Weight distributions of $\mathbb{Z}_{p^s}$-additive simplex and MacDonald codes

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Outline

Motivation

Since the publication in 1994 of the famous paper

A.R. Hammons, P.V. Kumar, A.R. Calderbank, N.J.A. Sloane and P. Solé, The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals and related codes, *IEEE Trans. Inform. Theory*, **40** *(1994), 301–319.*

the study of linear codes over rings has received a lot of attention.

Simplex and MacDonald codes over $\mathbb{Z}_{2^s}$ have been studied in

M.K. Gupta, M.C. Bhandari, and A.K. Lal: On some linear codes over $\mathbb{Z}_{2^s}$. *Designs, Codes and Cryptography, 36, no. 3, pp. 227–244 (2005).*

C. Fernández-Córdoba, C. Vela and M. Villanueva: Nonlinearity and Kernel of $\mathbb{Z}_{2^s}$-Linear simplex and MacDonald Codes. *IEEE Transactions on Information Theory, 68, no. 11, pp. 7174-7183 (2021).*

Our goal is to generalize and study these families of codes over $\mathbb{Z}_{p^s}$ for an arbitrary prime p and $s \geq 1$.

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$\mathbb{Z}_{p^s}$-additive codes

A **code** over $\mathbb{Z}_{p^s}$ of length n is a non-empty subset $\mathcal{C}$ of $\mathbb{Z}_{p^s}^n$.

If $\mathcal{C}$ has group structure, then it is called a $\mathbb{Z}_{p^s}$-**additive code**. In this case, $\mathcal{C}$ is a subgroup of $\mathbb{Z}_{p^s}^n$, so it is isomorphic to an abelian structure $\mathbb{Z}_{p^s}^{t_1} \times \mathbb{Z}_{p^{s-1}}^{t_2} \times \cdots \times \mathbb{Z}_{p^2}^{t_{s-1}} \times \mathbb{Z}_{p}^{t_s}$, and we say that $\mathcal{C}$ is of **type** $(n; t_1, \dots, t_s)$.

A matrix with rows that are generators of a code is called a **generator matrix**.

There is a generator matrix of $\mathcal{C}$ (with minimum number of rows) having t_i rows of order $p^{s-(i-1)}$ for all $i \in \{1, \dots, s\}$.

Example

In $\mathbb{Z}_{27}$,

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 9 & 12 & 15 \\ 0 & 0 & 9 & 18 & 9 \end{pmatrix}$$

generates a linear code of type $(5; 1, 1, 1)$.

Weights for $\mathbb{Z}_{p^s}^n$

Consider $c = (c_1, \dots, c_n) \in \mathbb{Z}_{p^s}^n$. The **Hamming weight** of c is defined as

$$w_H(c) = |\{1 \leq i \leq n \mid c_i \neq 0\}|.$$

The **homogeneous weight** of c_i is defined as

$$w_{Hom}(c_i) = \begin{cases} 0 & \text{if } c_i = 0, \\ p^{s-1} & \text{if } c_i \neq 0 \text{ and } c_i \in \langle p^{s-1} \rangle, \\ (p-1)p^{s-2} & \text{otherwise,} \end{cases}$$

and $w_{Hom}(c) = \sum_{i=1}^n w_{Hom}(c_i)$.

Example

Consider $\mathbb{Z}_9 = \mathbb{Z}_{3^2}$ and $c = (0, 1, 2, 3, 4)$.
Then, $w_H(c) = 4$ and $w_{Hom}(c) = 3 \cdot 2 + 1 \cdot 3 = 9$.

These weights w define **distance functions** by taking $d(x, y) = w(x - y)$.

Order and valuation

We define the **order** of $x \in \mathbb{Z}_{p^s}^n$, denoted by $\mathrm{o}(x)$, as the smallest natural number m such that $mx = \mathbf{0}$.

We define the **valuation** of $x \in \mathbb{Z}_{p^s}^n \setminus \{\mathbf{0}\}$, denoted by $\nu(x)$, as the largest natural number k such that $x = p^k y$, where y is another nonzero element of $\mathbb{Z}_{p^s}^n$. For $x = \mathbf{0}$, we define its valuation as $\nu(\mathbf{0}) = \infty$.

For $x \neq \mathbf{0}$, $\mathrm{o}(x) = p^{s-\nu(x)}$.

Example

For $x = (0,1) \in \mathbb{Z}_4^2$, we have that $\mathrm{o}(x) = 4$ and $\nu(x) = 0$.
For $y = (2,2) \in \mathbb{Z}_4^2$, we have that $\mathrm{o}(y) = 2$ and $\nu(y) = 1$.

A Griesmer-type bound for $\mathbb{Z}_{p^s}$-additive codes

The minimum Hamming distance of a $\mathbb{Z}_{p^s}$-additive code $\mathcal{C}$ is

$$d_H(\mathcal{C}) = \min\{w_H(c) \mid c \in \mathcal{C}\backslash\{\mathbf{0}\}\}.$$

Theorem

Let $\mathcal{C}$ be a $\mathbb{Z}_{p^s}$-additive code of type $(n; t_1, \dots, t_s)$ and minimum Hamming distance $d_H(\mathcal{C})$, and let $k = \sum_{j=1}^{s} t_j$. Then, we have that

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H(\mathcal{C})}{p^i} \right\rceil.$$

When this bound is met, the code $\mathcal{C}$ is optimal for the Griesmer-type bound.

K. Shiromoto and L. Storme, A Griesmer bound for linear codes over finite quasi-Frobenius rings. *Discrete Applied Mathematics*, 128, no. 1 pp. 263-274.

The generalized Gray map

The **generalized Gray map** $\Phi_s : \mathbb{Z}_{p^s} \longmapsto \mathbb{Z}_p^{p^{s-1}}$ is defined as follows:

$$\Phi_s(u) = (u_0, \ldots, u_{s-2}, u_{s-1})M_{s-1},$$

where $[u_0, u_1, \ldots, u_{s-1}]_p$ is the p-ary expansion of $u \in \mathbb{Z}_{p^s}$, $u = \sum_{i=0}^{s-1} u_i p^i$ with $u_i \in \{0, \ldots, p-1\}$, and

$$M_{s-1} = \begin{pmatrix} Y_{s-1} \\ \mathbf{1} \end{pmatrix}, \tag{1}$$

where Y_{s-1} is the matrix of size $s-1 \times p^{s-1}$ over $\mathbb{Z}_p$ such that its columns are all the elements of $\mathbb{Z}_p^{s-1}$. We define Φ_s over $\mathbb{Z}_{p^s}^n$ component-wise.

Example

The generalized Gray map for $\mathbb{Z}_8$ is $\Phi_3 : \mathbb{Z}_8 \longrightarrow \mathbb{Z}_2^4$:

$$M_2 = \begin{pmatrix} 0011 \\ 0101 \\ 1111 \end{pmatrix}$$

$$\begin{array}{lcl@{\qquad}lcl} 0 = [000]_2 & \longmapsto & 0000 & 4 = [001]_2 & \longmapsto & 1111 \\ 1 = [100]_2 & \longmapsto & 0011 & 5 = [101]_2 & \longmapsto & 1100 \\ 2 = [010]_2 & \longmapsto & 0101 & 6 = [011]_2 & \longmapsto & 1010 \\ 3 = [110]_2 & \longmapsto & 0110 & 7 = [111]_2 & \longmapsto & 1001 \end{array}$$

Outline

Construction of $\mathbb{Z}_{p^s}$-additive simplex codes: type α

Let G_k^α be the matrix consisting of all possible distinct columns.

$$G_1^\alpha = \begin{pmatrix} 0 & 1 & 2 & \cdots & p^s-1 \end{pmatrix} \quad \text{and}$$

$$G_k^\alpha = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \cdots & \mathbf{p^s-1} \\ G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & \cdots & G_{k-1}^\alpha \end{pmatrix}.$$

The code generated by G_k^α, denoted by S_k^α, is called a $\mathbb{Z}_{p^s}$**-additive simplex code of type** $\boldsymbol{\alpha}$ with $k \geq 1$ generators.

Example

Consider $\mathbb{Z}_3$. Then, we have that

$$G_1^\alpha = (012), \quad G_2^\alpha = \begin{pmatrix} 000 & 111 & 222 \\ 012 & 012 & 012 \end{pmatrix}.$$

Construction of $\mathbb{Z}_{p^s}$-additive simplex codes: type β

Let G_k^β be the matrix constructed recursively as follows:

$$G_1^\beta = (1)\,, \quad G_2^\beta = \left(\begin{array}{ccccc} \mathbf{1} & 0 & p & \cdots & p^s - p \\ 0\,1\,2\,\ldots\,p^s-1 & 1 & 1 & \cdots & 1 \end{array} \right),$$

$$G_k^\beta = \left(\begin{array}{cccc} \mathbf{1} & \mathbf{0} & \cdots & \mathbf{p^s - p} \\ G_{k-1}^\alpha & G_{k-1}^\beta & \cdots & G_{k-1}^\beta \end{array} \right).$$

The code generated by G_k^β, denoted by S_k^β, is called a $\mathbb{Z}_{p^s}$-**additive simplex code of type $\boldsymbol{\beta}$** with $k \geq 1$ generators.

Example

Consider $\mathbb{Z}_9$. Then, we have that

$$G_1^\beta = (1)\,,$$

$$G_2^\beta = \left(\begin{array}{cccc} 111111111 & 0 & 3 & 6 \\ 012345678 & 1 & 1 & 1 \end{array} \right).$$

Results for $\mathcal{S}_k^\alpha$

Theorem

- S_k^α *has type* $(p^{sk}; k, 0, \ldots, 0)$.
- *For all* $c \in S_k^\alpha \backslash \{\mathbf{0}\}$, $w_H(c) = p^{sk} - p^{\nu(c)} p^{s(k-1)}$, *so*

  $$d_H(S_k^\alpha) = (p-1)p^{sk-1}.$$

  For every valuation $j \in \{0, 1, \ldots, s-1\}$, *there are* $p^{k(s-j)} - p^{k((s-j)-1)}$ *codewords of weight* $p^{sk} - p^j p^{s(k-1)}$.
- *For all* $c \in S_k^\alpha \backslash \{\mathbf{0}\}$, $w_{Hom}(c) = p^{s(k+1)-2}(p-1)$, *so*

  $$d_{Hom}(S_k^\alpha) = p^{s(k+1)-2}(p-1).$$

- $\mathcal{S}_k^\alpha$ *is not optimal for the Griesmer-type bound, for any prime* p *and* $s \geq 1$.

Results for $\mathcal{S}_k^\beta$

Theorem

- $\mathcal{S}_k^\beta$ *has type* $(p^{(s-1)(k-1)}\frac{p^k-1}{p-1}; k, 0, \dots, 0)$.
- *For all* $c \in S_k^\beta \backslash \{\mathbf{0}\}$, $w_H(c) = p^{(s-1)(k-1)}\frac{p^k-1}{p-1} - p^{\nu(c)}(p^{(s-1)(k-2)}\frac{p^{k-1}-1}{p-1})$, *so*
$$d_H(\mathcal{S}_k^\beta) = p^{s(k-1)}.$$
For every valuation $j \in \{0, 1, \dots, s-1\}$, *there are* $p^{k(s-j)} - p^{k((s-j)-1)}$ *codewords of weight* $p^{(s-1)(k-1)}\frac{p^k-1}{p-1} - p^j(p^{(s-1)(k-2)}\frac{p^{k-1}-1}{p-1})$.
- *For* $c \in \mathcal{S}_k^\beta$, $w_{Hom}(c) = \begin{cases} p^{sk-1} & \text{if } \nu(c) = s-1, \\ p^{sk-k-1}(p^k-1) & \text{if } \nu(c) < s-1, \\ 0 & \text{otherwise}, \end{cases}$
so
$$d_{Hom}(\mathcal{S}_k^\beta) = p^{sk-k-1}(p^k-1).$$
- $\mathcal{S}_k^\beta$ *is optimal for the Griesmer-type bound, for every prime* p *and* $s \geq 1$.

Results for $\Phi_s(\mathcal{S}_k^\alpha)$ and $\Phi_s(\mathcal{S}_k^\beta)$

The generalized Gray map is an isometry $\Phi_s : (\mathbb{Z}_{p^s}^n, d_{Hom}) \longmapsto (\mathbb{Z}_p^{p^{s-1}n}, d_H)$. Moreover, generalized Gray map images of any linear code are distance invariant, as $d(\Phi_s(x), \Phi_s(y)) = w_H(\Phi_s(x-y))$ for any $x, y \in \mathbb{Z}_{p^s}^n$.

Theorem

- *The code $\Phi_s(\mathcal{S}_k^\alpha)$ is a $(p^{s(k+1)-1}, p^{sk}, p^{s(k+1)-2}(p-1))$ code over $\mathbb{Z}_p$ having all codewords of the same Hamming weight equal to $p^{s(k+1)-2}(p-1)$, except the all-zero codeword.*
- *The code $\Phi_s(\mathcal{S}_k^\beta)$ with $k \geq 2$ is a $(p^{(s-1)k}\frac{p^k-1}{p-1}, p^{sk}, p^{sk-k-1}(p^k-1))$ code over $\mathbb{Z}_p$, having p^k-1 codewords of Hamming weight $p^{sk}-1$, $p^{sk}-p^k$ codewords of Hamming weight $p^{sk-k-1}(p^k-1)$ and one of Hamming weight zero.*

Construction of $\mathbb{Z}_{p^s}$-additive MacDonald codes: type α

Let $k \geq 1$ and $1 \leq u \leq k-1$. We define $G^\alpha_{k,u}$ as follows:

$$G^\alpha_{k,u} = \left(\begin{array}{ccc} G^\alpha_k & \backslash & \frac{\mathbf{0}}{G^\alpha_u} \end{array} \right) = \left(\begin{array}{cccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \cdots & \mathbf{p^s-1} \\ G^\alpha_{k-1,u} & G^\alpha_{k-1} & G^\alpha_{k-1} & G^\alpha_{k-1} & \cdots & G^\alpha_{k-1} \end{array} \right),$$

where $(A \backslash B)$ is the matrix A by deleting the columns of B.

The code generated by $G^\alpha_{k,u}$ is denoted by $M^\alpha_{k,u}$ and is called a **$\mathbb{Z}_{p^s}$-additive MacDonald code of type α** with $k \geq 1$ generators.

Example

Take $\mathbb{Z}_4$. Then, we have that

$$G^\alpha_{2,1} = \left(\begin{array}{cccc} \cancel{0000} & 1111 & 2222 & 3333 \\ \cancel{0123} & 0123 & 0123 & 0123 \end{array} \right).$$

Construction of $\mathbb{Z}_{p^s}$-additive MacDonald codes: type β

Let $k \geq 2$ and $1 \leq u \leq k-1$. We define $G^{\beta}_{k,u}$ as follows:

$$G^{\beta}_{k,u} = \left(\begin{array}{ccc} G^{\beta}_{k} & \backslash & \frac{\mathbf{0}}{G^{\beta}_{u}} \end{array} \right) = \left(\begin{array}{cccccc} \mathbf{1} & \mathbf{0} & \mathbf{p} & \mathbf{2p} & \cdots & \mathbf{p^s - p} \\ G^{\alpha}_{k-1} & G^{\beta}_{k-1,u} & G^{\beta}_{k-1} & G^{\beta}_{k-1} & \cdots & G^{\beta}_{k-1} \end{array} \right),$$

The code generated by $G^{\beta}_{k,u}$ is denoted by $M^{\beta}_{k,u}$ and is called a **$\mathbb{Z}_{p^s}$-additive MacDonald code of type β** with $k \geq 2$ generators.

Example

Consider $\mathbb{Z}_4$. Then, we have that

$$G^{\beta}_{3,1} = \left(\begin{array}{cccccc} 1111 & 1111 & 1111 & 1111 & 0000\cancel{0}0 & 222222 \\ 0000 & 1111 & 2222 & 3333 & 1111\cancel{0}2 & 111102 \\ 0123 & 0123 & 0123 & 0123 & 0123\cancel{1}1 & 012311 \end{array} \right).$$

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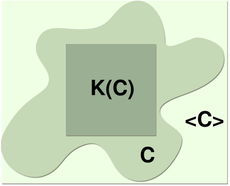
4. Future research

Future research

- Generalize the results for simplex codes defined over any finite chain ring.
- Study the weight distributions of $M^{\alpha}_{k,u}$ and $M^{\beta}_{k,u}$.
- Study the linearity of $\Phi_s(S^{\alpha}_k)$, $\Phi_s(S^{\beta}_k)$, as well as their kernel and rank.



Thank you for your attention!