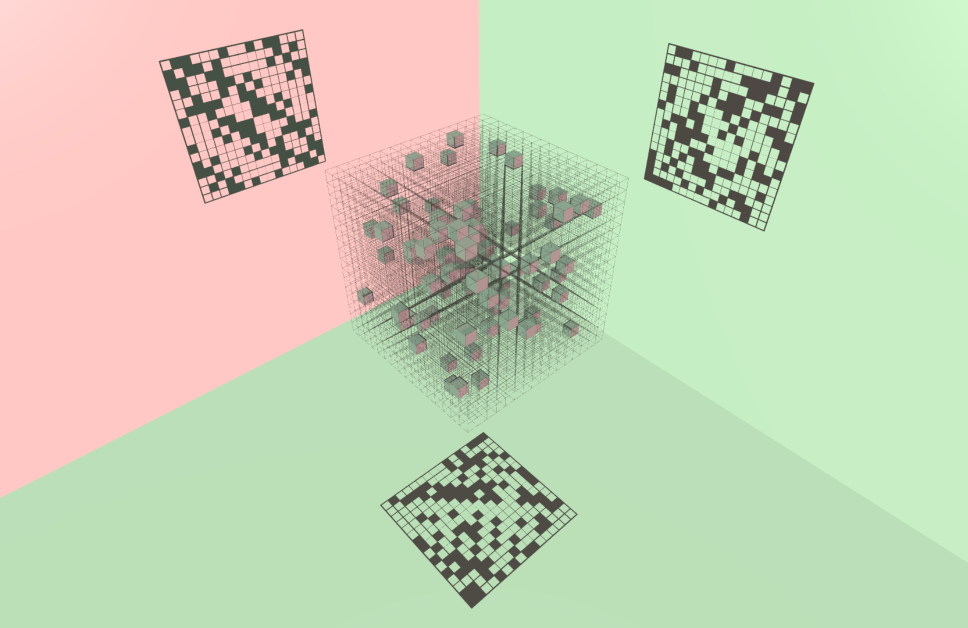


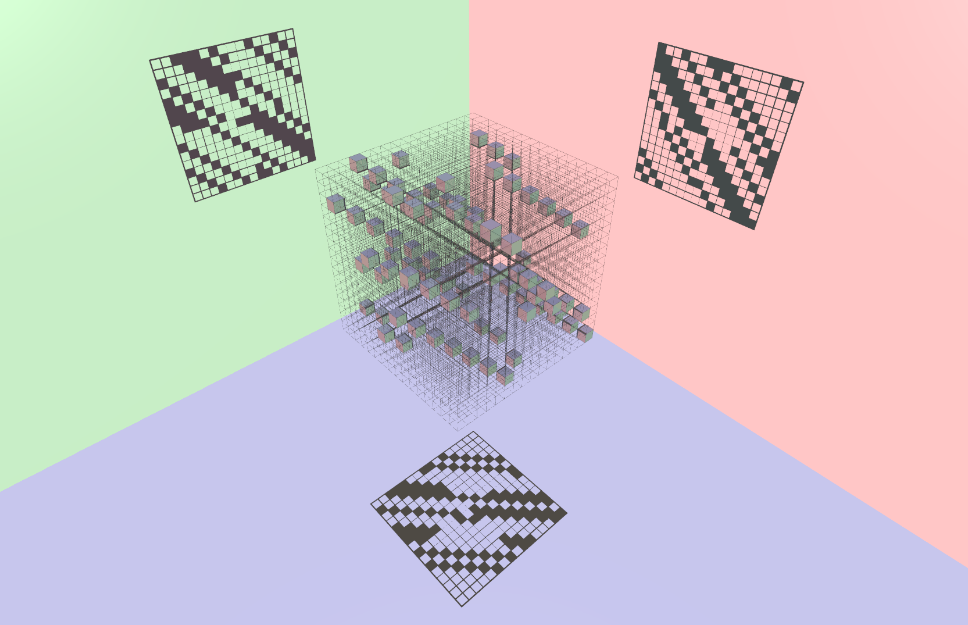
$(16, 6, 2)$ projection 3-cubes

Proposition

There are exactly 1076 *inequivalent* $\mathcal{P}^3(16, 6, 2)$*-cubes with an autotopy of order* 8 *acting in two cycles on each coordinate.*

$\rightsquigarrow$ 152 of them have three non-isomorphic projections

$(16, 6, 2)$ projection 3-cubes



The End

Thank you for your attention!