Complex projective 4-designs as orbits of Clifford-Weil groups

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Low Rank Matrix Recovery

$X \in \mathcal{H}_2$ Hermitian of low rank

$$\mathcal{A}(X) + \varepsilon = \begin{pmatrix} 4.57782762 \\ 3.30686302 \\ 3.30686302 \end{pmatrix}$$

$$\mathcal{A} : \mathcal{H}_2 \to \mathbb{R}^3,\ Y \mapsto \sum_{j=1}^{3} \operatorname{tr}(YA_j)e_j \text{ with } A_j \in \mathcal{H}_2\ (j \in \underline{3})$$

Low Rank Matrix Recovery

$X \in \mathcal{H}_n$ Hermitian of low rank

$$\mathcal{A}(X) + \varepsilon = b$$

$$\mathcal{A} : \mathcal{H}_n \to \mathbb{R}^m,\ Y \mapsto \sum_{j=1}^{m} \mathrm{tr}(YA_j)e_j \text{ with } A_j \in \mathcal{H}_n \ (j \in \underline{m})$$

$$\underset{Z \in \mathcal{H}_n}{\text{minimize}} \|Z\|_* \text{ subject to } \|\mathcal{A}(Z) - b\|_{l_2} \leq \eta$$

Q: suitable measurements

Theorem (R. Kueng, H. Rauhut, U. Terstiege)

- $D := \{\varphi_i, w_i\}_{i=1}^N$ weighted 4-design
- $A_j := \sqrt{n(n+1)} a_j a_j^*$, a_j drawn from D, $j \in \underline{m}$
- $X \in \mathcal{H}_n, \mathrm{rk}(X) \leq r$

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$X^\#$ solution of

$$\underset{Z \in \mathcal{H}_n}{\text{minimize}} \|Z\|_* \text{ subject to } \|\mathcal{A}(Z) - b\|_{l_2} \leq \eta$$

If

$$m \geq C_4 nr \log(n),$$

then $\|X - X^\#\|_2 \leq \frac{C_6 \eta}{\sqrt{m}}$ with probability at least $1 - e^{-C_5 m}$

Q: suitable measurements

Weighted complex projective **t**-designs
$\{\varphi_1, ..., \varphi_N\} \subset \Omega_n$, $\{w_1, ..., w_N\} \subset \mathbb{R}^N_{\geq 0}$ s.t. $\sum_{i=1}^N w_i = 1$

$$\sum_{i=1}^{N} w_i p(\varphi_i) = \int_{\Omega_n} p(\varphi) d\varphi \ \forall p \in \mathrm{Hom}_{(t,t)(\mathbb{C}^n)}$$

where $\mathrm{Hom}\,(t,t)(\mathbb{C}^n) \leq \mathbb{C}[x_1, ..., x_n, \overline{x}_1, ..., \overline{x}_n]$

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$G < U(n)$ finite, $\varphi \in \Omega_n$

$$\sum_{\psi \in G\varphi} \frac{1}{|G\varphi|} p(\psi) = \frac{1}{|G|} \sum_{g \in G} p(g\varphi) = \left(\frac{1}{|G|} \sum_{g \in G} p^g \right) (\varphi).$$

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A.Roy

$$\mathrm{Hom}_{(t,t)}(\mathbb{C}^n) = \mathop{\perp}_{i=0}^{t} Z^i \, \mathrm{Harm}_{(t-i,t-i)}(\mathbb{C}^n),$$

$Z : \mathbb{C}^n \to \mathbb{R}, \mathbf{x} \mapsto x_1\overline{x}_1 + \ldots + x_n\overline{x}_n,$
Laplacian $\Delta := \sum \frac{\delta^2}{\delta x_i \delta \overline{x}_i}$
inner product $\langle f, g \rangle_c := \int_{\Omega_n} \overline{f(\varphi)} g(\varphi) d\varphi.$

$p(\varphi) = 0 \ \forall p \in \mathrm{Harm}_{(i,i)}(\mathbb{C}^n)^G, i \in \underline{t}$

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Gleason 1970

Types I,**II**,III $\quad \mathrm{hwe}(C) \in \mathrm{Inv}(G_T)$

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Nebe, Rains, Sloane 2006

Types $\rho \qquad \mathrm{Inv}(\mathcal{C}_m(\rho)) = <\mathrm{cwe}_m(C) | C \text{ of Type } \rho >$

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Grassl, Gross, Kueng, Zhu 2016 & Bannai, Oura, Zhao

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Nebe, Rains, Sloane 2006

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Grassl, Gross, Kueng, Zhu 2016 & Bannai, Oura, Zhao

$$\mathrm{II}^{(\mathrm{id},^{-})} \qquad \mathrm{Inv}^{(\mathrm{id},^{-})}(\chi_m) =< \mathrm{ccwe}_m(C) | C \text{ of Type } \mathrm{II}^{(\mathrm{id},^{-})} >$$

$$\mathrm{Harm}_{(4,4)}^{\chi_m} =< H_m >$$

$p(\varphi) = 0 \; \forall p \in \mathrm{Harm}_{(i,i)}(\mathbb{C}^n)^G, i \in \underline{t}$

Grassl, Gross, Kueng, Zhu 2016 & Bannai, Oura, Zhao

$\mathrm{II}^{(\mathrm{id},\overline{\ })}$ $\quad \mathrm{Inv}^{(\mathrm{id},\overline{\ })}(\chi_m) =< \mathrm{ccwe}_m(C) | C$ of Type $\mathrm{II}^{(\mathrm{id},\overline{\ })} >$

$\mathrm{Harm}_{(4,4)}^{\chi_m} =< H_m >$

Nebe, S. 2023

$\rho^{(a_1,\ldots,a_N)}$ $\quad K[\mathbf{x}^{(m)} \circ \Gamma]_{d_1,\ldots,d_n}^{\mathcal{C}_m(\rho)} =< ccwe_m(C) \mid C \in R >$

$\mathbb{F}_5$ $(1, 2)$

Euclidean selfdual Codes over $\mathbb{F}_5$

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$K := \mathbb{Q}(\zeta_5)$, $\alpha_1 = \mathrm{id}, \alpha_2$ induced by $\zeta_5 \mapsto \zeta_5^2$.

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The corresponding codes $V \leq \mathbb{F}_5^{(N_1+N_2)}$ satisfy

$$V = \left\{ y \in \mathbb{F}_5^{(N_1+N_2)} \middle| \sum_{i=1}^{N_1} y_i v_i + \sum_{j=N_1+1}^{N_2} 2 y_j v_j = 0 \ \forall v \in V \right\}$$

and we consider

$$\mathbb{Q}(\zeta_5)[x_i, x_i^{\alpha_2} | i \in \mathbb{F}_5].$$

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Further recommended sights

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Kneser

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Homotopy Continuation

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Kneser

Homotopy Continuation

The Clifford group fails gracefully to be a unitary 4-design
Grassl, Gross, Kueng, Zhu 2016

$$\begin{pmatrix} 0.999 + 0.000\,i & -0.999 - 1.809e - 06\,i \\ -0.999 - 1.809e - 06\,i & 0.999 + 0.000e + 00\,i \end{pmatrix}$$

$$\begin{pmatrix} 0.999 + 0.000\,i & -0.999 - 1.809e - 06\,i \\ -0.999 - 1.809e - 06\,i & 0.999 + 0.000e + 00\,i \end{pmatrix}$$

https://git-ce.rwth-achen.de/leonie.scheeren/t_DesignsFromOrbits.jl

/lowRankMatrixRecovery.jl